Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 16, 1991, 199–210

ON THE ESSENTIAL MAXIMALITY OF LINEAR OPERATORS IN A HILBERT SPACE

Bernhard Steinig

Introduction

In this paper we want to define and study the essential maximality of a linear (not necessarily bounded) operator in a Hilbert space. The concept of essential maximality is a generalization of the essential self-adjointness of a symmetric operator in the sense that an essentially maximal symmetric operator is essentially self-adjoint. To the author's knowledge, the essential maximality has been previously defined and studied only by R.A. Goldstein [1], P. Hess [2], and J. Tervo [7] for linear partial differential operators (cf. also I.S. Louhivaara and C.G. Simader [4]; for a survey of results on the essential self-adjointness of differential operators of mathematical physics we refer to H. Kalf, U.-W. Schmincke, J. Walter and R. Wüst [3]).

In 1 we shall define the essential maximality of a linear operator in view of a second one. We also give a method for verifying the essential maximality of given operators. From a corollary of this result it follows that the methods developed in the literature for the verification of the essential self-adjointness of symmetric operators can be applied to the proof of the essential maximality of linear operators.

In 2 the methods of 1 will be applied for a tensor product of linear operators, and we shall prove the relation

$$(T_1 \otimes T_2)^* = \overline{T_1^* \otimes T_2^*}$$

for two densely defined closable operators T_1 and T_2; in this relation the right side means the closure of the tensor product $T_1^* \otimes T_2^*$ in the Hilbert space $H_1 \widehat{\otimes} H_2$ which is defined as the completion of the tensor product $H_1 \otimes H_2$. K. Vala [8] has proven this result in the case of bounded operators, and he has also showed $T_1^* \otimes T_2^* \subset (T_1 \otimes T_2)^*$ for general linear operators. The result is also known at least for unbounded self-adjoint operators; cf. J. Weidmann [9], pp. 259–268, who has proved the result using the spectral theorem. We do not use the spectral theorem in our considerations.

In 3 the results of 1 and 2 will be formulated for linear partial differential operators.

doi:10.5186/aasfm.1991.1627

On this occasion I would like to express my gratitude to Professor I.S. Louhivaara of Freie Universität Berlin for his interest and his extensive support during my work on the manuscript. I also want to thank Professor R. Wüst of Technische Universität Berlin for his valuable suggestions which clarified the ideas of this paper.

1. A characterization of the essential maximality of linear operators in a Hilbert space

Let T and T' be two densely defined linear operators in the complex Hilbert space H (with the domains $D(T)$ and $D(T')$) such that the relation

$$(T\varphi, \psi) = (\varphi, T'\psi)$$

is valid for all $\varphi \in D(T)$ and all $\psi \in D(T')$ (i.e. the operators T and T' are formal adjoints of each other). The operators T and T' are closable, because the adjoint operators T^* and T'^* of T and T' in H are densely defined (e.g. $T' \subset T^*$), and we have the closures $\overline{T} = T^{**}$ and $\overline{T'} = T'^{**}$.

We say that the operator T *essentially maximal in view of the operator* T' if the relation

$$T^* = \overline{T'}$$

is valid. Of course, in this case also T' is essentially maximal in view of T (since the above relation implies $\overline{T} = T'^*$).

A closable densely defined operator T in a Hilbert space is of course essentially maximal in view of the adjoint operator T^*.

If the relation $\overline{T} = T^*$ holds for a symmetric operator T, we call T *essentially self-adjoint.*

We shall refer to the following result several times (for the proof cf. e.g. F. Riesz and B. Sz.-Nagy [5], pp. 322–323):

Theorem 1.1. *Let H be a complex Hilbert space. Let T: $D(T) \to H$, $D(T) \subset H$, be a densely defined closed linear operator. Then the linear operator $T^*T + I$,*

$$(T^*T + I)\colon D(T^*T) \to H,$$
$$D(T^*T) := \{x \in D(T) \mid Tx \in D(T^*)\},$$

is densely defined, bijective and self-adjoint.

