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ON THE CONFORMAL MODULUS DISTORTION UNDER QUASIMÖBIUS MAPPINGS

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0. Introduction

In this paper we shall study some properties of the topological embeddings $f\colon \Sigma \to \overline{R}^n$, Σ being a compact in $\overline{R}^n$, under which the distortion of conformal moduli of rings in Σ is of a bounded character. Such mappings have been termed ω-BMD embeddings, where ω denotes a bound for modulus distortion. Although the class of ω-BMD embeddings of continua in $\overline{R}^n$ is essentially equivalent to that of ω^*-quasimöbius embeddings, there are some problems concerning the modulus distortion function ω. Does the sequence of ω-BMD embeddings $f_k\colon \Sigma_k \to \overline{R}^n$ converge to BMD-embeddings with the same bound ω for the distortion of moduli? In Sections 2–3 the affirmative answer will be given in the case where Σ is a locally equiconnected sequence of continua or the limit continuum is a Jordan arc in $\overline{R}^n$. In Section 4 we give a counterexample for the negative answer in a general case. Section 5 aims to get an analogue of Liouville's theorem for BMD-embeddings.

1. BMD and QM embeddings

We equip the Möbius space $\overline{R}^n$ with the chordal distance $[xy]$. The conformal invariant characteristic $r(T)$ of a quadruplet (an ordered quadruple of distinct points) $T = abcd$ in $\overline{R}^n$ is defined by

$$(1.1)\qquad r(T) = [ab][cd]/\big([ac][bd]\big).$$

1.2. Definition ([Vä] or [As1]). Let $\omega\colon [0,+\infty) \to [0,+\infty)$ be a homeomorphism. An embedding $f\colon \Sigma \to \overline{R}^n$ of $\Sigma \subset \overline{R}^n$ into $\overline{R}^n$ is said to be a ω-QM (quasimöbius) embedding if $r(fT) \le \omega\big(r(T)\big)$ for all quadruplets T in Σ.

Given a pair of compact sets $E, F \subset \overline{R}^n$ and a domain $\mathcal{D} \subset \overline{R}^n$, let $M(E;F;\mathcal{D})$ denote the conformal modulus of the family of all arcs joining E to F in $\mathcal{D}$. A pair of disjoint continua E, F in $\overline{R}^n$ is called a ring. We set $M(E,F) = M(E,F;\overline{R}^n)$.

1.3. Definition ([As1]). An embedding $f\colon \Sigma \to \overline{R}^n$ of $\Sigma \subset \overline{R}^n$ is said to be ω-BMD (of bounded modulus distortion) if

$$(1.4)\qquad \omega^{-1}\big(M(E,F)\big) \le M(fE,fF) \le \omega\big(M(E,F)\big)$$

for all rings (E,F) on Σ.

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The connection between BMD and QM classes of embeddings has been ascertained [As1, Theorem 5.6, p. 22, Theorem 4.3, p. 12] as follows.

1.5. Theorem. (i) *Every* ω-QM *embedding* $f\colon \Sigma \to \overline{R}^n$ *is also* ω^*-BMD, *where* ω^* *depends only on* ω *and* n. (ii) *Every* ω-BMD *embedding* $f\colon \Sigma \to \overline{R}^n$ *of a continuum* $\Sigma \subset \overline{R}^n$ *is also* $\tilde{\omega}$-QM, *where* $\tilde{\omega}$ *depends only on* ω *and* n.

1.6. Remarks. The concept of QM embeddings of subsets in the plane is actually emplyed in S. Rickman's paper [R, p. 389]. These embeddings were termed as "quasimöbius" by J. Väisälä [Vä] and the author [As2] in 1984. The notion of BMD embeddings had been offered for investigation by P. Belinskij in 1976 and was introduced in [AsV1].

2. Convergence theorems

For a compact metric space $\mathcal{X}$ we shall denote by $\operatorname{Cont}\mathcal{X}$ the space of all continua in $\mathcal{X}$ equipped with Hausdorff distance between compact subsets of $\mathcal{X}$ (see [K, Chapter 2, Section 21]). The compactness of $\operatorname{Cont}\mathcal{X}$ [K, Chapter 4, Section 42] will be employed throughout the paper. An embedding $f\colon \Sigma \to \overline{R}^n$ of a continuum $\Sigma \subset \overline{R}^n$ may be associated with its graph in $\overline{R}^n \times \overline{R}^n$

$$\Gamma f = \{(x,y) \in \overline{R}^n \times \overline{R}^n : x \in \Sigma, y = fx\}$$

and thereby be considered an element of $\operatorname{Cont}(\overline{\mathrm{R}}^n \times \overline{\mathrm{R}}^n)$. We assume the convergence $f_k \to f$ of embeddings $f_k\colon \Sigma_k \to \overline{R}^n$ to be equivalent to the convergence $\Gamma f_k \to \Gamma f$ in the metric space $\operatorname{Cont}(\overline{\mathrm{R}}^n \times \overline{\mathrm{R}}^n)$. Since the characteristic $r(T)$ is continuous on the space of quadruplets in $\overline{R}^n$, we have the following property.

2.1. If $f_k \to f$ as $k \to \infty$, f_k being ω-QM, the limit embedding f is also ω-quasimöbius with the same distortion bound ω.

2.2. Definition [As3]. A family of embeddings $\mathcal{M} = \{f_\alpha\colon \Sigma_\alpha \to \overline{R}^n; \Sigma_\alpha \in \operatorname{Cont}\overline{\mathrm{R}}^n\}$ is called compact (in the class of embeddings) if any sequence in $\mathcal{M}$ has a subsequence converging to an embedding. The family $\mathcal{M}$ is termed normal if any sequence $\{f_k\} \subset \mathcal{M}$ has a subsequence $\{f_{k_s}\}$ such that $\Gamma f_{k_s} \to \Gamma$ in $\operatorname{Cont}(\overline{\mathrm{R}}^n \times \overline{\mathrm{R}}^n)$, Γ being either the graph of an embedding or a compact set containing none triple of points with distinct projections.

2.3. Theorem [As1, Theorem 6.1, p. 23.] *Given a homeomorphism* $\omega\colon [0,+\infty) \to [0,+\infty)$, *the family* $\mathcal{M} = \{f\colon \Sigma \to \overline{R}^n; \Sigma \in \operatorname{Cont}\overline{\mathrm{R}}^n\}$ *of all* ω-*quasimöbius embeddings is normal. Moreover, any subfamily* $\mathcal{M}' \subset \mathcal{M}$ *of* ω-BMD *embeddings with a common triple of fixed points is compact.*

According to 2.1 and Theorem 1.5, this immediately implies the following statement.

