

A CLASS OF MEROMORPHIC FUNCTIONS WITH NO WANDERING DOMAINS

Gwyneth M. Stallard

Abstract. We consider the class of meromorphic functions with finitely many fixed points, that is the class of functions of the form $f(z) = z + R(z)e^{g(z)}$, where R is rational and g is entire. We show that, in the case where g is a polynomial, such a function has certain properties which allow us to apply the method of proof of [2, Theorem 6.1] to show that its set of normality can have no wandering domains.

1. Introduction

In the following we will let $f: \mathbf{C} \rightarrow \hat{\mathbf{C}}$ denote a non-linear meromorphic function and f^n , $n \in \mathbf{N}$, the n -th iterate of f . The set of normality, $N(f)$, is defined to be the set of points $z \in \mathbf{C}$ such that the sequence (f^n) is defined, meromorphic and forms a normal family in some neighbourhood of z . It is easy to see that $N(f)$ is open and has the property of complete invariance under f , that is $z \in N(f)$ if and only if $f(z) \in N(f)$. The complement $J(f)$ of $N(f)$ is called the Julia set. This set is clearly closed and contains all the poles of f and their pre-images. More details of these and other basic properties of the sets $N(f)$ and $J(f)$ can be found in [8] and [9] for entire functions and in [3] for meromorphic functions with at least one pole.

For any meromorphic function f it can easily be seen that a component U of the set of normality, $N(f)$, must be mapped by f into a component V of $N(f)$. If $f^{n+m}(U) \cap f^n(U) \neq \emptyset$ for some integers $n \geq 0$, $m > 0$ then we say that $f^n(U)$ is a periodic domain and, if this fails to hold for $n = 0$, U is a pre-periodic domain. If, however, $f^n(U) \cap f^m(U) = \emptyset$ for all integers $n, m \geq 0$, $n \neq m$, then U is called a wandering domain of f . The situation where U is periodic is well understood, and such domains can arise in only a small number of ways. In the case where f is a rational function Sullivan [13, 14] proved the following important result thus solving a problem which had been open since the time of Fatou and Julia.

Theorem A. *A rational function whose degree is at least 2 has no wandering domains.*

This result cannot, however, be extended to cover all meromorphic functions. Indeed several examples of transcendental entire functions with wandering domains are known. One such example is the function $f(z) = z - 1 + 2\pi i + e^{-z}$ quoted by Sullivan in [13]. It is, however, possible to extend the result to certain classes of meromorphic functions.

One such class is the class of transcendental functions whose inverse functions have only finitely many singularities as proved by Eremenko and Lyubich in [7], and by Goldberg and Keen in [10]. Goldberg and Keen obtain their result by a method which is based on Bers' version [6] of Sullivan's proof of Theorem A. This proof, however, involves the theory of Teichmüller spaces. A similar result has been obtained by Baker [2, Theorem 6.1]. His method of proof avoids the use of Teichmüller spaces by constructing an explicit family of quasiconformal maps. The precise form of his theorem is as follows.

Theorem B. *Suppose that the transcendental entire function f has the following properties:*

- (i) *there are finitely many singularities of f^{-1} ;*
- (ii) *the components of $N(f)$ are simply connected;*
- (iii) *there is a constant $K > 1$ and a finite set of real parameters $X_1, \dots, X_{n(K,f)}$ such that every entire function of the form $f_\Phi = \Phi f \Phi^{-1}$, where Φ is K -quasiconformal, and Φ fixes $0, 1, \infty$, can be expressed uniquely in terms of $X_1, \dots, X_{n(K,f)}$;*
- (iv) *if in (iii) $\Phi = \Phi(t, z)$ depends on a system of parameters t which vary in an open neighbourhood \mathcal{I} of the origin of \mathbf{R}^M , for some M , in such a way that $\Phi(0, z) = z$ and $\Phi(t, z) \in C^1(\mathcal{I})$ for fixed z , then in some open subset of \mathcal{I} the parameters X_i are also C^1 -functions of t .*

Then f has no wandering domains.