Theorem 1.2. *Let T and T' be two densely defined operators in a Hilbert space H with the dense domains $D(T)$ and $D(T')$ so that T and T' are formal adjoints to each other. Then the operator T is essentially maximal in view of T' if and only if the condition*

$$R(\overline{T'}\,\overline{T} + I) = H$$

is satisfied.

Proof. A. Let T be essentially maximal in view of T'. Then by Theorem 1.1 one has

$$R(\overline{T'}\,\overline{T}+I)=R(T^*\overline{T}+I)=H.$$

B. Let now $R(\overline{T'}\,\overline{T}+I)=H$. Because $\overline{T'}\subset T^*$, one has

$$\overline{T'}\,\overline{T}+I\subset T^*\overline{T}+I.$$

Since by Theorem 1.1 the operator $(T^*\overline{T}+I)$: $D(T^*\overline{T})\to H$ is bijective and $R(\overline{T'}\,\overline{T}+I)=H$, it follows that

$$\overline{T'}\,\overline{T}+I=T^*\overline{T}+I$$

and consequently

$$\overline{T'}\,\overline{T}=T^*\overline{T}.$$

Because of the self-adjointness of $T^*\overline{T}$ the operator $\overline{T'}\,\overline{T}$ is densely defined and we have

$$(\overline{T'}\,\overline{T})^*=\overline{T'}\,\overline{T}.$$

From the relation

$$\overline{T'}\,\overline{T}\subset T^*T'^*\subset(\overline{T'}\,\overline{T})^*=\overline{T'}\,\overline{T}$$

we get

$$\overline{T'}\,\overline{T}=T^*T'^*. \tag{1.1}$$

Take $y\in D(T^*)$. By Theorem 1.1 there exists an element $v\in D(T'^*\overline{T'})$ with

$$y=(T'^*\overline{T'}+I)v,$$

which means

$$y-v=T'^*\overline{T'}v.$$

Because $D(\overline{T'})\subset D(T^*)$, the difference $y-v$ is an element of $D(T^*)$, and we have by (1.1) the following equations

$$\begin{aligned}T^*(y-v)&=T^*(T'^*\overline{T'}v)\\&=(T^*T'^*)(\overline{T'}v)\\&=(\overline{T'}\,\overline{T})(\overline{T'}v)\\&=\overline{T'}(\overline{T}\,\overline{T'}v).\end{aligned}$$

Since one has $D(\overline{T})\subset D(T'^*)$, we get

$$T^*(y-v)=\overline{T'}(T'^*\overline{T'}v)=\overline{T'}(y-v).$$

Therefore the difference $y - v$ is an element of $D(\overline{T'})$. Because $v \in D(\overline{T'})$, it follows $y \in D(\overline{T'})$ and one has $D(T^*) \subset D(\overline{T'})$. Thus we have shown

$$D(T^*) = D(\overline{T'}) \qquad \text{and} \qquad T^* = \overline{T'}.$$

Corollary 1.3. *Let T and T' be two densely defined operators, formal adjoints to each other, in a Hilbert space H as in Theorem 1.2.*

Each of the following conditions implies the essential maximality:

(i) $\overline{R(T'T + I)} = H$,

(ii) $T'T$ *is essentially self-adjoint.*

Thereby the domain $D(T'T)$ is defined by

$$D(T'T) := \{x \in D(T) \mid Tx \in D(T')\}.$$

Proof. In both cases one can easily show that the relation

$$R(\overline{T'}\,\overline{T} + I) = H$$

is valid ([6], p. 19).

2. The adjoint operator of the tensor product of operators

For two complex Hilbert spaces H_1 and H_2 one defines the (algebraic) tensor product $H_1 \otimes H_2$ equipped with the usual scalar product. The completion of $H_1 \otimes H_2$ in the topology induced by this scalar product will be denoted by $H_1 \widehat{\otimes} H_2$. The tensor product $T_1 \otimes T_2$ of two operators T_1, T_2 (T_j: $D(T_j) \to H_j$, $j = 1,2$) is defined as an operator in $H_1 \widehat{\otimes} H_2$ by

$$D(T_1 \otimes T_2) := D(T_1) \otimes D(T_2)$$

and

$$(T_1 \otimes T_2)(x_1 \otimes x_2) := T_1 x_1 \otimes T_2 x_2 \qquad \text{for } x_j \in D(T_j).$$

For the complete definitions we refer to J. Weidmann [9], pp. 47–49, 259–268.