2.4. Theorem. *Given a homeomorphism* $\omega\colon [0,+\infty) \to [0,+\infty)$, *the family* $\mathcal{M} = \{f\colon \Sigma \to \overline{R}^n; \Sigma \in \operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}, \mathrm{f} \in \omega\text{-BMD}\}$ *is normal. Moreover, any subfamily* $\mathcal{M}' \subset \mathcal{M}$ *of* ω-BMD *embeddings with a common triple of fixed points is compact. For a convergent sequence* $\{f_k\} \subset \mathcal{M}$ *the limit embedding* f *is also* ω^*-BMD, ω^* *depending only on* ω *and* n.

2.5. Now the question arises whether the limit embedding in the above theorem is actually ω-BMD with the same distortion bound ω. We shall answer in the affirmative in the two special cases and give a counterexample for the general situation.

2.6. Given $\Sigma \in \operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$, one may consider $\operatorname{Cont}\Sigma$ to be a continuum in $\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$ (see [K, Chapter 5, Section 47.7, Theorem 3]) as well as a point in the metric space $\operatorname{Cont}\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$.

2.7. Theorem. *Let* ω-BMD *embeddings* $f_k\colon \Sigma_k \to \overline{R}^n$, *where* $\Sigma_k \in \operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$ *for* $k = 1,2,\dots$, *approach an embedding* $f\colon \Sigma \to \overline{R}^n$. *If* $\operatorname{Cont}\Sigma_{\mathrm{k}} \to \operatorname{Cont}\Sigma$ *in* $\operatorname{Cont}\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$ *when* $k \to \infty$, *then* f *is also* ω-BMD *with the same distortion bound* ω.

Proof. Let (E,F) be an arbitrary ring on Σ. The convergence $\operatorname{Cont}\Sigma_{\mathrm{k}} \to \operatorname{Cont}\Sigma$ immediately implies that each subcontinuum $E \subset \Sigma$, while being a point in $\operatorname{Cont}\Sigma$, may be approached in $\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$ with a sequence E_{k_s} $(s = 1,2,\dots)$ of subcontinua $E_{k_s} \subset \Sigma_{k_s}$. Since $\operatorname{Cont}\Sigma_{\mathrm{k_s}} \to \operatorname{Cont}\Sigma$ in $\operatorname{Cont}\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$ and $f_{k_s} \to f$ as $s \to \infty$, we may assume the subsequence f_{k_s} to be the initial sequence f_k. Since the space $\operatorname{Cont}(\overline{\mathrm{R}}^{\mathrm{n}} \times \overline{\mathrm{R}}^{\mathrm{n}})$ is compact, the sequence f_k may be replaced once more with a subsequence so as to provide the convergence $\Gamma f_k \mid E_k \to \Gamma f \mid E$, and so the convergence $f_k E_k \to fE$ as $k \to \infty$. The same argument gives a subsequence $F_{k_j} \in \operatorname{Cont}\Sigma_{\mathrm{k_j}}$ such that $F_{k_j} \to F$ and $f_{k_j}F_{k_j} \to fF$ in $\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$. Since the convergences $E_{k_j} \to E$ and $f_{k_j}E_{k_j} \to fE$ have been preserved, we may assume the subsequence f_{k_j} to be the initial one. Thus we have gained a sequence (E_k, F_k) of rings on Σ_k and the convergences $(E_k,F_k) \to (E,F)$, $(f_kE_k, f_kF_k) \to (fE, fF)$ of rings in $\overline{R}^n$. The continuity theorem for the conformal capacity of rings in $\overline{R}^n$ (see [G, Theorem 5, p. 228; Theorem 1, p. 222]) implies

$$\lim_{k\to\infty} M(E_k,F_k) = M(E,F),$$

$$\lim_{k\to\infty} M(f_kE_k, f_kF_k) = M(fE, fF).$$

Letting $k \to \infty$ in

$$\omega^{-1}\bigl(M(E_k,F_k)\bigr) \le M(f_kE_k, f_kF_k) \le \omega\bigl(M(E_k,F_k)\bigr)$$

yields the desired estimate (1.4) for the embedding f. □

2.8. Corollary. *Let a sequence $f_k\colon \Sigma_k \to \overline{R}^n$ of ω-BMD embeddings with $\Sigma_k \in \operatorname{Cont}\overline{\mathrm{R}}^n$ converge to $f\colon \Sigma \to \overline{R}^n$. If $\Sigma_{k+1} \subset \Sigma_k$ for all $k = 1, 2, \ldots$, then f is ω-BMD.*

Proof. If a sequence $E_k \in \operatorname{Cont}\Sigma_k$ converges to E in $\operatorname{Cont}\overline{\mathrm{R}}^n$, then $E \subset \Sigma$, E being a continuum. Thus $\liminf_{k\to\infty} \operatorname{Cont}\Sigma_k \subset \operatorname{Cont}\Sigma$. Since $\Sigma = \lim_{k\to\infty}\Sigma_k = \bigcap_k \Sigma_k$

$$\operatorname{Cont}\Sigma = \bigcap_{\mathrm{k}} \operatorname{Cont}\Sigma_{\mathrm{k}} = \lim_{\mathrm{k}\to\infty} \operatorname{Cont}\Sigma_{\mathrm{k}} = \liminf_{\mathrm{k}\to\infty} \operatorname{Cont}\Sigma_{\mathrm{k}}.$$

These inclusions imply that $\operatorname{Cont}\Sigma = \lim_{\mathrm{k}\to\infty} \operatorname{Cont}\Sigma_{\mathrm{k}}$. Hence (see [K, Chapter 5, Section 9, 42.2, Remark 1]) $\operatorname{Cont}\Sigma_{\mathrm{k}} \to \operatorname{Cont}\Sigma$ in $\operatorname{Cont}\operatorname{Cont}\overline{\mathrm{R}}^n$ as $k \to \infty$. The assertion now follows from Theorem 2.7. □

3. Special cases of convergence

3.1. Theorem. *Let a sequence $f_k\colon \Sigma_k \to \overline{R}^n$ of ω-BMD embeddings of $\Sigma_k \in \operatorname{Cont}\overline{\mathrm{R}}^n$ converge to an embedding $f\colon \Sigma \to \overline{R}^n$. If Σ is a Jordan arc (a topological image of a closed interval), then f is ω-BMD with the same distortion bound ω.*