In a recent series of papers Baker, Kotus and Lü Yinian have proved several results concerning the iteration of meromorphic functions. In [5] they show that the results on wandering domains can be extended to cover the class of meromorphic functions which are not rational functions of degree less than 2 and whose inverse functions have only finitely many singularities.

By examining the conditions of Theorem B and its proof and the recent work on meromorphic functions we are led to consider meromorphic functions with finitely many fixed points. Such functions are of the form $f(z) = z + R(z)e^{g(z)}$ where R is a rational function and g is entire. We are able to prove the following result.

Theorem C. *A function f of the form $f(z) = z + R(z)e^{Q(z)}$, where R is a rational function, Q is a polynomial, and f is not a rational function of degree less than 2, has no wandering domains.*

The class of functions in Theorem C with Q constant is simply the class of rational functions of degree at least 2 for which the result is already known. The

remaining work is devoted to the proof of Theorem C for the class of functions where Q is not a constant. We denote this class by F .

We claim that if f is a function in F that is not entire then the set

$$O^-(\infty) = \{z : f^n(z) = \infty \text{ for some } n \in \mathbf{N}\}$$

contains infinitely many points and hence, from [3, Lemma 1], $J(f) = O^-(\infty)'$. For suppose that there exists a function g in F which is not entire and for which the set $O^-(\infty)$ is finite. It follows from Picard's theorem that g must be of the form

$$g(z) = \alpha + (z - \alpha)^{-k} e^{h(z)}$$

where h is an entire function and k is a positive integer. Thus $g(z) = \infty$ only at $z = \alpha$ and $g(z) = \alpha$ has no roots. As g is in the class F , it also follows that $g(z) = z$ has only finitely many roots. Thus the function

$$G(z) = \frac{g(z) - z}{\alpha - z}$$

takes each of the values 0, 1, ∞ at only finitely many points. It then follows by Picard's theorem that G and hence g are rational functions, thus giving a contradiction.

We also note that, for any function f in F , f^n is analytic in $N(f)$ for each $n \in \mathbf{N}$.

We will show that a slightly adapted form of the proof of Theorem B can be used to prove Theorem C. We first consider the effect of allowing f to be a meromorphic function instead of just entire. The proof of Theorem B uses the fact that, if f is entire, the fixed points of f are dense in $J(f)$. It was shown in [3, Theorem 1] that if f is meromorphic and $O^-(\infty)$ is infinite then the fixed points of f are dense in $J(f)$. As $O^-(\infty)$ is infinite for all functions in the class F , the argument still holds. The rest of the proof follows through easily for meromorphic functions satisfying conditions (i) to (iv) with 'entire' replaced by 'meromorphic' in condition (iii) and in a few places in the proof. In the following work we consider how each of the four conditions of Theorem B are used in the proof and show that functions in the class F satisfy similar conditions which are sufficient to allow this method of proof to be used.

2. Singularities of f^{-1}

We first consider the condition (i) of Theorem B and note that this is needed only to ensure that if f has a wandering domain U then $f^n(U)$ contains no singularities of f^{-1} , for large $n \in \mathbf{N}$. We show that this is in fact the case for all functions in the class F . The finite singularities of f^{-1} arise in one of two ways. Firstly, suppose that $z \in \mathbf{C}$ and $f'(z) = 0$. Then z is defined to be a critical

point of f and its image $f(z)$ to be a critical value of f . The critical values of f are clearly singularities of f^{-1} . The other type of singularity is known as a transcendental singularity. A point α is said to be a transcendental singularity of f^{-1} if there is a curve Γ in \mathbf{C} such that, as $z \rightarrow \infty$ on Γ , $f(z) \rightarrow \alpha$ on $f(\Gamma)$. For a function in the class F we first show that there are no problems due to its critical values by showing that all but finitely many of its critical points are contained in invariant domains of f and hence in invariant components of $N(f)$.