We shall prove the following theorem.

Theorem 2.1. *Let H_1 and H_2 be two Hilbert spaces. By T_j we denote a densely defined closable linear operator from $D(T_j)$ ($\subset H_j$) into H_j ($j = 1,2$). Then the operator $T_1 \otimes T_2$ in the Hilbert space $H_1 \widehat{\otimes} H_2$ is essentially maximal in view of the operator $T_1^* \otimes T_2^*$:*

$$(T_1 \otimes T_2)^* = \overline{T_1^* \otimes T_2^*}.$$

Remark 2.2. (i) The statement of the theorem is reasonable since by a direct use of the definition of the tensor product of operators we have

$$\big((T_1 \otimes T_2)\varphi, \psi\big)_{H_1 \widehat{\otimes} H_2} = \big(\varphi, (T_1^* \otimes T_2^*)\psi\big)_{H_1 \widehat{\otimes} H_2} \tag{2.1}$$

for all $\varphi \in D(T_1 \otimes T_2)$ and all $\psi \in D(T_1^* \otimes T_2^*)$.

(ii) Through induction it is of course possible to prove a corresponding result for the tensor product of r operators ($r \in \mathbf{N}$), since the tensor product is associative.

Proof. A. By a direct calculation one easily verifies the relation

$$\overline{T_1} \otimes \overline{T_2} \subset \overline{T_1 \otimes T_2}.$$

B. As the second step we shall consider the operator

$$V := T_1^* \overline{T_1} \otimes T_2^* \overline{T_2}.$$

It is in $H_1 \widehat{\otimes} H_2$ densely defined with the domain

$$D(V) = D(T_1^* \overline{T_1}) \otimes D(T_2^* \overline{T_2}),$$

and V as a tensor product of two symmetric operators is symmetric in $H_1 \widehat{\otimes} H_2$ (cf. [8], p. 9).

We shall now prove the relation

$$V = T_1^* \overline{T_1} \otimes T_2^* \overline{T_2} = (T_1^* \otimes T_2^*)(\overline{T_1} \otimes \overline{T_2}). \tag{2.2}$$

Let us take two elements $u_j \in D(T_j^* \overline{T_j})$ ($j = 1, 2$). We get

$$\begin{aligned}(T_1^* \overline{T_1} \otimes T_2^* \overline{T_2})(u_1 \otimes u_2) &= T_1^* \overline{T_1} u_1 \otimes T_2^* \overline{T_2} u_2 \\ &= (T_1^* \otimes T_2^*)(\overline{T_1} u_1 \otimes \overline{T_2} u_2) \\ &= (T_1^* \otimes T_2^*)(\overline{T_1} \otimes \overline{T_2})(u_1 \otimes u_2).\end{aligned}$$

Since the corresponding relation is valid also for all finite linear combinations of elements of the form $u_1 \otimes u_2$, we get the relation (2.2).

C. Now we have to prove the statement that the operator

$$W := (\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2})$$

is symmetric. Since $\overline{T_1 \otimes T_2}$ is densely defined and closed, the operator

$$(T_1 \otimes T_2)^* (\overline{T_1 \otimes T_2})$$

is self-adjoint by Theorem 1.1. From

$$(\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2}) \subset (T_1 \otimes T_2)^* (\overline{T_1 \otimes T_2})$$

the statement then follows.