Proof. According to 2.7, it is sufficient to obtain the convergence $\operatorname{Cont}\Sigma_{\mathrm{k}} \to \operatorname{Cont}\Sigma$ in $\operatorname{Cont}\operatorname{Cont}\overline{\mathrm{R}}^n$. Since $\limsup_{k\to\infty}\Sigma_k \subset \operatorname{Cont}\Sigma$, it suffices to derive the inclusion

$$\operatorname{Cont}\Sigma \subset \liminf_{\mathrm{k}\to\infty} \operatorname{Cont}\Sigma_{\mathrm{k}}. \tag{3.2}$$

Let $\varphi\colon [0,1] \to \Sigma$ be a parametrisation of the arc Σ. Every nongenerated continuum $\tau \subset \Sigma$ may be represented as $\tau = \varphi[t_1, t_2]$, where $0 \le t_1 < t_2 \le 1$. Let $P_1 = \varphi(t_1)$, $P_2 = \varphi(t_2)$, $\tau_1 = \varphi[0, t_1]$, $\tau_2 = \varphi[t_2, 1]$. For a set $A \subset \overline{R}^n$ denote by $A(\varepsilon)$ its closed ε-neighbourhood in $\overline{R}^n$. Given $\varepsilon > 0$, there exists $e_1 \in (0, \varepsilon]$ such that $\tau_1(\varepsilon_1) \cap \tau_2(\varepsilon_1) = \emptyset$. Hence we may choose $\delta > 0$ such that

$$\gamma_i = \varphi\big([0,1] \cap (t_i - \delta, t_i + \delta)\big) \subset P_i(\varepsilon_1),$$

where $i = 1, 2$. Since the closed arcs $\sigma_1 = \tau_1 \setminus \gamma_1$, $\sigma = \tau \setminus (\gamma_1 \cup \gamma_2)$, $\sigma_2 = \tau_2 \setminus \gamma_2$ are mutually disjoint, this is also true for $\sigma_1(\varepsilon_2)$, $\sigma(\varepsilon_2)$ and $\sigma_2(\varepsilon_2)$ when $\varepsilon_2 \in (0, \varepsilon_1]$ is sufficiently small. Because $\Sigma_k \to \Sigma$, there exists an integer k_0 such that $\Sigma_k \subset \Sigma(\varepsilon_2)$, $E_k = \Sigma_k \cap P_1(\varepsilon_1) \neq \emptyset$ and $F_k = \Sigma_k \cap P_2(\varepsilon_1) \neq \emptyset$ for all $k \ge k_0$. We shall next show that $\Sigma_k \cap \tau(\varepsilon_1)$ is connected between E_k and F_k (see [K, Chapter 5, Section 46.4]). Assume that the statement is false. Then $\Sigma_k \cap \tau(\varepsilon_1)$ is a union $\mathcal{E} \cup \mathcal{F}$ of two disjoint closed sets $\mathcal{E}$ and $\mathcal{F}$ such that $E_k \subset \mathcal{E}$, $F_k \subset \mathcal{F}$. Consider the closed nonempty subsets $\mathcal{E} \cup \big(\Sigma_k \cap \sigma_1(\varepsilon_2)\big)$ and $\mathcal{F} \cup \big(\Sigma_k \cap \sigma_2(\varepsilon_2)\big)$ of Σ_k. Since

(i) $\mathcal{E} \cap \mathcal{F} = \emptyset$,
(ii) $(\Sigma_k \cap \sigma_1(\varepsilon_2)) \cap (\Sigma_k \cap \sigma(\varepsilon_2)) \subset \sigma_1(\varepsilon_2) \cap \sigma_2(\varepsilon_2) = \emptyset$,
(iii) $\mathcal{E} \cap (\Sigma_k \cap \sigma_2(\varepsilon_2)) \subset (P_1(\varepsilon_1) \cap \sigma_2(\varepsilon_2)) \cup (\sigma(\varepsilon_2) \cap \sigma_2(\varepsilon_2)) = P_1(\varepsilon_1) \cap \sigma_2(\varepsilon_2) = \emptyset$,
(iv) $\mathcal{F} \cap (\Sigma_k \cap \sigma_1(\varepsilon_2)) \subset (P_2(\varepsilon_1) \cap \sigma_1(\varepsilon_2)) \cup (\sigma(\varepsilon_2) \cap \sigma_1(\varepsilon_2)) = P_2(\varepsilon_1) \cap \sigma_1(\varepsilon_2) = \emptyset$,

$\mathcal{E} \cup (\Sigma_k \cap \sigma_1(\varepsilon_2))$ and $\mathcal{F} \cup (\Sigma_k \cap \sigma_2(\varepsilon_2))$ are disjoint. Nevertheless, their union is Σ_k:

$$\begin{aligned}&\Big[\mathcal{E} \cup (\Sigma_k \cap \sigma_1(\varepsilon_2))\Big] \cup \Big[\mathcal{F} \cup (\Sigma_k \cap \sigma_2(\varepsilon_2))\Big]\\ &= (\mathcal{E} \cup \mathcal{F}) \cup (\sigma_1(\varepsilon_2) \cap \Sigma_k) \cup (\sigma_2(\varepsilon_2) \cap \Sigma_k)\\ &= \Sigma_k \cap \Big[\tau(\varepsilon_1) \cup \sigma_1(\varepsilon_2) \cup \sigma_2(\varepsilon_2)\Big] = \Sigma_k \cap \Sigma(\varepsilon_2) = \Sigma_k.\end{aligned}$$

This contradicts the connection of Σ_k.

Since $\Sigma_k \cap \tau(\varepsilon_1)$ is connected between E_k and F_k, there exists a continuum $\gamma_k \subset \Sigma_k \cap \tau(\varepsilon_1)$ joining E_k to F_k (see [K, Chapter 5, Section 47.2, Theorem 3; Section 47.1, Theorem 6]). Letting $\varepsilon = 1/s$ for $s = 1, 2, \ldots$, we obtain the increasing sequence k_s and continua $\gamma_k \subset \Sigma_k \cap \tau(1/s)$ for $k_s \le k \le k_{s+1}$. Obviously, $\gamma_k \to \tau$ as $k \to \infty$, and hence $\tau \in \liminf_{k\to\infty} \operatorname{Cont} \Sigma_k$. Thus (3.2) is proved. □

3.3. A family $\mathcal{F}$ of continua in $\overline{R}^n$ is called locally equiconnected if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that every pair of points $x, y \in \Sigma \in \mathcal{F}$ with $[xy] < \delta$ can be joined by a continuum $\gamma \subset \Sigma$ of spherical diameter $\le \varepsilon$.