Theorem 2.1. *If f is in the class F then all but finitely many of its critical points are contained in invariant domains.*

Proof. We first observe that f has an infinite number of critical points. It follows from Nevanlinna's theory of meromorphic functions (see, for example, [11, Theorem 3.3], that if g is a transcendental meromorphic function and if $g(z) = 0$ has only finitely many solutions then $g'(z) = c$, where c is a constant, $c \neq 0$, has infinitely many solutions. We apply this to the function $f(z) - z$ and deduce that $f'(z) = 0$ has an infinite number of solutions.

As the result we require is invariant under conjugation we may write $f(z) = z + \lambda S(z)e^{Q(z)}/T(z)$ where λ is a non-zero constant and S, T, Q are polynomials having leading coefficients equal to 1. We write $R(z) = S(z)/T(z)$, $(\deg S - \deg T) = n$ and $\deg Q = m$. Differentiating, we see that

$$(2.1) \quad f'(z) = 1 + \lambda[R'(z) + R(z)Q'(z)]e^{Q(z)}.$$

For the sake of simplicity we will write $U(z) = R'(z) + R(z)Q'(z)$.

We now put $t = Q(z)$. In a neighbourhood of ∞ the branches of the inverse function $z = Q^{-1}(t)$ have an expansion of the form

$$(2.2) \quad z = t^{1/m} + c_0 + c_1 t^{-1/m} + \dots$$

where c_0, c_1 are constants. So $Q^{-1}(t)$ has m branches in the region $A = \{t : |t| > M, t \notin \mathbf{R}^+\}$, where M is a sufficiently large constant. A particular branch $Q_j^{-1}(t)$ is defined by taking $\arg t^{1/m}$ in the interval $(2\pi(j-1)/m, 2\pi j/m)$ where j is an integer, $1 \leq j \leq m$. For sufficiently large M , Q_j^{-1} maps A univalently onto a region B_j .

From (2.1) it is clear that the critical points, z_c , of f satisfy

$$(2.3) \quad \lambda U(z_c)e^{Q(z_c)} = -1.$$

Putting $m + n - 1 = k$ we see that, as $|z| \rightarrow \infty$,

$$U(z) = mz^k[1 + O(1/z)]$$

and so taking the logarithm of the modulus of both sides of (2.3) gives

$$(2.4) \quad \operatorname{Re} Q(z_c) = -\log |\lambda m z_c^k| + O(1/|z_c|)$$

as $|z_c| \rightarrow \infty$. Putting $t_c = Q(z_c)$, and noting that

$$|Q(z)| = |z^m|(1 + O(1/|z|))$$

as $z \rightarrow \infty$, it easily follows that $|\operatorname{Im} t_c| \rightarrow \infty$ as $|z_c| \rightarrow \infty$. Hence, for $|z_c|$ sufficiently large, z_c is one of the values $Q_j^{-1}(t_c)$ and lies inside some B_j .

We now consider all such z_c in one particular region B_j . From now on $t^{1/m}$ refers to the j -th branch. From (2.2) and (2.4) we see that

$$\operatorname{Re} t_c = -\log |\lambda m t_c^{k/m}| + O(1/|t_c|^{1/m})$$

as $|t_c| \rightarrow \infty$. So for any $M_0 > M$, $\alpha > 0$, all such t_c , with $|t_c|$ sufficiently large, lie in the region X where

$$X = \{t : |t| > M_0, \operatorname{Re} t < \alpha - \log (|\lambda| m |t|^{k/m})\}.$$