D. We will further prove that the operator $W := (\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2})$ is closed. Therefore we choose a sequence

$$\{u_k\} \subset D\big((\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2})\big)$$

with elements $u,\ f \in H_1 \widehat{\otimes} H_2$ having the properties

$$\|u_k - u\|_{H_1 \widehat{\otimes} H_2} \to 0$$

and

$$\big\|(\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2})u_k - f\big\|_{H_1 \widehat{\otimes} H_2} \to 0.$$

By (2.1) it follows

$$\big\|(\overline{T_1 \otimes T_2})(u_k - u_l)\big\|^2_{H_1 \widehat{\otimes} H_2} = \big((\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2})(u_k - u_l), u_k - u_l\big)_{H_1 \widehat{\otimes} H_2} \to 0.$$

This means that $\big\{(\overline{T_1 \otimes T_2})u_k\big\}_{k \in \mathbf{N}}$ is a Cauchy sequence in $H_1 \widehat{\otimes} H_2$, and this sequence has a limit element $v \in H_1 \widehat{\otimes} H_2$. Since the operators $\overline{T_1 \otimes T_2}$ and $\overline{T_1^* \otimes T_2^*}$ are closed, we see that

$$(\overline{T_1 \otimes T_2})u = v, \qquad (\overline{T_1^* \otimes T_2^*})v = f$$

and, furthermore,

$$(\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2})u = f.$$

Thus the operator $(\overline{T_1^* \otimes T_2^*})(\overline{T_1 \otimes T_2})$ is closed.

E. Now we introduce two (densely defined) operators in $H_1 \widehat{\otimes} H_2$:

$$A := T_1^* \overline{T_1} \otimes I_2 \quad \text{with the domain } D(A) := D(V),$$

$$B := I_1 \otimes T_2^* \overline{T_2} \quad \text{with the domain } D(B) := D(V).$$

(We denote the identity operator of H_j $(j = 1, 2)$ by I_j and the identity operator of $H := H_1 \widehat{\otimes} H_2$ by I_H.)

Let us note that A is here defined as the restriction of the symmetric operator $T_1^* \overline{T_1} \otimes I_2$ with the maximal domain $D(T_1^* \overline{T_1}) \otimes H_2$ to the domain $D(V)$ of V, and B is defined analogously. As restrictions of symmetric operators the operators A and B are symmetric on H.

F. The operators A, B and V fulfill for $u \in D(A) = D(B) = D(V)$ the relations

$$(2.3) \qquad (Au, u)_H \geq 0, \qquad (Bu, u)_H \geq 0, \qquad (Vu, u)_H \geq 0$$

and

$$(2.4) \qquad (Au, Vu)_H \geq 0, \qquad (Bu, Vu)_H \geq 0.$$

Only the latter relation will proved here.

An element $u \in D(V)$ has a representation

$$u = \sum_{r=1}^{m} u_{1,r} \otimes u_{2,r}$$

with $u_{j,r} \in D(T_j^* \overline{T_j})$. By the bilinearity of the scalar product one gets the following estimation:

$$\begin{aligned}
(Bu, Vu)_H &= \Big(\sum_{r=1}^{m} u_{1,r} \otimes T_2^*\overline{T_2}u_{2,r}, \sum_{s=1}^{m} T_1^*\overline{T_1}u_{1,s} \otimes T_2^*\overline{T_2}u_{2,s}\Big)_H \\
&= \sum_{r,s=1}^{m} \big(u_{1,r} \otimes T_2^*\overline{T_2}u_{2,r}, T_1^*\overline{T_1}u_{1,s} \otimes T_2^*\overline{T_2}u_{2,s}\big)_H \\
&= \sum_{r,s=1}^{m} \big(u_{1,r}, T_1^*\overline{T_1}u_{1,s}\big)_{H_1} \big(T_2^*\overline{T_2}u_{2,r}, T_2^*\overline{T_2}u_{2,s}\big)_{H_2} \\
&= \sum_{r,s=1}^{m} \big(\overline{T_1}u_{1,r}, \overline{T_1}u_{1,s}\big)_{H_1} \big(T_2^*\overline{T_2}u_{2,r}, T_2^*\overline{T_2}u_{2,s}\big)_{H_2} \\
&= \sum_{r,s=1}^{m} \big(\overline{T_1}u_{1,r} \otimes T_2^*\overline{T_2}u_{2,r}, \overline{T_1}u_{1,s} \otimes T_2^*\overline{T_2}u_{2,s}\big)_H \\
&= \Big(\sum_{r=1}^{m} \overline{T_1}u_{1,r} \otimes T_2^*\overline{T_2}u_{2,r}, \sum_{s=1}^{m} \overline{T_1}u_{1,s} \otimes T_2^*\overline{T_2}u_{2,s}\Big)_H \\
&\geq 0.
\end{aligned}$$