3.4. Theorem [As4, Theorem 2.1, p. 19]. *Let a sequence* $f_k\colon \Sigma_k \to \overline{R}^n$ *of* ω-BMD *embeddings of continua* $\Sigma_k \subset \overline{R}^n$ *converge to an embedding* $f\colon \Sigma \to \overline{R}^n$. *If the family* $\{\Sigma_k : k = 1, 2, \ldots\}$ *is locally equiconnected,* $f \in \omega$-BMD *with the same distortion bound* ω.

Proof. Let Σ_{k_s} be an arbitrary subsequence of Σ_k. Since the family $\{\Sigma_{k_s} : s = 1, 2, \ldots\}$ remains locally equiconnected, it follows from [As4, Lemma 1.2, p. 17] that $\operatorname{Cont} \Sigma \subset \limsup_{s\to\infty} \operatorname{Cont} \Sigma_{k_s}$. Because of the arbitrary choice of a subsequence Σ_{k_s} we obtain by [K, Chapter 2, Section 29.5 (1)]

$$\operatorname{Cont} \Sigma \subset \bigcap \limsup_{s\to\infty} \operatorname{Cont} \Sigma_{k_s} = \liminf_{k\to\infty} \operatorname{Cont} \Sigma_k \subset \limsup_{k\to\infty} \operatorname{Cont} \Sigma_k \subset \operatorname{Cont} \Sigma,$$

where the intersection expands over all subsequences Σ_{k_s} of Σ_k. Thus the equality $\operatorname{Cont} \Sigma = \lim_{k\to\infty} \operatorname{Cont} \Sigma_k$ holds and the desired result follows from Theorem 2.7.

3.5. *Question.* Does Theorem 3.1 remain true if Σ is replaced by a Jordan curve (a topological image of a circle)?

4. Counterexample

All the considerations throughout this section will refer to the extended complex plane $\overline{\mathbf{C}}$, $z = x + iy = \varrho e^{i\varphi}$ being a complex variable.

4.1. A continuous mapping $f\colon \Sigma \to \overline{\mathbf{C}}$, where $\Sigma \subset \overline{\mathbf{C}}$, is termed circular with respect to a point z_0 if $|fz - fz_0| = |z - z_0|$ for all $z \in \Sigma$.

4.2. Lemma. *Let (E, F) be a ring in $\mathbf{C}$ and $f\colon E \cup F \to \mathbf{C}$ a circular mapping with respect to z_0. Denote by $\alpha(a,b)$ the acute angle between segments $z_0 a$ and $z_0 b$ while $a, b \in \mathbf{C} \setminus \{z_0\}$. If*

$$\alpha(fa, fb) \begin{cases} \le \alpha(a,b) & \text{as } a \in E, b \in F, \\ \ge \alpha(a,b) & \text{as } a, b \in E \text{ or } a, b \in F, \end{cases}$$

then $M(fE, fF) \ge M(E, F)$.

Proof. Since the distance $|a - b|$ between the points $a, b \in \mathbf{C} \setminus \{z_0\}$ with fixed $|a - z_0|$ and $|b - z_0|$ is increasing to $\alpha(a, b)$, the estimates

$$|fa - fb| \begin{cases} \le |a - b| & \text{as } a \in E, b \in F, \\ \ge |a - b| & \text{as } a, b \in E \text{ or } a, b \in F \end{cases}$$

hold. By [AV, Theorem 2, p. 8; Theorem 1, p. 7] we have the inequality $\mathrm{md}(E, F) \ge \mathrm{md}(fE, fF)$ for transfinite 2-moduli. Thus by Bagby's theorem [B, Theorem 5, p. 325] the same inequality holds for conformal moduli of these condensers. The connection between the conformal moduli and the conformal capacity of condensers gives the desired estimate. □

4.4. In the case where the distortion bound ω of ω-BMD embedding is of the form $\omega(t) = kt$, $k \ge 1$, the coefficient k will be termed the distortion coefficient of f and denoted by $k[f]$.

4.5. *Question* (P.P. Belinskij). Is it true that every BMD-embedding f of a continuum has a finite distortion coefficient? For a brief discussion of the problem see [AsV2]. In this connection also see [AsT, Theorem 5.2, p. 547].

4.6. The following construction is a mere modification of the example from [AsV3, p. 14] (the paper contains a lot of misprints). For some fixed $\varepsilon \in (0, \pi/8)$ set $l_1 = \{z \in \overline{\mathbf{C}} : \arg z = \varepsilon + \frac{1}{2}\pi\}$, $l_2 = \{z \in \overline{\mathbf{C}} : \arg z = -\varepsilon + \frac{1}{2}\pi\}$, $l_3 = \{z \in \overline{\mathbf{C}} : \arg z = 0\}$, $\Sigma = l_1 \cup l_2 \cup l_3$. The embedding $f\colon \Sigma \to \overline{\mathbf{C}}$ is defined by the formula

$$f(z) = \begin{cases} z & \text{as } z \in l_3, \\ i\bar{z} & \text{as } z \in l_1 \cup l_2. \end{cases}$$

For $k = 1, 2, \ldots$ we set $l_{1k} = \{z \in l_1 : |z| \le k\}$, $l_{2k} = \{z \in l_2 : |z| \ge 1/k\}$, $\Sigma_k = l_{1k} \cup l_{2k} \cup l_3$ and $f_k = f|\Sigma_k$. Obviously $f_k \to f$ when $k \to \infty$. We are going to show that

$$k[f_k] \le \frac{8}{7}\Big(\frac{\pi}{2\varepsilon} + 9\Big) \tag{4.7}$$

for all $k = 1, 2, \ldots$.

Let a pair of disjoint continua in Σ_k be denoted by E, F so as to have E between F and the endpoint $ike^{i\varepsilon}$ of Σ_k. Note that, for any pair $i, j \in \{1, 2, 3\}$ and continua $E_i = E \cap l_i$, $F_j = F \cap l_j$ (possibly empty), the circular mapping $f\colon E_i \cup F_j \to \overline{\mathbf{C}}$ with respect to 0 preserves angles on E_i and F_j separately and does not increase angles between E_i and F_j. By Lemma 4.2 we obtain the inequality $M(E_i, F_j) \le M(fE_i, fF_j)$ for each pair i, j. Hence

$$M(E,F) \le \sum_{i,j} M(E_i, F_j) \le \sum_{i,j} M(fE_i, fF_j) \le 9M(fE, fF).$$

Thus

$$M(E,F)/9 \le M(f_k E, f_k F) \tag{4.8}$$

holds for all rings (E, F) on Σ_k.

In order to obtain an upper estimate for $M(f_k E, f_k F)$ we consider the following five cases.