Now $f|_{B_j}$ is conjugate to the function $F_j = QfQ_j^{-1}|_A$. To prove our theorem it is sufficient to show that all the values t_c lie in invariant domains of F_j for $|t_c|$ sufficiently large. We see that

$$(2.5) \quad F_j(t) = Q\{Q_j^{-1}(t) + \lambda R(Q_j^{-1}(t))e^t\} = Q\{Q_j^{-1}(t) + \lambda e^t t^{n/m} [1 + O(t^{-1/m})]\}$$

as $|t| \rightarrow \infty$. In X we have

$$(2.6) \quad \begin{aligned} \left| \lambda e^t t^{n/m} [1 + O(t^{-1/m})] \right| &\leq \frac{|\lambda| e^\alpha}{|\lambda| m} |t|^{(n-k)/m} [1 + O(|t|^{-1/m})] \\ &= \frac{e^\alpha}{m} |t|^{(1-m)/m} [1 + O(|t|^{-1/m})] \end{aligned}$$

as $|t| \rightarrow \infty$. Recalling that $Q_j^{-1}(t) = t^{1/m} [1 + O(t^{-1/m})]$ as $|t| \rightarrow \infty$, we deduce from (2.5) and (2.6) that, for $t \in X$,

$$F_j(t) = t + \lambda m t^{(m-1+n)/m} e^t [1 + O(t^{-1/m})] = t + \lambda m t^{k/m} e^t [1 + O(t^{-1/m})]$$

as $|t| \rightarrow \infty$. We also have from (2.6) that

$$|F_j(t) - t| \leq e^\alpha [1 + O(|t|^{-1/m})]$$

as $|t| \rightarrow \infty$. Hence, given an $\varepsilon > 0$, we choose α, M_0 to ensure that

$$(2.7) \quad |F_j(t) - t| < 1 + \varepsilon$$

for all t in X .

We now put $H_j(t) = \lambda m t^{k/m} e^t$ and observe that

$$(2.8) \quad \arg H_j(t) = C_j + \operatorname{Im} t + \frac{k}{m} \arg t, \quad 0 < \arg t < 2\pi,$$

where C_j is a constant, gives a continuous definition of $\arg H_j(t)$ for t in A . We consider the curves Γ_K in X on which $\arg H_j(t)$ is constant. Writing $t = re^{i\theta}$ we have

$$(2.9) \quad \Gamma_K = \left\{ t \in X : r \sin \theta + \frac{k}{m} \theta = K - C_j \right\}$$

for some constant K . Writing $t = x + iy$ this becomes

$$(2.10) \quad \Gamma_K = \left\{ t \in X : y + \frac{k}{m} \tan^{-1}(y/x) = K - C_j \right\}.$$

It follows from (2.9) and (2.10) that there exists $R > 0$ such that

$$(2.11) \quad K > R \Rightarrow \Gamma_K \subset \{z : \operatorname{Im} z > 0\} \text{ and } K < -R \Rightarrow \Gamma_K \subset \{z : \operatorname{Im} z < 0\}.$$

We find that on a curve Γ_K

$$\frac{dy}{dx} = \frac{(k/m)y}{x^2 + y^2 + (k/m)x}.$$

If $k = 0$ then it is clear that $dy/dx = 0$. If $k \neq 0$ then by choosing α , M_0 appropriately we can ensure that

$$(2.12) \quad \left| \frac{dy}{dx} \right| < \frac{|2(k/m)y|}{x^2 + y^2} < \frac{2|k|}{|t|} < \frac{2|k|}{M_0} < 1/100.$$

We now consider the vertical separation of two curves $\Gamma_K, \Gamma_{K'}$, in X where $\delta K = K - K' > 0$, and either $K, K' > R$ or $K, K' < -R$. Take points t, t' lying on $\Gamma_K, \Gamma_{K'}$ respectively such that the real part of each point is equal to x . Then from (2.9), (2.10) and (2.11) the vertical separation of the two points is given by

$$\delta y + (k/m)\delta\theta = \delta K, \quad \text{where } |\delta\theta| < \pi/2.$$