G. For the symmetric operator

$$S := V + A + B + I_H$$

with the domain $D(S) := D(V)$ the relations (2.3) imply

$$(2.5) \qquad (Su, u)_H \geq (u, u)_H \qquad \text{for all } u \in D(S).$$

We will show here that

$$(2.6) \qquad \overline{R(S)} = H.$$

Let h_j be an arbitrary element in H_j $(j = 1, 2)$. By Theorem 1.1 there exists an element $u_j \in D(T_j^* \overline{T_j})$ with

$$h_j = (T_j^* \overline{T_j} + I_j) u_j.$$

One further gets

$$\begin{aligned} h_1 \otimes h_2 &= (T_1^* \overline{T_1} u_1) \otimes (T_2^* \overline{T_2} u_2) + u_1 \otimes (T_2^* \overline{T_2} u_2) + (T_1^* \overline{T_1} u_1) \otimes u_2 + u_1 \otimes u_2 \\ &= S(u_1 \otimes u_2). \end{aligned}$$

As the completion of the linear hull of all elements $h_1 \otimes h_2$ is the space H, we get the statement (2.6).

From (2.5) and (2.6) it follows that already

$$R(\overline{S}) = H. \tag{2.7}$$

H. By the relations (2.3) and (2.4) we get for all $u \in D(V)$

$$\begin{aligned} \|Su\|_H \, \|(V + I_H)u\|_H &\geq \left|(Su, (V + I_H)u)_H\right| \\ &= (Vu + Au + Bu + u, Vu + u)_H \\ &\geq \|(V + I_H)u\|_H^2 \end{aligned} \tag{2.8}$$

and furthermore

$$\|Su\|_H \geq \|(V + I_H)u\|_H .$$

The last relation implies

$$D(\overline{S}) \subset D(\overline{V + I_H}).$$

I. It follows from part F of this proof that the symmetric operator $V + I_H$ fulfills the relation

$$((V + I_H)u, u)_H \geq \|u\|_H^2 \tag{2.9}$$

for all $u \in D(V)$. We shall show that

$$\overline{R(V + I_H)} = H.$$

Let $z \in H$ be an element with

$$(z, (V + I_H)u)_H = 0 \qquad \text{for all } u \in D(V). \tag{2.10}$$

By (2.7) there exists an element $v \in D(\overline{S}) \subset D(\overline{V + I_H})$ with $z = \overline{S}v$. The relation (2.10) implies

$$(\overline{S}v, (\overline{V + I_H})v)_H = 0.$$

Since (2.8) is valid also for the closures $\overline{S}$ and $\overline{V + I_H}$, we get

$$(\overline{V + I_H})v = 0.$$

Finally, the relation (2.9) implies $v = 0$. Thus the element $z \in H$ must be the zero element, and the relation $\overline{R(V + I_H)} = H$ is proved.

J. Since we have

$$(V + I_H)u = (T_1^*\overline{T_1} \otimes T_2^*\overline{T_2} + I_H)u$$

for $u \in D(V)$, we can conclude that the range

$$R(T_1^*\overline{T_1} \otimes T_2^*\overline{T_2} + I_H) = R\big((T_1^* \otimes T_2^*)(\overline{T_1} \otimes \overline{T_2}) + I_H\big)$$

is dense in the space H.

K. Now we can apply Corollary 1.2 to the operator

$$\overline{T_1} \otimes \overline{T_2},$$

and we get the essential maximality of $\overline{T_1} \otimes \overline{T_2}$ in view of $T_1^* \otimes T_2^*$; because $\overline{\overline{T_1} \otimes \overline{T_2}} = \overline{T_1 \otimes T_2}$, Theorem 2.1 is proved.

Finally, let us remark that the above proof could be somewhat shorter if we had used the well-known corresponding result for self-adjoint operators.