Case 1. Let $E \subset l_3 \cup l_2$. Then $F \subset l_3 \cup l_2$. The embedding $f|(l_3 \cup l_2)$ extends to a quasiconformal mapping $g_1\colon \varrho e^{i\varphi} \mapsto \varrho e^{i\beta(\varphi)}$, where $\beta(0) = 0$, $\beta(-\varepsilon + \frac{1}{2}\pi) = \varepsilon$, $\beta(2\pi) = 2\pi$, the function β being linear on $[0, -\varepsilon + \frac{1}{2}\pi]$ and $[-\varepsilon + \frac{1}{2}\pi, 2\pi]$. Since $\varepsilon < \pi/8$, the dilatation of g_1 is $(\pi - 2\varepsilon)/2\varepsilon$. Hence

$$M(f_k E, f_k F) = M(g_1 E, g_1 F) \le (-1 + \pi/2\varepsilon) M(E, F).$$

Case 2. Let $F \subset l_1 \cup l_3$. Then $E \subset l_1 \cup l_3$. The restriction $f \mid (l_1 \cup l_3)$ extends to a quasiconformal mapping $g_2\colon \varrho e^{i\varphi} \mapsto \varrho e^{-i\beta(\varphi)}$, where $\beta(0) = 0$, $\beta(\varepsilon + \frac{1}{2}\pi) = \varepsilon$, $\beta(2\pi) = 2\pi$, the function β being linear on $[0, \varepsilon + \frac{1}{2}\pi]$ and $[\varepsilon + \frac{1}{2}\pi, 2\pi]$. Since the dilatation of g_2 is $(\pi + 2\varepsilon)/2\varepsilon$, the estimate

$$M(f_k E, f_k F) = M(g_2 E, g_2 F) \le (1 + \pi/2\varepsilon) M(E, F)$$

holds.

Case 3. Let $E \subset l_1 \cup l_3$ and $F \subset l_3 \cup l_2$. The circular mapping $g_3(z) = \{z \text{ on } l_3 \cup l_2; \bar{z} \text{ on } l_1 \cup l_3\}$ preserves angles on E as well as on F and does not decrease angles between E and F. By Lemma 4.2 $M(\tilde{E}, F) \le M(E, F)$, where $\tilde{E} = g_3 E$. The mapping $g_4\colon \varrho e^{i\varphi} \mapsto \varrho e^{i\beta(\varphi)}$, where $\beta(-\pi) = -\pi$, $\beta(-\varepsilon - \frac{1}{2}\pi) = -\varepsilon$, $\beta(-\varepsilon + \frac{1}{2}\pi) = \varepsilon$, $\beta(0) = 0$, $\beta(\pi) = \pi$, β being linear on the segments $[-\pi, -\varepsilon - \frac{1}{2}\pi]$, $[-\varepsilon - \frac{1}{2}\pi, 0]$, $[0, -\varepsilon + \frac{1}{2}\pi]$, $[-\varepsilon + \frac{1}{2}\pi, \pi]$, transforms $\tilde{E}$ into fE and F into fF. Since $\varepsilon < \pi/8$, the dilatation of g_4 is equal to $(\pi + 2\varepsilon)/2\varepsilon$. Hence

$$M(f_k E, f_k F) = M(fE, fF) \le (1 + \pi/2\varepsilon) M(\tilde{E}, F) \le (1 + \pi/2\varepsilon) M(E, F).$$

Case 4. Let $l_3 \subset E$. Then $F \subset l_2$. Denote $E_2 = E \cap l_2$ and $S = l_3 \cup f l_1$. The mapping $g_5\colon \varrho e^{i\varphi} \mapsto \varrho e^{i\beta(\varphi)}$, where $\beta(0) = 0$, $\beta(\varepsilon) = \varepsilon$, $\beta(2\pi - \varepsilon) = 2\pi$, β being linear on segments $[0, \varepsilon]$, $[\varepsilon, 2\pi - \varepsilon]$, has dilatation $K[g_5] = (2\pi - \varepsilon)/2(\pi - \varepsilon) < 8/7$ and transforms the domain $\mathcal{D} = \{z : 0 < \arg z < 2\pi - \varepsilon\}$ into $\overline{\mathbf{C}} \setminus l_3$. Hence

$$\begin{aligned} M(fE, fF) &\leq M(S \cup fE_2, fF) = M(S \cup fE_2, fF; \mathcal{D}) \\ &\leq (8/7) M(l_3 \cup fE_2, fF; g_5\mathcal{D}) = (8/7) M\big(g_1(l_3 \cup l_2), g_1 F\big) \\ &\leq (8/7)(-1 + \pi/2\varepsilon) M(l_3 \cup E_2, F) \leq (8/7)(-1 + \pi/2\varepsilon) M(E, F). \end{aligned}$$

Thus

$$M(f_k E, f_k F) \leq (8/7)(-1 + \pi/2\varepsilon) M(E, F).$$

Case 5. Let $l_3 \subset F$. Then $E \subset l_1$. Denote $F_1 = F \cap l_1$, $S' = l_3 \cup f l_2$. The mapping $g_6(z) = \overline{g_5(\bar{z})}$ transforms the domain $\mathcal{D}' = \{z : \varepsilon < \arg z < 2\pi\}$ into $\overline{\mathbf{C}} \setminus l_3$ and has the same dilatation as g_5. That is, $K[g_6] = K[g_5] < 8/7$. Hence

$$\begin{aligned} M(fE, fF) &\leq M(fE, S' \cup fF_1) = M(fE, S' \cup fF_1; \mathcal{D}') \\ &\leq (8/7) M\big(fE, fF_1 \cup l_3; g_6(\mathcal{D}')\big) = (8/7) M\big(fE, f(F_1 \cup l_3)\big) \\ &= (8/7) M\big(g_2 E, g_2(F_1 \cup l_3)\big) \leq (8/7)(1 + \pi/2\varepsilon) M(E, F_1 \cup l_3) \\ &\leq (8/7)(1 + \pi/2\varepsilon) M(E, F). \end{aligned}$$

Thus

$$M(f_k E, f_k F) \leq (8/7)(1 + \pi/2\varepsilon) M(E, F).$$

The estimates in Cases 1–5 together give the inequality

$$M(f_k E, f_k F) \leq (8/7)(1 + \pi/2\varepsilon) M(E, F)$$

for all rings (E, F) on Σ_k. On the strength of (4.8) it implies the announced upper bound (4.7).