If we consider the case when $K, K' > R$ and $\operatorname{Re} t > 0$ then, putting $L = \min(|t|, |t'|)$, we see from Figure 2.1 that

$$|\delta\theta| < \frac{\pi |\sin \delta\theta|}{2} < \frac{\pi S}{2L} < \frac{\pi |\delta y|}{2L}.$$

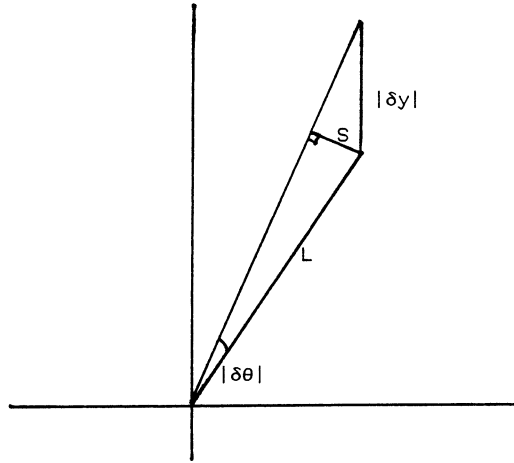


Figure 2.1.

By following the same line of argument we see that the above also holds if $\text{Re } t \leq 0$ and, similarly, if $K, K' < -R$. Thus, by choosing M_0 and hence L sufficiently large, it follows that $\delta y > 0$ and indeed

$$(2.13) \quad \delta y \pi / 3 > \delta y \left(1 + \frac{\pi |k|}{2mL} \right) > \delta K > 0.$$

We now consider the points t_c , and recall that they satisfy $\lambda U(Q_j^{-1}(t_c))e^{t_c} = -1$ (see (2.3)), where $U(Q_j^{-1}(t)) = mt^{k/m} [1 + O(t^{-1/m})]$ as $|t| \rightarrow \infty$. By definition, $H_j(t) = \lambda mt^{k/m} e^t$ and so

$$H_j(t_c) = -1 + O(t_c^{-1/m})$$

as $|t_c| \rightarrow \infty$. Hence, for large $|t_c|$, $\arg H_j(t_c)$, defined as in (2.8), lies in an interval $((2r + 1)\pi - \pi/6, (2r + 1)\pi + \pi/6)$ for some integer $r(t_c)$. As t_c lies on the curve Γ_K where $K = \arg H_j(t_c)$ it follows that t_c lies in the domain V_r in X bounded by the curves $\Gamma_{K_1}, \Gamma_{K_2}$ where $K_1 = (2r + 1)\pi + \pi/3, K_2 = (2r + 1)\pi - \pi/3$. For large r it is clear that V_r is contained in A as $M_0 > M$.

From previous arguments we see that for a suitable choice of α, M_0 we have

- i) $|F_j(t) - t| < 3/2, t \in X$, (see (2.7)),
- ii) $\arg (F_j(t) - t) \in (K - \pi/100, K + \pi/100), t \in \Gamma_K$, (as $\arg H_j(t) = K$ on Γ_K and $F_j(t) - t = H_j(t)[1 + O(t^{-1/m})]$),
- iii) $|dy|/|dx| < 1/100$ on each curve Γ_K in X (see (2.12)),
- iv) $\delta y > (3/\pi)\delta K$ for large K (see (2.13) for details).

Using these results it is not difficult to see that, for large $r, F_j(V_r) \subset V_r$. The case where $r > 0, k < 0$ is illustrated in Figure 2.2.

