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THE THREE-SEPARATED-ARC PROPERTY OF FUNCTIONS IN A DISK

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Let D be the open unit disk and Γ be the unit circle in the complex plane, and suppose that $f(z)$ is a single-valued function in D with values on the Riemann sphere. If $\zeta \in \Gamma$ and if Λ is an arc at ζ, then $C_\Lambda(f,\zeta)$ denotes the cluster set of f at ζ along Λ. If there exist three arcs Λ_1, Λ_2, Λ_3 at ζ such that

$$\mathbf{C}_{\Lambda_1}(f,\zeta) \cap \mathbf{C}_{\Lambda_2}(f,\zeta) \cap \mathbf{C}_{\Lambda_3}(f,\zeta) = \emptyset,$$

then f is said to have the three-arc property at ζ. If the three arcs can be taken to be mutually exclusive, we say that f has the three-separated-arc property at ζ; if they can be taken to be chords at ζ, f is said to have the three-chord property at ζ.

There exists [3] a normal holomorphic function in D that has the three-separated-arc property at every point of Γ. It follows from [1, Theorem 4], however, that the set of points of Γ at each of which a normal meromorphic function in D, and hence, in particular, a bounded holomorphic function in D, has the three-chord property, is of measure zero and first category. There does exist [6, Theorem 2] a meromorphic function in D that has the three-chord property at each point of a perfect subset of Γ. The set of points on Γ at each of which a meromorphic function of bounded characteristic in D has the three-arc property is of measure zero [2, Theorem 2]; this holds then, in particular, for any function that is holomorphic and bounded in D. A consequence of [1, Theorem 3] is that there is a bounded holomorphic function in D having the three-chord property at each point of an enumerable subset of Γ.

There is [5, Theorem 4] a bounded holomorphic function in D, in the form of a Blaschke product, that has the three-arc property at each point of a perfect subset of Γ. It has been shown recently [4] that if a meromorphic function in D has the three-arc property at a point of Γ, then it has the three-separated-arc property at that point. Hence

There exists a Blaschke product that has the three-separated-arc property at each point of a perfect subset of Γ.

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C.L. Belna has asked (in written communications):

Does there exist a bounded holomorphic function in D having the three-chord property at each point of a nonenumerable subset of Γ?

If a function is continuous in D and has the three-arc property at a point $\zeta \in \Gamma$, does it have the three-separated-arc property at ζ?

We prove:

There exists a three-valued function in D that has the three-arc property at a point $\zeta \in \Gamma$ but does not have the three-separated-arc property at ζ.

It is easier to describe the construction of the function in a half-plane instead of a disk, so we define our function in the lower half of the complex plane, and take the point ζ to be the origin.

Let Λ_1 be the segment extending from $-1-i$ to ζ, and Λ_2 be the segment extending from $1-i$ to ζ. Denote by (a_n) and (b_n) sequences of points on Λ_1, Λ_2, respectively, with imaginary parts $-1/n$ $(n=1,2,3,\dots)$. Define (c_n) to be the sequence of points on Λ_2 with imaginary parts $\frac{1}{2}(-(1/2n)-1/(2n+1))$ $(n=1,2,3\dots)$. Take the point d to be $-i$. We let Λ_3 be the arc at ζ consisting of the segments

$$da_2,\ a_2c_1,\ c_1a_4,\ a_4c_2,\ c_2a_6,\dots,a_{2n}c_n,\ c_na_{2n+2},\dots.$$

Define the function $f(z)$, for z in the lower half-plane, to have one of the three values 1, 2, 3 at z, in the following way.

For z on each of the open segments $a_{2n-1}a_{2n}$ $(n=1,2,3,\dots)$, $f(z)=1$. For $z=a_1$ and for z on each of the closed segments $a_{2n}a_{2n+1}$ $(n=1,2,3,\dots)$, $f(z)=2$. Then $f(z)$ is defined on Λ_1, and $C_{\Lambda_1}(f,\zeta)=\{1,2\}$.