Provided ε is sufficiently small, we can show that the limit embedding $f\colon \Sigma \to \overline{\mathbf{C}}$ is not of the class ω-BMD with the same distortion bound $\omega(t) = (8/7)(9 + \pi/2\varepsilon)t$ as that of f_k. Consider the continua $E = l_1 \cup l_2$, $F(\delta) = \{z \in l_3 : \delta < |z| < \delta^{-1}\}$, where $\delta \in (0, 1)$. It is easy to get the crude estimates for the capacities of rings $\big(E, F(\delta)\big)$ and $\big(fE, fF(\delta)\big)$

$$\begin{aligned} M\big(E, F(\delta)\big) &\leq \big[(-\varepsilon + \tfrac{1}{2}\pi)^{-1} + (-\varepsilon + 3\pi/2)^{-1}\big] 2 \log 1/\delta + \frac{8(\pi - \varepsilon)}{\pi - 2\varepsilon} \\ &= \frac{16(\pi - \varepsilon)}{(\pi - 2\varepsilon)(3\pi - 2\varepsilon)} \log \frac{1}{\delta} + \frac{8(\pi - \varepsilon)}{\pi - 2\varepsilon}; \end{aligned}$$

$$M\big(fE, fF(\delta)\big) \geq M\left(fE, fF(\delta); \{\delta < |z| < \delta^{-1}\}\right) = (4/\varepsilon) \log 1/\delta.$$

Since

$$\lim_{\delta\to 0}\frac{M(fE,fF(\delta))}{M(E,F(\delta))}\geq\frac{(\pi-2\varepsilon)(3\pi-2\varepsilon)}{4\varepsilon(\pi-\varepsilon)}$$

and

$$\frac{(\pi-2\varepsilon)(3\pi-2\varepsilon)}{4\varepsilon(\pi-\varepsilon)}\approx\frac{3\pi}{4\varepsilon}>\frac{4\pi}{7\varepsilon}\approx\frac{8}{7}\Big(9+\frac{\pi}{2\varepsilon}\Big)$$

as $\varepsilon\to 0$, there exists, for a sufficiently small ε, a $\delta=\delta(\varepsilon)<1$ such that the strict inequality

$$M\big(fE,fF(\delta)\big)>(8/7)(9+\pi/2\varepsilon)M\big(E,F(\delta)\big)$$

holds. It shows that for such ε the embedding f is not of the class ω-BMD with $\omega(t)=(8/7)(9+\pi/2\varepsilon)t$.

5. id-BMD embeddings

According to Definition 1.3, the embedding $f\colon\Sigma\to\overline{R}^n$ is id-BMD if

$$M(fE,fF)=M(E,F)\tag{5.1}$$

for any ring (E,F) on Σ. We use the term Möbius embedding for any id-quasimöbius embedding. Note that every Möbius embedding in $\overline{R}^n$ may be transformed by a suitable Möbius mapping into an isometric embedding and hence it may be extended to an isometry over all $\overline{R}^n$. So in order to obtain a Möbius extension of an id-BMD embedding one only needs to prove that it is a Möbius embedding.

5.2. Conjecture (P.P. Belinskij; see the final remark in [As5, p. 1529]). Every id-BMD embedding of a continuum into $\overline{R}^n$ is a Möbius embedding.

We will commence with a two-dimensional case.

5.3. Theorem (see also [As6]). *If $\Sigma\subset\overline{R}^2$ has a positive topological dimension at each point of a dense subset $\Sigma'\subset\Sigma\subset\overline{\Sigma}'$, then every id-BMD embedding $f\colon\Sigma\to\overline{R}^2$ is a Möbius one.*

Proof. Choose a decreasing sequence $\delta_k\searrow 0$. By [K, Chapter 5, Section 47.2, Theorem 9] there exists at each point $a\in\Sigma'$ a continuum $\gamma_k\subset\Sigma$ such that $a\in\gamma_k$ and $0<\operatorname{diam}\gamma_k<\delta_k$. Hence for an arbitrarily given quadruplet $a_1a_2a_3a_4$ in Σ' we may contruct a sequence of continua γ_{ik} $(i=1,2,3,4;\ k=1,2,\dots)$ such that $a_i\subset\gamma_{ik}\subset\Sigma$ and $0<\operatorname{diam}\gamma_{ik}<\delta_k$. By [AV, Theorem 5, p. 14] and [B, Theorem 5, p. 325] we have

$$\lim_{k\to\infty}\tau(\gamma_{ik})\tau(\gamma_{jk})\exp\operatorname{mod}(\gamma_{ik},\gamma_{jk})=|a_i-a_j|^2,\tag{5.3.1}$$

where $\tau(E)$ denotes the transfinite diameter of E in $\overline{R}^2$ and $\operatorname{mod}(\gamma_{ik}, \gamma_{jk}) = 2\pi/M(\gamma_{ik}, \gamma_{jk})$. Thus the characteristic $r(T)$ of the quadruplet $T = a_1a_2a_3a_4$ may be derived as follows:

$$r(T)^2 = \lim_{k\to\infty} \exp\left[\operatorname{mod}(\gamma_{1k}, \gamma_{2k}) + \operatorname{mod}(\gamma_{3k}, \gamma_{4k}) - \operatorname{mod}(\gamma_{1k}, \gamma_{3k}) - \operatorname{mod}(\gamma_{2k}, \gamma_{4k})\right].$$

The same arguments for the condensers $(f\gamma_{ik}, f\gamma_{jk})$ give the following expression of $r(fT)$:

$$r(fT)^2 = \lim_{k\to\infty} \exp\left[\operatorname{mod}(f\gamma_{1k}, f\gamma_{2k}) + \operatorname{mod}(f\gamma_{3k}, f\gamma_{4k}) - \operatorname{mod}(f\gamma_{1k}, f\gamma_{3k}) - \operatorname{mod}(f\gamma_{2k}, f\gamma_{4k})\right].$$

Since $\operatorname{mod}(f\gamma_{ik}, f\gamma_{jk}) = \operatorname{mod}(\gamma_{ik}, \gamma_{jk})$, it follows that $r(T) = r(fT)$ for any quadruplet T in Σ', so that $f \mid \Sigma' \in$ id-QM. Hence the continuous extension $\tilde{f}$ of $f \mid \Sigma'$ over $\overline{\Sigma}'$ is also a Möbius embedding.

