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FACTORS FOR $|\overline{N}, p_n|_k$ SUMMABILITY OF INFINITE SERIES

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Abstract. In this paper we prove a theorem on $|\overline{N}, p_n|_k$ summability factors, which generalizes a result of Sulaiman [3] on $|C,1|$ summability factors.

1. Introduction. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series of complex numbers with the sequence of partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \quad \text{as} \quad n \to \infty \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu, \qquad n \geq 0, \tag{1.2}$$

defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the series $\sum a_n$, generated by the sequence of coefficients (p_n) (see [2]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, where $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.3}$$

In the special case when $p_n = 1$ for all values of n (respectively $k = 1$), $|\overline{N}, p_n|_k$ summability is the same as $|C,1|_k$ (respectively $|\overline{N}, p_n|$) summability.

For any sequence (λ_n) we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Sulaiman [3] has proved the following theorem for $|C,1| = |C,1|_1$ summability factors of infinite series.

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Theorem A. *Let $(X_n)_{n\geq 0}$ be a given sequence of positive numbers and let*

$$s_n = O(X_n) \qquad \text{as} \quad n \to \infty. \tag{1.4}$$

If $(\lambda_n)_{n\geq 0}$ is a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} \frac{X_n|\lambda_n|}{n} < \infty, \tag{1.5}$$

$$\sum_{n=0}^{\infty} X_n|\Delta\lambda_n| < \infty, \tag{1.6}$$

then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $|C,1|$.

2. The aim of this paper is to generalize Theorem A for $|\overline{N},p_n|_k$ summability. Now, we shall prove the following theorem.

Theorem. *Let $(X_n)_{n\geq 0}$ be a given sequence of positive numbers and let condition (1.4) of Theorem A be satisfied. If $(\lambda_n)_{n\geq 0}$ is a sequence of complex numbers such that*

$$\sum_{n=0}^{\infty}(p_n/P_n)\big(|\lambda_n|X_n\big)^k < \infty \tag{2.1}$$

and the condition (1.6) of Theorem A is satisfied, then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $|\overline{N},p_n|_k$, where $k \geq 1$.

Remark. If we take $p_n = 1$ for all values of n in this theorem, then we get $|C,1|_k$ summability of the series $\sum a_n\lambda_n$; in particular, the case $k = 1$ yields Theorem A.

Proof. Let (T_n) be the sequence of the $(\overline{N},p_n)$ means of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{z=0}^{\nu} a_z\lambda_z = \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu\lambda_\nu, \qquad n \geq 0.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{\nu=0}^{n} P_{\nu-1}a_\nu\lambda_\nu.$$

Using Abel's transformation we get

$$\begin{aligned} T_n - T_{n-1} &= -\frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} p_\nu s_\nu \lambda_\nu + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} P_\nu s_\nu \Delta\lambda_\nu + \frac{p_n s_n \lambda_n}{P_n} \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say.} \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,z}|^k < \infty \qquad \text{for} \quad z = 1, 2, 3.$$

First, applying Hölder's inequality and using the fact that

$$\sum_{n=\nu+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O(1/P_\nu),$$

we get

$$\begin{aligned} \sum_{n=1}^{m+1} (P_n/p_n)^{k-1} |T_{n,1}|^k &\le \sum_{n=1}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \Big\{ \sum_{\nu=0}^{n-1} p_\nu |s_\nu|\, |\lambda_\nu| \Big\}^k \\ &= O(1) \sum_{n=1}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \Big\{ \sum_{\nu=0}^{n-1} p_\nu X_\nu |\lambda_\nu| \Big\}^k \\ &= O(1) \sum_{n=1}^{m+1} \frac{p_n}{P_n P_{n-1}} \Big\{ \sum_{\nu=0}^{n-1} p_\nu \big(X_\nu |\lambda_\nu|\big)^k \Big\} \times \Big\{ \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} p_\nu \Big\}^{k-1} \\ &= O(1) \sum_{\nu=0}^{m} p_\nu \big(X_\nu |\lambda_\nu|\big)^k \sum_{n=\nu+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{\nu=0}^{m} (p_\nu/P_\nu) \big(X_\nu |\lambda_\nu|\big)^k = O(1) \end{aligned}$$

as $m \to \infty$, by virtue of (1.4) and (2.1). Again, we have

$$\sum_{n=1}^{m+1}(P_n/p_n)^{k-1}|T_{n,2}|^k \leq \sum_{n=1}^{m+1} \frac{p_n}{P_n P_{n-1}^k}\Big\{\sum_{\nu=0}^{n-1} P_\nu |s_\nu|\,|\Delta\lambda_\nu|\Big\}^k$$
$$= O(1)\sum_{n=1}^{m+1} \frac{p_n}{P_n P_{n-1}^k}\Big\{\sum_{\nu=0}^{n-1} P_\nu X_\nu |\Delta\lambda_\nu|\Big\}^k$$
$$= O(1)\sum_{n=1}^{m+1} \frac{p_n}{P_n P_{n-1}}\Big\{\sum_{\nu=0}^{n-1} P_\nu X_\nu |\Delta\lambda_\nu|\Big\} \times \Big\{\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1} P_\nu X_\nu |\Delta\lambda_\nu|\Big\}^{k-1}$$
$$= O(1)\sum_{\nu=0}^{m} X_\nu |\Delta\lambda_\nu| = O(1)$$

as $m \to \infty$, by (1.4) and (1.6). Finally, we get

$$\sum_{n=1}^{m}(P_n/p_n)^{k-1}|T_{n,3}|^k = O(1)\sum_{n=1}^{m}(p_n/P_n)\big(X_n|\lambda_n|\big)^k = O(1)$$

as $m \to \infty$, by virtue of (1.4) and (2.1). This completes the proof of the theorem.

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