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ON SOME THEOREMS OF LITTLEWOOD AND SELBERG III

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1. Introduction

In paper II with the same title [2] we proved some unconditional results about $\zeta(s)$ in a more general set up. In this paper we continue these investigations. As before we begin by stating the final result of this paper as follows.

Theorem 1. *Let* $s = \sigma + it$ *and*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1} \tag{1}$$

where p *runs over all primes and* $\omega(p)$ *are arbitrary complex numbers (independent of* s*) with absolute value not exceeding* 1. *Suppose* α *and* δ *are positive constants satisfying* $\frac{1}{2} \le \alpha \le 1 - \delta$ *and that in* $\{\sigma \ge \alpha - \delta, T - H \le t \le T + H\}$, $F(s)$ *can be continued analytically and there* $|F(s)| < T^A$. *Here* A *is a positive constant,* $T \ge T_0$, $H = C \log\log\log T$ *where* T_0 *and* C *are large positive constants. Let* $F(s) \ne 0$ *in* $\{\sigma > \alpha, T - H \le t \le T + H\}$. *Then for* $\alpha \le \sigma \le \alpha + C_1(\log\log T)^{-1}$, *and* $T - \frac{1}{2}H \le t \le T + \frac{1}{2}H$, *we have,*

(a) $\log|F(\sigma + it)|$ *lies between* $C_2(\log T)(\log\log T)^{-1}$ *and* $-C_3(\log T)(\log\log T)^{-1} \log\{C_4((\sigma - \alpha)\log\log T)^{-1}\}$ *and*

(b) $|\arg F(\sigma + it)| \le C_5(\log T)(\log\log T)^{-1}$, *where* C_1, C_2, C_3, C_4 *and* C_5 *are certain positive constants.*

Corollary 1. *For* $\alpha + C_1(\log\log T)^{-1} \le \sigma \le 1 - \delta$, $t = T$, *we have,*

$$|\log F(\sigma + it)| \le C_6(\log T)^{\theta}(\log\log T)^{-1}$$

where $\theta = (1 - \sigma)/(1 - \alpha)$ *and* C_6 *is a positive constant.*

Corollary 2. *For* $\alpha \le \sigma \le 1 - \delta$, $t = T$, *we have,*

$$|F(\sigma + it)| \le \exp(C_7(\log T)^{\theta}(\log\log T)^{-1}),$$

where θ *is as before and* C_7 *is a positive constant.*

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Remark 1. The application of Theorem 1 to $\zeta(s)$ is clear by density results. Under the conditions of the theorem we can also prove density theorems for $F(s)$.

Remark 2. We now indicate the proof of the corollaries. Corollary 1 is already proved in [2] for the σ-range $\alpha+\delta \le \sigma \le 1-\delta$. The theorem above gives an upper bound $|\log F(\sigma+it)| \le C_8(\log T)(\log\log T)^{-1}$ for $\sigma=\sigma_1=\alpha+(\log\log T)^{-1}$ and $T-H/3 \le t \le T+H/3$ (where $C_8>0$ is a constant). We have already an upper bound $|\log F(\sigma+it)| \le C_8(\log T)^{\theta}(\log\log T)^{-1}$ for $\sigma=\sigma_2=1-\delta$ and the same t-range. We now apply maximum modulus principle to the function (for suitable $X>0$)

$$\varphi(w)=\big(\log F(s+w)\big)X^{w}\exp\left(\left(\sin\frac{w}{100}\right)^2\right).$$

According to this its absolute value at $w=0$, namely $|\log F(s)|$ is majorised by its maximum modulus on the boundary of the rectangle $\{\sigma_1-\sigma \le \operatorname{Re} w \le \sigma_2-\sigma, -H/10 \le \operatorname{Im} w \le H/10\}$. Corollary 1 follows by a proper choice of X as a suitable power of $\log T$. (The bound for $|\log F(s+w)|$ namely $O\big((\log T)^{20}\big)$, needed on the horizontal sides of the rectangle can be obtained by Borel–Carathéodory theorem). This completes the proof of Corollary 1. Corollary 2 follows from Corollary 1 and the part (a) of the theorem.