5.4. Definition (cf. [As7, p. 201]). A continuum $\gamma \subset \overline{R}^n$ is said to be raylike at a point $a \in \gamma \cap \overline{R}^n$ if for any stretching sequence $\mu_a[t_k]$: $x \mapsto a + t_k(x-a)$, $\mu_a[t_k]$: $\infty \mapsto \infty$, $t_k \to \infty$ of Möbius self-mappings in $\overline{R}^n$ the limit set $\lim_{k\to\infty} \mu_a[t_k]\gamma$ in $\operatorname{Cont}\overline{\mathrm{R}}^n$, if any, is a ray origined at a. □

5.4.1. Remark. The raylike property of a continuum γ at $a \in \gamma$ does not imply the existence of a tangent ray at a point a. The counterexample was communicated to me by V.A. Vasilenko in 1986.

5.5. Lemma. *Let Jordan arcs $\gamma_1, \gamma_2 \in \overline{R}^n$ be raylike at points $x_1 \in \gamma_1$ and $x_2 \in \gamma_2$, respectively. Then for any sequences $\{\gamma_{ik} \subset \gamma_i : k = 1, 2, \dots\}$ of subarcs such that $x_i \in \gamma_{ik}$ $(i = 1, 2)$ and $\delta_{ik} = \max\{|x_i - x| : x \in \gamma_{ik}\} \to 0$ as $k \to \infty$ the equality*

$$\lim_{k\to\infty} \delta_{1k}\delta_{2k} \exp\operatorname{mod}(\gamma_{1k}, \gamma_2 k) = \lambda_n |x_1 - x_2|^2 \tag{5.5.1}$$

holds. Here λ_n denotes the Grötzsch constanst in $\overline{R}^n$.

Proof. We may assume $x_2 = x_1 + e$, where $|e| = 1$. Denote by $\psi(t)$ the conformal modulus of the Teichmüller ring in $\overline{R}^n$. Given $\varepsilon > 0$, since $\log\lambda_n t^2 - \psi(t^2 - 1)$ decreases to 0 as $t \to \infty$ [G, (a), (c), p. 225], there exists $\alpha = \alpha(\varepsilon) > 1$ such that

$$0 < \log\lambda_n\alpha^2 - \psi(\alpha^2 - 1) < \varepsilon. \tag{5.5.2}$$

When k is sufficiently large, the spheres $S_{ik} = \{x : |x_i - x| = \alpha\delta_{ik}\}$ $(i = 1, 2)$ are disjoint. For any line segments τ_{ik} of length δ_{ik} origined at x_i $(x = 1, 2)$ we have (see [V, Lemma 5.53, p. 66])

$$\operatorname{mod}(\tau_{ik}, S_{ik}) = 2^{-1}\psi(\alpha^2 - 1), \qquad i = 1, 2. \tag{5.5.3}$$

After a suitable subsequence has been chosen and relabelled, it may be assumed by the raylikeness condition that $\mu_{x_i}[1/\delta_{ik}]\gamma_{ik} \to \gamma_{i0}$ in $\operatorname{Cont}\overline{\mathrm{R}}^n$, γ_{i0} being a unit line segment origined at x_i. The uniform convergence of rings

$$\big(\mu_{x_i}[1/\delta_{ik}]\gamma_{ik}, \mu_{x_i}[1/\delta_{ik}]S_{ik}\big) \to \big(\gamma_{i0}, S_i = \{x : |x_i - x| = \alpha\}\big)$$

combined with the continuous property of ring moduli [G, Theorem 5, p. 228] and the equality (5.5.3) together imply

$$\text{(5.5.4)} \qquad \operatorname{mod}(\gamma_{1k}, S_{1k}) + \operatorname{mod}(\gamma_{2k}, S_{2k}) = \psi(\alpha^2 - 1) + O_1$$

with $O_1 \to 0$ as $k \to \infty$. Since the ring (S_{1k}, S_{2k}) may be transformed by a Möbius map into a spherical ring $\{x : 1 < |x| < T_k\}$ with $T_k = \exp\operatorname{mod}(S_{1k}, S_{2k}) \to \infty$ as $k \to \infty$, the direct calculation yields

(5.5.5)
$$\begin{aligned}\operatorname{mod}(S_{1k}, S_{2k}) &= \log T_k \\ &= \log\big(1 - \alpha^2(\delta_{1k} - \delta_{2k})^2\big) - \log\alpha^2\delta_{1k}\delta_{2k} - 2\log(1 + T_k^{-1}) \\ &= \log(1/\delta_{1k}\delta_{2k}) - \log\alpha^2 + O_2,\end{aligned}$$

where $O_2 \to 0$ as $k \to \infty$. The extremal property of the Teichmüller ring [G, Section 2, Theorem 4, p. 226] and the asymptotics for its conformal modulus [G, Section 2, (c), p. 225] imply the estimate

$$\text{(5.5.6)} \qquad \operatorname{mod}(\gamma_{1k}, \gamma_{2k}) \le \psi\Big(\frac{1 + \delta_{1k} + \delta_{2k}}{\delta_{1k}\delta_{2k}}\Big) = \log\frac{\lambda_n}{\delta_{1k}\delta_{2k}} + O_3,$$

where $O_3 \to 0$ as $k \to \infty$. It follows from (5.5.4), (5.5.5) and

$$\operatorname{mod}(\gamma_{1k}, \gamma_{2k}) \ge \operatorname{mod}(\gamma_{1k}, S_{1k}) + \operatorname{mod}(S_{1k}, S_{2k}) + \operatorname{mod}(\gamma_{2k}, S_{2k})$$

that

$$\operatorname{mod}(\gamma_{1k}, \gamma_{2k}) \ge \psi(\alpha^2 - 1) - \log\lambda_n\alpha^2 + \log(\lambda_n/\delta_{1k}\delta_{2k}) + O_4,$$

where $O_4 \to 0$ as $k \to \infty$. The latter estimate, together with (5.5.2), (5.5.6), implies the double bound

$$\log\lambda_n + O_4 - \varepsilon \le \operatorname{mod}(\gamma_{1k}, \gamma_{2k}) + \log\delta_{1k}\delta_{2k} \le \log\lambda_n + O_3.$$

Letting $k \to \infty$ and $\varepsilon \to 0$ yields

$$\lim_{k\to\infty} \delta_{1k}\delta_{2k}\exp\operatorname{mod}(\gamma_{1k}, \gamma_{2k}) = \lambda_n$$

as desired. □

5.6. Lemma. *Let $f\colon \Sigma \to \overline{R}^n$ be an* id-BMD *embedding. A point $a \in \Sigma$ will be regarded as a regular point for f if there exists an arc $\gamma \subset \Sigma$ origined at a such that both γ and $f\gamma$ are raylike at points a and fa, respectively. If the set Σ' of all regular points for f is dense in Σ, then f is a Möbius embedding.*