3. The essential maximality of linear partial differential operators

Let G be an arbitrary domain in $\mathbf{R}^n$. We now consider in G a linear partial differential operator

$$L(\cdot, D) = \sum_{|\alpha| \leq m} a_\alpha(\cdot) D^\alpha$$

of order $m \in \mathbf{N}$. Here we have defined $D := (D_1, \dots, D_n)$ with $D_j = -i\partial/\partial x_j$ and $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n = \big(\mathbf{N} \cup \{0\}\big)^n$ and $|\alpha| := \alpha_1 + \dots + \alpha_n$. We suppose that, for the complex-valued coefficient functions a_α, all derivatives $D^\nu a_\alpha$ for $\nu \in \mathbf{N}_0^n$ with $\nu \leq \alpha$ (i.e. for all $\nu = (\nu_1, \dots, \nu_n) \in \mathbf{N}_0^n$ with $\nu_1 \leq \alpha_1$, ..., $\nu_n \leq \alpha_n$) are continuous in G. On this condition we have for the formal adjoint linear partial differential operator $L'(\cdot, D)$ defined through the equation

$$\big(L(\cdot, D)\varphi, \psi\big)_0 = \big(\varphi, L'(\cdot, D)\psi\big)_0 \qquad \text{for all } \varphi, \psi \in C_0^\infty(G)$$

(with the $L^2(G)$ scalar product $(\cdot, \cdot)_0$) the expression

$$\begin{aligned} L'(x, D)\psi(x) &= \sum_{|\alpha| \leq m} D^\alpha \big(\overline{a_\alpha(x)}\psi(x)\big) \\ &= \sum_{|\alpha| \leq m} \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \big(D^\nu \overline{a_\alpha(x)} D^{\alpha - \nu}\psi(x)\big) \quad \text{for } \psi \in C_0^\infty(G), \end{aligned}$$

where

$$\binom{\alpha}{\nu} := \binom{\alpha_1}{\nu_1} \cdots \binom{\alpha_n}{\nu_n}.$$

The differential operators $L(\cdot, D)$ and $L'(\cdot, D)$ induce in the Hilbert space $L^2(G)$ the linear operators L and L' defined by

$$\begin{aligned} D(L) &:= C_0^\infty(G), & D(L') &:= C_0^\infty(G), \\ L\varphi &:= L(\cdot, D)\varphi & &\text{for } \varphi \in D(L), \\ L'\psi &:= L'(\cdot, D)\psi & &\text{for } \psi \in D(L'). \end{aligned}$$

The operators L and L' are closable as in the abstract setting in 1. In the literature normally the closures $\overline{L}$ and $\overline{L'}$ of L and L' are called the minimal realizations of $L(\cdot, D)$ and $L'(\cdot, D)$, and the adjoints L'^* and L^* of L' and L the maximal realizations of $L(\cdot, D)$ and $L'(\cdot, D)$ (namely, e.g. $L \subset \overline{L} \subset L'^*$).

Since in our present case the domain $D(L')$ of the operator L' coincides with the domain $D(L)$ of L in a canonical way, we do not need the words "in view of the operator L'" (which we needed in the abstract setting) in the following definition of the essential maximality of the differential operator $L(\cdot, D)$ (cf. e.g. [2], [4], [6]).

Definition 3.1. *The differential operator $L(\cdot, D)$ is called essentially maximal if the relation $L^* = \overline{L'}$ is valid.*

We now have the following results as corollaries of Theorem 1.2 and of Theorem 2.1:

Theorem 3.2. *For the essential maximality of the differential operator $L(\cdot, D)$ it is necessary and sufficient that the range of the operator $\overline{L'}\,\overline{L} + I$ is the whole space $L^2(G)$.*

Theorem 3.3. *Let $G_j \subset \mathbf{R}^{n_j}$ ($1 \le j \le r$, $r \in \mathbf{N}$) be domains in $\mathbf{R}^{n_j}$. The linear partial differential operators*

$$L_j(\cdot, D) = \sum_{|\alpha^j| \le m_j} a_{\alpha^j}(\cdot) D^{\alpha^j}, \qquad \alpha^j \in \mathbf{N}_0^{n_j}, 1 \le j \le r,$$

are defined in the domains G_j. We suppose that the derivatives $D^{\nu^j} a_{\alpha^j}(x^j)$ for all $\nu^j \in \mathbf{N}_0^{n_j}$ with $\nu^j \le \alpha^j$ exist and are continuous functions in G_j.