By a modification of our proof of Theorem 1 we can prove

Theorem 2. *Let $1-\delta \le \alpha \le 1-10(\log\log T)^{-1}$. Then for $\alpha \le \sigma \le \alpha+(\log\log T)^{-1}$ (in place of $\big(\alpha \le \sigma \le \alpha+C_1(\log\log T)^{-1}\big)$) the assertions* (a) *and* (b) *hold, provided $F(s)\ne 0$ in the region mentioned in Theorem 1.*

From Theorems 1 and 2 we can prove by the methods of [3] theorems like

Theorem 3. *Let λ_0 $(> 7/12)$ be a constant. Then for any fixed k, the number of lattice points $(n_1, n_2, \ldots, n_k)$ in the first quadrant of the k dimensional Euclidean space such that*

$$X \le n_1 \cdots n_k \le X + X^{\lambda_0}$$

is given by

$$X^{\lambda_0}(\log X)^{k-1} + O(X^{\lambda_0}(\log X)^{k-2}).$$

Remark. In fact we can prove an asymptotic formula valid uniformly for $k \le \varepsilon(\log\log X)(\log\log\log X)^{-1}$. These and similar results will form the subject matter of another paper. Theorem 3 is due to M.N. Huxley and C. Hooley (unpublished).

In what follows we will prove only Theorem 1.

2. Notation

We use $z = x + iy$, $w = u + iv$ and $s = \sigma + it$ in various contexts and we hope that this does not cause confusion. For any analytic function $F(s)$ we write $(F'/F)(s)$ for $F'(s)/F(s)$. The symbol $\equiv$ denotes a definition.

3. Proof of Theorem 1

It suffices to prove the theorem for $t = T$. Because we can consider a larger H, and every point of the smaller interval $T - \frac{1}{2}H \le t \le T + \frac{1}{2}H$ will be a mid point τ of a bigger interval of the type $\tau - \frac{1}{2}H \le t \le \tau + \frac{1}{2}H$ contained in $[T - H, T + H]$. We split the proof into three parts. The first two parts deal with an upper bound for a positive quantity J_0 in the form $O\big((\log T)(\log\log T)^{-1}\big)$. The third part deals with an application of this result to the proof of the theorem.

Lemma 1. *Let $z = x + iy$ be a complex variable and*

$$F(z) = \sum_{n=1}^{\infty} a_n n^{-z} = \prod_p \left(1 - \omega(p) p^{-z}\right)^{-1} \tag{2}$$

where p runs over all the primes and $\omega(p)$ are complex numbers independent of z with $|\omega(p)| \le 1$. Let $F(z)$ be regular in $\{x \ge \alpha - \delta, T - H \le y \le T + H\}$ and there $|F(z)| < T^A$ where $A > 0$ is a constant. Here $T \ge T_0$, $\frac{1}{2} \le \alpha \le 1 - \delta$, $H = C \log\log\log T$ where δ is a small positive constant, α is a constant and T_0 and C are large positive constants. Put $z_0 = 2 + iy$. Then for $z_1 = x_0 + iy_0$ where $T - \frac{1}{2}H \le y_0 \le T + \frac{1}{2}H$ with $\alpha - \delta_1 \le x_0 \le 2$, we have

$$\left| \frac{F'}{F}(z_1) - \sum_{\rho \in D} \frac{1}{z_1 - \rho} \right| \ll \log T, \tag{3}$$

where ρ runs over all the zeros of $F(z)$ in the disc $D = D(z_0, 2 - \alpha + 2\delta_1)$ defined by

$$|z - z_0| \le 2 - \alpha + 2\delta_1. \tag{4}$$

Here δ_1 is any positive constant. We will suppose $11\delta_1 < \delta$.

Proof. This is Lemma 3 of [2].