Proof. To prove this assertion we just need to modify the arguments from the proof of Theorem 5.3 slightly as follows. We may think of γ_{ik} as subarcs in γ_i such that $a_i \in \gamma_{ik}$ and $\delta_{ik} = \max_{x\in\gamma_{ik}} |x - a_i|$, while γ_i is just the arc mentioned in the above definition of a regular point for f. Because of the raylikeness of γ_i and $f\gamma_i$ at the respective points, Lemma 5.5 yields the asymptotics

$$\lim_{k\to\infty} \delta_{ik}\delta_{jk} \exp \operatorname{mod}(\gamma_{ik}, \gamma_{jk}) = \lambda_n |a_i - a_j|^2, \tag{5.6.1}$$

$$\lim_{k\to\infty} \delta'_{ik}\delta'_{jk} \exp \operatorname{mod}(f\gamma_{ik}, f\gamma_{jk}) = \lambda_n |fa_i - fa_j|^2,$$

where $\delta'_{ik} = \max |x - fa_i|$ over $f\gamma_{ik}$. The asymptotics (5.3.1) and similar asymptotics for $\operatorname{mod}(f\gamma_{ik}, f\gamma_{jk})$ should now be replaced by (5.6.1). □

5.7. Theorem. *If $\Sigma \subset \overline{R}^n$ is a circle or circular arc, every* id-BMD *embedding $f\colon \Sigma \to \overline{R}^n$ is a Möbius embedding.*

Proof. If Σ is a circle, the theorem has been proved in [As5] by arguments quite similar to [G, Section 5, p. 241–243]. Thus we may assume Σ to be a ray origined at 0 and the points 0, ∞ to be fixed under f. Let $a \in \Sigma \setminus \{0, \infty\}$ and $\gamma_a \subset \Sigma$ be the ray origined at a. In order to prove that $f\gamma_a$ is raylike at fa we consider an arbitrary stretching sequence $\{\mu_{fa}[t_k]\}$ such that $\mu_{fa}[t_k] f\gamma_a \to \gamma$ in $\operatorname{Cont}\overline{\mathrm{R}}^n$ as $k \to \infty$. Let $\tilde{z}_k = fz_k$ be a point at the Jordan arc $\tau \subset f\Sigma$ with endpoints 0 and fa such that $|\mu_{fa}[t_k]\tilde{z}_k - fa| = |fa|$. Since $\tilde{z}_k \to fa$ as $k \to \infty$, there may be found an increasing sequence $t'_k \to \infty$ such that $a + t'_k(z_k - a) = \frac{1}{2}a$. The sequence of rays $\Sigma_k = \mu_a[t'_k]\Sigma$ converges to a circle $\Sigma_0 \subset \overline{R}^n$ in $\operatorname{Cont}\overline{\mathrm{R}}^n$ as $k \to \infty$. Since the sequence $\{\nu_k = \mu_{fa}[t_k] \circ f \circ \mu_a^{-1}[t'_k]\colon \Sigma_k \to \overline{R}^n\}$ of id-BMD embeddings is normed by the conditions $\nu_k(a) = f(a)$, $\nu_k(\infty) = \infty$, $|\nu_k(\frac{1}{2}a) - fa| = |fa|$, it is a normal family. If we apply Theorem 2.3 to choose a subsequence ν_{k_s} that converges to an BMD embedding $\nu\colon \Sigma_0 \to \overline{R}^n$ of a circle $\Sigma_0 \subset \overline{R}^n$, then, because $t'_{k+1} > t'_k$ and $\Sigma_k \subset \Sigma_{k+1}$, we get the equality $M(E, F) = M(\nu E, \nu F)$ for all rings (E, F) on $\Sigma_0 \setminus \{\infty\} = \cup_k \Sigma_k$. Since $\nu|\Sigma_0 \setminus \{\infty\}$ is an id-BMD embedding of the line $\Sigma_0 \setminus \{\infty\}$, we have the situation as in [As5, Lemma 4]. Thus ν is a Möbius embedding of a line $\Sigma_0 \setminus \{\infty\}$ and hence $\gamma = \nu\gamma_a$ is a ray origined at fa. Thus $f\gamma_a$ is raylike at fa and the point a is a regular point for f. So by Lemma 5.6 f is a Möbius embedding. □

5.8. Lemma. *Let an arc $\gamma \subset \overline{R}^n$ be raylike at a point $a \in \gamma$. Then for any* id-BMD *embedding $f\colon \gamma \to \overline{R}^n$ its image $f\gamma$ is also raylike at fa.*

Proof. Consider an arbitrary stretching sequence $\mu_{fa}[t_k]$ such that $t_k \to \infty$ and $\mu_{fa}[t_k]f\gamma \to \gamma'$ in $\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$. For every k there exists a point $\tilde{z}_k = fz_k \in f\gamma$ such that $\left|\mu_{fa}[t_k]\tilde{z}_k - fa\right| = 1$. Since $\tilde{z}_k \to fa$ as $k \to \infty$, we have $z_k \to a$ and may consider $\nu_k = \mu_a[1/|a - z_k|]$ a stretching sequence for γ. Choosing a suitable relabelled subsequence gives the convergence $\nu_k\gamma \to \gamma_0$ in $\operatorname{Cont}\overline{\mathrm{R}}^{\mathrm{n}}$, where γ_0 is a ray origined at a. The sequence $\left\{\tau_k = \mu_{fa}[t_k] \circ f \circ \nu_k^{-1}\colon \nu_k\gamma \to \overline{R}^n\right\}$ of id-BMD embeddings is normed by conditions $\tau_k a = fa$, $\tau_k(\infty) = \infty$ and $|\tau_k b_k - fa| = 1$ for $b_k = \nu_k z_k$ with $|b_k - a| = 1$. By Theorem 2.3 we may assume the convergence $\tau_k \to \tau\colon \gamma_0 \to \gamma'$ of id-BMD embeddings τ_k to a BMD embedding τ of a ray γ_0. By Theorem 3.1, τ is also an id-BMD embedding. Hence by Theorem 5.7 it is a Möbius embedding and $\gamma' = \tau\gamma_0$ is a ray. Thus the raylikeness of $f\gamma$ at the point fa has been proved. □

5.9. Theorem. *Let $\Sigma \subset \overline{R}^n$ have a dense subset Σ' such that for every point $a \in \Sigma'$ there exists an arc $\gamma_a \subset \Sigma$ which is raylike at a. Then every id-BMD embedding $f\colon \Sigma \to \overline{R}^n$ is a Möbius one.*

Proof. By Lemma 5.8 the situation satisfies the conditions of Lemma 5.6, which implies the desired assertion. □

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