Then also the formal adjoint differential operators $L'_j(\cdot, D)$ are defined.

We further suppose that the differential operators $L_j(\cdot, D)$ are essentially maximal in $L^2(G_j)$.

Then the product operator

$$L(x^1, \dots, x^r, D) = \sum_{|\alpha^1| \le m_1, \dots, |\alpha^r| \le m_r} a_{\alpha^1}(x^1) \cdots a_{\alpha^r}(x^r) D^{(\alpha^1, \dots, \alpha^r)}$$

is essentially maximal in $L^2(G_1 \times \cdots \times G_r)$. *(For the differential operator* $L(x^1, \ldots, x^r, D)$ *the formal adjoint operator* $L'(x^1, \ldots, x^r, D)$ *is defined in* $G_1 \times \cdots \times G_r$.*)*

Proof. By Theorem 2.1 we get

$$(L_1 \otimes \cdots \otimes L_r)^* = \overline{L_1^* \otimes \cdots \otimes L_r^*}$$

and, since $L_j(\cdot, D)$ is essentially maximal, also

$$\begin{aligned} &= \overline{\overline{L'}_1 \otimes \cdots \otimes \overline{L'_r}} \\ &= \overline{L'_1 \otimes \cdots \otimes L'_r} \\ &= \overline{(L_1 \otimes \cdots \otimes L_r)'}. \end{aligned}$$

(Here we used the fact that the closure of the tensor product of closures of operators coincides with the closure of the tensor product of operators and that the tensor product of formal adjoints coincides with the formal adjoint of the tensor product of the original operators.)

Bibliography

[1] GOLDSTEIN, R.A.: Equality of minimal and maximal extensions of partial differential operators in $L_p(R^n)$. - Proc. Amer. Math. Soc. 17, 1966, 1031–1033.

[2] HESS, P.: Über die wesentliche Maximalität gleichmäßig stark elliptischer Operatoren in $L^2(R^n)$. - Math. Z. 107, 1968, 67–70.

[3] KALF, H., U.-W. SCHMINCKE, J. WALTER, and R. WÜST: On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials. - Spectral theory and differential equations, Proceedings of the Symposium held at Dundee, Scotland, 1–19 July, 1974. Lecture Notes in Mathematics 448, 182–226. Springer-Verlag, Berlin–Heidelberg–New York, 1975.

[4] LOUHIVAARA, I.S., and C.G. SIMADER: Über nichtelliptische lineare partielle Differentialoperatoren mit konstanten Koeffizienten. - Ann. Acad. Sci. Fenn. Ser. A I Math. 513, 1972, 1–22.

[5] RIESZ, F., and B. SZ.-NAGY: Vorlesungen über Funktionalanalysis. - [Vierte, durchgesehene Auflage.] Hochschulbücher für Mathematik 27. Deutscher Verlag der Wissenschaften, Berlin, 1982.

[6] STEINIG, B.: Remarks on the essential maximality of linear partial differential operators. - Inauguraldissertation. Freie Universität Berlin, Fachbereich Mathematik, Berlin, 1989.

[7] TERVO, J.: Zur Theorie der koerzitiven linearen partiellen Differentialoperatoren. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 45, 1983.

[8] VALA, K.: Sur le produit tensoriel des espaces hilbertiens. - Ann. Acad. Sci. Fenn. Ser. A I Math. 267, 1959.

[9] WEIDMANN, J.: Linear operators in Hilbert spaces. - Graduate Texts in Mathematics 68. Springer-Verlag, New York–Heidelberg–Berlin, 1980.

Freie Universität Berlin
Institut für Mathematik I
Arnimallee 2–6
DW-1000 Berlin 33
Bundesrepublik Deutschland

Received 6 August 1990