Lemma 2. *Under the conditions of Lemma 1, we have,*

$$\operatorname{Re} \frac{F'}{F}(x_0 + iy_0) = \sum_{\rho \in D} \frac{x_0 - \beta}{(x_0 - \beta)^2 + (y_0 - \gamma)^2} + O(\log T) \tag{5}$$

where we have written $\rho = \beta + i\gamma$. This holds in particular for $\alpha \le x_0 \le 2$.

4. Another expression for the left-hand side of (5)

Lemma 3. *Let* $s = \sigma + it$ *where* $\alpha + (\log X)^{-1} = \sigma_1 \leq \sigma \leq 1 - \delta$ *and* $X = (\log T)^{\lambda}$ *with some positive constant* $\lambda < 1$. *Let* $B(\geq 100000)$ *be a constant. Then*

$$\text{(6)}\qquad \begin{aligned} I &\equiv \frac{1}{2\pi i}\int_{u=2} \frac{F'}{F}(s+w)\Big(\frac{X^{2w}-X^{w}}{w^2\log X}\Big)\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\,dw \\ &= O\big((\log T)^{\lambda}(\log\log T)^{-1}\big). \end{aligned}$$

Proof. We have,

$$\frac{1}{2\pi i}\int_{u=2}\Big(\frac{X}{n}\Big)^{w}\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w^2\log X} = (\log X)^{-1}\log\frac{X}{n} + O\Big(\frac{n}{X\log X}\Big),$$

if $n \leq X$ and

$$\frac{1}{2\pi i}\int_{u=2}\Big(\frac{X}{n}\Big)^{w}\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w^2\log X} = O\Big(\frac{X}{n\log X}\Big),$$

if $n \geq X$.

Hence for any constant ε $(0 < \varepsilon < 1)$, we have,

$$\frac{1}{2\pi i}\int_{u=2}\Big(\Big(\frac{X^2}{n}\Big)^{w} - \Big(\frac{X}{n}\Big)^{w}\Big)\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w^2\log X} = 1 + O\Big(\frac{n}{X\log X}\Big),$$

if $n \leq X$ and

$$\frac{1}{2\pi i}\int_{u=2}\Big(\Big(\frac{X^2}{n}\Big)^{w} - \Big(\frac{X}{n}\Big)^{w}\Big)\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w^2\log X} = O_{\varepsilon}\Big(\Big(\frac{X^2}{n}\Big)^{\varepsilon}(\log X^{-1})\Big),$$

if $X \leq n \leq X^2$ and

$$\frac{1}{2\pi i}\int_{u=2}\Big(\Big(\frac{X^2}{n}\Big)^{w} - \Big(\frac{X}{n}\Big)^{w}\Big)\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w^2\log X} = O\Big(\frac{X^2}{n\log X}\Big),$$

if $n \geq X^2$.

Thus if $1 - \delta + \varepsilon < 1$ we obtain

$$\begin{aligned} I &= O\Big(\sum_{n\leq X}\frac{\Lambda(n)}{n^{\sigma}} + \sum_{X\leq n\leq X^2}\Big(\frac{X^2}{n}\Big)^{\varepsilon}\frac{\Lambda(n)}{n^{\sigma}\log X} + \sum_{n\geq X^2}\frac{X^2\Lambda(n)}{n^{1+\sigma}\log X}\Big) \\ &= O(X^{2-2\sigma}(\log X)^{-1}), \end{aligned}$$

where we have used $\Lambda(n) = \log p$ if $n = p^m, 0$ otherwise. This proves the lemma since $2\lambda(1-\sigma) \leq \lambda$.

Lemma 4. *Let $3V$ be asymptotic to H and $|v| \leq V'$ (V' will be chosen to be asymptotic to V). Then for (fixed $s = \sigma + it$ and all $w = u + iv$), $u + \sigma \geq \alpha - \delta_1$, we have,*

$$\left| \frac{F'}{F}(s+w) - \sum_{\rho} \frac{1}{s+w-\rho} \right| \ll \log T, \tag{7}$$

where ρ runs over all the zeros of $F(z)$ in the disc $D = D(z_0, 2 - \alpha + 2\delta_1)$ defined by $|z - z_0| \leq 2 - \alpha + 2\delta_1$ where $z_0 = 2 + it + iv$.