For z on each of the closed segments $b_{2n-1}b_{2n}$ $(n=1,2,3,\dots)$, $f(z)=1$. For z on each of the open segments $b_{2n}b_{2n+1}$ $(n=1,2,3,\dots)$, $f(z)=3$. Then $f(z)$ is defined on Λ_2, and $C_{\Lambda_2}(f,\zeta)=\{1,3\}$.

For $z=d$, for z on each of the open segments $a_{2n}c_n$ $(n=1,2,3,\dots)$, for z on each of the open segments c_na_{2n+2} $(n=1,2,3,\dots)$, and for z on the open segment da_2, $f(z)=3$. Then $f(z)$ is defined on Λ_3, and (note that $f(a_{2n})=2$, $(n=1,2,3,\dots)$) $C_{\Lambda_3}(f,\zeta)=\{2,3\}$.

For z on the open segment a_1d, $f(z)=1$; for z on the open segment db_1, $f(z)=3$. For z on each of the open segments $a_{2n}b_{2n}$ $(n=1,2,3,\dots)$, $f(z)=1$.

For z inside the triangle with vertices a_1, a_2, d, for z inside each of the triangles with vertices a_{2n}, b_{2n}, c_n $(n=1,2,3,\dots)$, and for z inside each of the triangles with vertices c_n, a_{2n+1}, a_{2n+2} $(n=1,2,3,\dots)$, $f(z)=1$.

For z inside each of the triangles with vertices c_n, a_{2n}, a_{2n+1} $(n=1,2,3,\dots)$, $f(z)=2$.

For every z inside the triangle Δ with vertices a_1, b_1, ζ at which f has not yet been defined, $f(z)=3$.

Suppose that $z = x + iy$ is a point in the lower half-plane in the exterior of Δ. If $y \leq -1$, $f(z) = 1$. Express the interval $(-1, 0)$ as the union of three disjoint sets A, B, C, each everywhere dense in the interval. If $y \in A$, $f(z) = 1$; if $y \in B$, $f(z) = 2$; if $y \in C$, $f(z) = 3$.

This completes the definition of $f(z)$ in the lower half-plane.

Since

$$C_{\Lambda_1}(f,\zeta) \cap C_{\Lambda_2}(f,\zeta) \cap C_{\Lambda_3}(f,\zeta) = \{1,2\} \cap \{1,3\} \cap \{2,3\} = \emptyset,$$

f has the three-arc property at ζ.

To show that f does not have the three-separated-arc property at ζ, suppose, to the contrary, that there exist three disjoint arcs Σ_1, Σ_2, Σ_3 at ζ such that

$$C_{\Sigma_1}(f,\zeta) \cap C_{\Sigma_2}(f,\zeta) \cap C_{\Sigma_3}(f,\zeta) = \emptyset. \tag{1}$$

We observe the following:

1. From the way in which f was defined, it is evident that f does not have an asymptotic value at ζ.

2. In order for (1) to hold, none of the three cluster sets in (1) can contain all three values 1, 2, 3.

3. Hence, each of the three cluster sets in (1) consists of two values, and for (1) to hold, the three cluster sets must be the sets $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.

4. If an arc at ζ has, in every neighborhood of ζ, a subarc lying to the left of Λ_1 or a subarc lying to the right of Λ_2, then the cluster set of f at ζ along that arc is $\{1,2,3\}$. Consequently, the arcs Σ_1, Σ_2, Σ_3 must eventually lie in the closure of Δ.

5. Suppose that $C_{\Sigma_1}(f,\zeta) = \{1,2\}$. Then Σ_1 must contain all but a finite number of the points a_{2n} $(n = 1,2,3,\ldots)$.

6. Suppose that $C_{\Sigma_2}(f,\zeta) = \{2,3\}$. Then Σ_2 must also contain all but a finite number of the points a_{2n} $(n = 1,2,3,\ldots)$.

7. It follows that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, contradicting the supposition that Σ_1 and Σ_2 are disjoint.

Thus our supposition is untenable, and our result is proved.

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