Proof. The proof follows from Lemma 1.

Lemma 5. *Let*

$$\mu(\rho) = \frac{2^{s+w-\rho} - 1}{(s+w-\rho)^2 \log 2} \tag{8}$$

and

$$\mu = \sum_{\rho} \mu(\rho), \tag{9}$$

where ρ runs over all the zeros of $F(z)$ in "the rectangle" R defined by

$$R: \left\{ \operatorname{Re} z \geq \alpha - 2\delta_1, |t - y| \leq 2V \right\}. \tag{10}$$

Then for $|v| \leq V'$ (V' will be chosen asymptotic to V) and $u + \sigma \geq \alpha - \delta_1$, we have,

$$\left| \frac{F'}{F}(s+w) - \mu \right| \ll \log T. \tag{11}$$

Proof. This is Lemma 6 of [2].

Lemma 6. *It is possible to choose V' (asymptotic to V) such that on $v = \pm V'$ and $u + \sigma \geq \alpha - 10\delta_1$, we have,*

$$\left| \sum_{\rho \in R} \mu(\rho) \right| \ll (\log T)^2. \tag{12}$$

Proof. This is Lemma 7 of [2].

Remark. From now on we assume that $F(z) \neq 0$ in $\{x > \alpha, T - H \leq y \leq T + H\}$.

Lemma 7. *We have,*

$$I = \frac{F'}{F}(s) + I_1 + I_2 + S + O\big((\log T)^{\lambda}(\log\log T)^{-1}\big), \tag{13}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{u=\alpha-\sigma-\frac{1}{2}\delta_1}\Big(\frac{F'}{F}(s+w) - \mu\Big)\Big(\frac{X^{2w} - X^{w}}{w^2\log X}\Big)\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\,dw \tag{14'}$$

$$I_2 = \frac{1}{2\pi i}\int_{u=\alpha-\sigma-10\delta_1}\mu\Big(\frac{X^{2w} - X^{w}}{w^2\log X}\Big)\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\,dw \tag{14''}$$

and

$$S = \sum_{\rho\in R}\Big(\frac{X^{2(\rho-s)} - X^{\rho-s}}{(\rho-s)^2\log X}\Big)\exp\Big(\Big(\sin\Big(\frac{\rho-s}{B}\Big)\Big)^2\Big). \tag{14'''}$$

The two integrals in (14) *are subject to* $|v| \le V'$.

Proof. The proof follows by Cauchy's theorem of residues.

Lemma 8. *For* $\alpha + (\log X)^{-1} = \sigma_1 \le \sigma \le 1-\delta$, *we have,*

$$I_1 = O\Big(\big(X^{2\alpha-2\sigma-\delta_1} + X^{\alpha-\sigma-\delta_1/2}\big)\frac{\log T}{\log\log T}\Big), \tag{15}$$

$$I_2 = O\Big(\big(X^{2\alpha-2\sigma-2\delta_1} + X^{\alpha-\sigma-10\delta_1}\big)\frac{\log T}{\log\log T}\Big) \tag{16}$$

and

$$S = \sum_{\rho\in D}\frac{X^{2\rho-2s} - X^{\rho-s}}{(\rho-s)^2\log X} + O\Big(\frac{\log T}{\log\log T}\Big). \tag{17}$$

Proof. The estimates (15) and (16) follow from (11) and the fact that on $u = \alpha-\sigma-10\delta_1$ we have $\mu = O(\log T)$ since $|s+w-\rho| \ge \delta_1$ on this line.

Lemma 9. *For* $\alpha + (\log X)^{-1} = \sigma_1 \le \sigma \le 1-\delta$, *we have,*

$$I_1 = O\Big(\frac{\log T}{\log\log T}\Big), \qquad I_2 = O\Big(\frac{\log T}{\log\log T}\Big) \tag{18}$$

and

$$S_0 = \omega\sum_{\rho\in D}\frac{X^{2\beta-2\sigma} + X^{\beta-\sigma}}{(\log X)\big((\sigma-\beta)^2 + (t-\gamma)^2\big)} \tag{19}$$

where S_0 *denotes the sum in* (17) *and* ω *is a complex number (depending on other parameters) with* $|\omega| \le 1$.

Proof. The proof follows from Lemma 8.

Lemma 10. *For* $\alpha + (\log X)^{-1} = \sigma_1 \le \sigma \le 1-\delta$, *we have,*

$$\sum_{\rho\in D}\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} = \omega' \sum_{\rho\in D}\frac{(X^{2\beta-2\sigma}+X^{\beta-\sigma})(\sigma_1-\alpha)}{(\sigma-\beta)^2+(t-\gamma)^2} + O(\log T) \tag{20}$$

where ω' *is real and* $|\omega'| \le 1$.

Proof. The proof follows from Lemmas 2, 3, 7, 8 and 9.

Lemma 11. *We have,*

$$J_0 \equiv \sum_{\rho\in D}\frac{\sigma_1-\beta}{(\sigma_1-\beta)^2+(t-\gamma)^2} = O(\log T). \tag{21}$$

Proof. Put $\sigma=\sigma_1$ in (20). We have $\beta-\sigma_1 \le \alpha-\sigma_1 = -(\log X)^{-1}$ and also $\sigma_1-\alpha \le \sigma_1-\beta$ and so

$$J_0 = \sum_{\rho\in D}\frac{\sigma_1-\beta}{(\sigma_1-\beta)^2+(t-\gamma)^2} = \omega''\Big(\frac{1}{e^2}+\frac{1}{e}\Big)J_0 + O(\log T),$$

where $|\omega''| \le 1$
This proves the lemma since $e^{-2}+e^{-1}<1$ and $|\omega''|\le 1$.

Lemma 12. *For* S_0 *defined by* (19) *we have,*

$$S_0 = w_1\sum_{\rho\in D}\Big(\frac{X^{2\beta-2\sigma}+X^{\beta-\sigma}}{(\sigma_1-\beta)^2+(t-\gamma)^2}\Big)(\sigma_1-\beta), \tag{22}$$

with $|\omega_1| \le 1$.

Proof. The lemma follows from $(\log X)^{-1} = \sigma_1-\alpha \le \sigma_1-\beta$ and also $\sigma \ge \sigma_1$.

Lemma 13. *With* $\sigma_1 = \alpha+(\log X)^{-1}$, *we have,*

$$\log F(\sigma_1+it) = O\Big(\frac{\log T}{\log\log T}\Big). \tag{23}$$

Proof. The lemma follows from

$$\int_{\sigma_1}^{1-\delta_1}\frac{F'}{F}(\sigma+it)\,dt = O\Big(\frac{\log T}{\log\log T}\Big) - \log F(\sigma_1+it)$$

and the fact that here the left-hand side is (by Lemmas 2, 3, 7, 8, 9, 11 and 12) $O\big((\log T)(\log\log T)^{-1}\big)$.

Lemma 14. *For $\alpha \le \sigma \le \sigma_1$, we have,*

$$(\sigma_1 - \sigma)\frac{F'}{F}(\sigma_1 + it) = O\Big(\frac{\log T}{\log\log T}\Big). \tag{24}$$

Proof. The proof follows from Lemmas 2, 3, 7, 8, 9, 11 and 12.

Lemma 15. *For $\alpha \le \sigma \le \sigma_1$ the quantity*

$$\int_\sigma^{\sigma_1} \operatorname{Re}\Big(\frac{F'}{F}(\sigma_1 + it) - \frac{F'}{F}(u + it)\Big)\, du \tag{25}$$

does not exceed $C_9(\log T)(\log\log T)^{-1}$, but exceeds

$$C_{10}(\log T)(\log\log T)^{-1} - C_{11}(\log T)(\log\log T)^{-1}\log\big((C_{12}(\log\log T)(\sigma-\alpha))^{-1}\big),$$

where C_9, C_{10}, C_{11} and C_{12} are positive constants.

Proof. By Lemma 2 the quantity in question is

$$\int_\sigma^{\sigma_1} \big(J(\sigma_1) - J(u)\big)du + O\Big(\frac{\log T}{\log\log T}\Big)$$

where

$$J(u) \equiv \sum_{\rho\in D} \frac{u-\beta}{(u-\beta)^2 + (t-\gamma)^2}.$$

Now

$$J(\sigma_1) - J(u) = \sum_{\rho\in D} \frac{(\sigma_1 - u)(t-\gamma)^2}{Y} + \sum_{\rho\in D} \frac{(\sigma_1-\beta)(u-\beta)(u-\sigma_1)}{Y}$$

where $Y = \big((u-\beta)^2 + (t-\gamma)^2\big)\big((\sigma_1-\beta)^2 + (t-\gamma)^2\big)$. Denote the two sums in $J(\sigma_1) - J(u)$ by $\sum_1$ and $\sum_2$. We have

$$\sum\nolimits_1 \le \sum_{\rho\in D} \frac{\sigma_1 - u}{(\sigma_1-\beta)^2 + (t-\gamma)^2}$$

and so

$$\begin{aligned}\int_\sigma^{\sigma_1} \sum\nolimits_1 du &\le \sum_{\rho\in D} \frac{\frac{1}{2}(\sigma_1-\sigma)^2}{(\sigma_1-\beta)^2 + (t-\gamma)^2} \\ &\le \tfrac{1}{2}(\log X)^{-1} \sum_{\rho\in D} \frac{\sigma_1 - \beta}{(\sigma_1-\beta)^2 + (t-\gamma)^2} \\ &= \tfrac{1}{2}J_0(\log X)^{-1} = O\big((\log T)(\log\log T)^{-1}\big).\end{aligned}$$

Now $\sum_2$ is negative and

$$-\int_{\sigma}^{\sigma_1} \sum\nolimits_2 du = \int_{\sigma}^{\sigma_1} \sum_{\rho\in D} \frac{(u-\beta)(\sigma_1-\beta)(\sigma_1-u)}{Y}\, du$$
$$\leq \int_{\sigma}^{\sigma_1} \sum_{\rho\in D} \Big(\frac{\sigma_1-u}{u-\alpha}\cdot\frac{(u-\beta)^2(\sigma_1-\beta)}{Y}\Big)\, du$$
$$\leq J_0(\log X)^{-1}\int_{\sigma}^{\sigma_1}\frac{du}{u-\alpha} = O\Big(\frac{\log T}{\log\log T}\log\frac{\sigma_1-\alpha}{\sigma-\alpha}\Big).$$

This proves the lemma.

Lemma 16. *For $\alpha \leq \sigma \leq \sigma_1$, we have,*

$$\int_{\sigma}^{\sigma_1} \operatorname{Im}\Big\{\frac{F'}{F}(\sigma_1+it) - \frac{F'}{F}(u+it)\Big\}\, du = O\Big(\frac{\log T}{\log\log T}\Big). \tag{26}$$

Proof. For $\alpha \leq \sigma \leq \sigma_1$ we see that the integrand is (apart from a term of the type $O(\log T)$), by Lemma 2,

$$\sum_{\rho\in D}\Big(\frac{-(t-\gamma)}{(\sigma_1-\beta)^2+(t-\gamma)^2} - \frac{-(t-\gamma)}{(u-\beta)^2+(t-\gamma)^2}\Big)$$
$$= \sum_{\rho\in D}\frac{(t-\gamma)((\sigma_1-\beta)^2-(u-\beta)^2)}{Y}$$

and hence its absolute value is

$$\leq \sum_{\rho\in D} |t-\gamma|(\sigma_1-u)(2\sigma_1-2\beta)Y^{-1}.$$

Hence the absolute value of the integral in question is

$$\leq \frac{1}{\log X}\sum_{\rho\in D}\Big\{\Big(\int_{\sigma}^{\sigma_1}\frac{|t-\gamma|}{(u-\beta)^2+(t-\gamma)^2}\, du\Big)\cdot$$
$$\cdot\Big(\frac{2(\sigma_1-\beta)}{(\sigma_1-\beta)^2+(t-\gamma)^2}\Big)\Big\}$$
$$\leq \frac{2}{\log X}\sum_{\rho\in D}\Big\{\Big(\int_{\beta}^{\infty}\frac{|t-\gamma|}{(u-\beta)^2+(t-\gamma)^2}\, du\Big)\cdot$$
$$\cdot\Big(\frac{\sigma_1-\beta}{(\sigma_1-\beta)^2+(t-\gamma)^2}\Big)\Big\}$$
$$= \frac{\pi J_0}{\log X} = O\Big(\frac{\log T}{\log\log T}\Big).$$

This proves the Lemma.

5. Proof of Theorem 1

From the results of Section 4 Theorem 1 follows from the identity (valid for $\alpha \le \sigma \le \sigma_1 = \alpha + (\log X)^{-1}$)

$$\log F(\sigma + it) = \log F(\sigma_1 + it) - (\sigma_1 - \sigma)\frac{F'}{F}(\sigma_1 + it) + \int_\sigma^{\sigma_1} K(u)\,du \tag{27}$$

where

$$K(u) \equiv \frac{F'}{F}(\sigma_1 + it) - \frac{F'}{F}(u + it),$$

just as in [1].

APPENDIX

1. We can in this paper replace $F(s)$ by any function

$$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s},$$

where $1 = \lambda_1 < \lambda_2 < \cdots$ is any increasing sequence of real numbers and $\{a_n\}$ with $a_1 = 1$ is any sequence of complex numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$$

is absolutely convergent at some point of the complex plane. The condition $\alpha \ge \frac{1}{2}$ is unimportant. Any α will do. The only change is in place of Lemma 3 we have (by fixing u to be large instead of $u = 2$) $I = O(X^{2u}(\log X)^{-1})$ since $(F'/F)(s+w)$ can be proved to be $O(1)$ for all u exceeding some suitable u_0. We can now choose $X = (\log T)^\lambda$, where λ is a sufficiently small positive constant. The rest of the proof is unaltered.

2. In our paper [2], in the condition $\alpha \ge \frac{1}{2}, \frac{1}{2}$ does not play any serious role, and the condition can be relaxed to any $\alpha \le 1 - \delta$.

3. In [2], we can (instead of the Euler product) work with the condition $F(1 + it) = O\big((\log T)^A\big)$ for $T - H \le t \le T + H$ where

$$H = C(\log\log T)(\log\log\log T).$$

For, it follows that in $(\sigma \ge 1, T - 3H/4 \le t \le T + 3H/4)$ we have $\operatorname{Re}\log F(s) \le C \log\log T$ (not the same C at all places) and by the Borel–Carathéodory theorem we can prove that in $(\sigma \ge 1 + 1/(\log\log T), T - \frac{1}{2}H \le t \le T + \frac{1}{2}H)$ there holds $\log F(s) = O\big((\log\log T)^2\big)$. Now we can apply convexity arguments to obtain a bound for $|\log F(s)|$ in $(\sigma \ge \alpha + C'/\log\log T, T - H/3 \le t \le T + H/3)$, not very different from the results proved in paper [2].

4. Without stating the most general results obtainable by the results of this paper, we can state results like this for example. Let

$$F(s) = \zeta(s) + \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where

$$\sum_{n \le x} a_n = O(x^{(1/2)-\delta})$$

where $\delta > 0$ is a constant. Let $T \ge T_0$. Then $F(s)$ has $\ge T^{1-\varepsilon}$ zeros in $(\sigma \ge \frac{1}{2} - C''/\log\log T, T \le t \le 2T)$ where C'' depends only on ε and other constant like δ. The same is also true of the function $F(s)$ defined in $\sigma > 0$ by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\lambda_n^s},$$

where $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and $\lambda_{n+1} - \lambda_n$ is both $\gg$ and $\ll 1$. (It may be noted that both a_n and λ_n can depend on T).

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