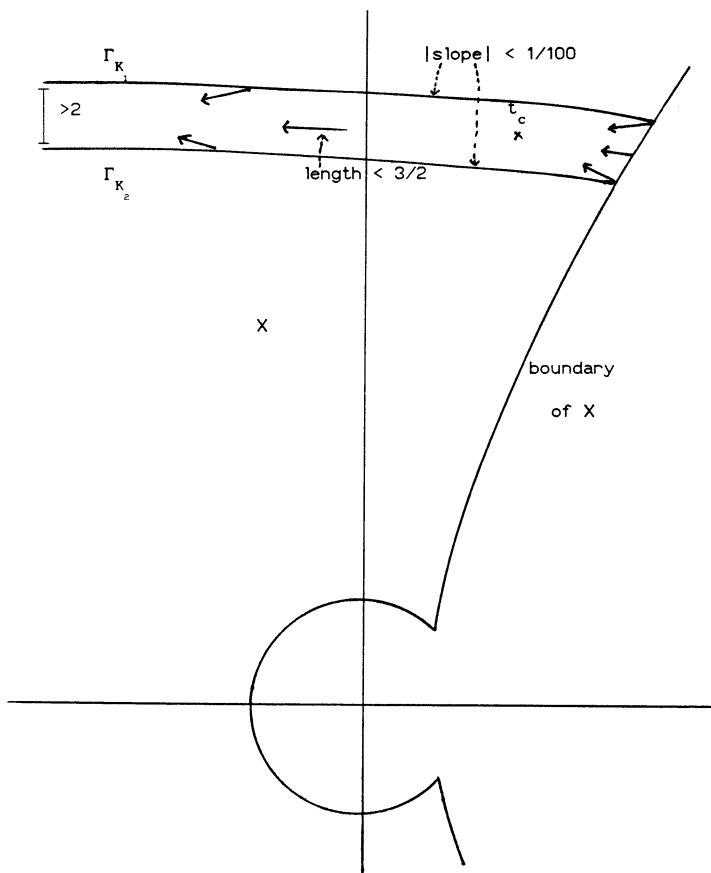


Figure 2.2.

We have shown that, for $|z_c|$ sufficiently large, there is a branch Q_j^{-1} of Q^{-1} such that $t_c = Q(z_c)$ lies in an invariant domain, V_r , of $F_j = QfQ_j^{-1}$ and $Q_j^{-1}(t_c) = z_c$. Q_j^{-1} is well defined in V_r and so $Q_j^{-1}(V_r)$ is an invariant domain of f containing z_c as required.

For any function f in F we now consider the transcendental singularities of f^{-1} . As in Theorem 2.1 we may assume that f is of the form $f(z) = z + \lambda R(z)e^{Q(z)}$ where λ is a constant, R is a rational function, Q is a polynomial and $R(z) = z^n [1 + o(1)]$, $Q(z) = z^m [1 + o(1)]$ as $z \rightarrow \infty$ for some $m, n \in \mathbf{Z}$, $m > 0$.

We now suppose that f^{-1} has a transcendental singularity at $\alpha \in \mathbf{C}$. Then by definition there is a path Γ in \mathbf{C} such that, as $z \rightarrow \infty$ on Γ , $f(z) \rightarrow \alpha$ on $f(\Gamma)$. So, writing $z = re^{i\theta}$, as $z \rightarrow \infty$ on Γ we have

$$(2.14) \quad \lambda R(z)e^{Q(z)} = \alpha - z + o(1) = -z[1 + o(1)],$$

$$(2.15) \quad |\lambda R(z)e^{Q(z)}| = |\lambda|r^n \exp \left\{ r^m [\cos(m\theta) + o(1)] \right\}.$$

For f to satisfy both (2.14) and (2.15) it is clear that as $z \rightarrow \infty$ on Γ we must have $\cos(m\theta) \rightarrow 0$ and hence $\theta \rightarrow \theta_0$ where $\sin(m\theta_0) = \pm 1$.

We can also see that as $z \rightarrow \infty$ on Γ

$$\arg \lambda R(z)e^{Q(z)} = r^m [\sin(m\theta_0) + o(1)] = r^m [\pm 1 + o(1)] \pmod{2\pi}.$$

It follows from (2.14), however, that as $z \rightarrow \infty$ on Γ

$$\arg \lambda R(z)e^{Q(z)} = \theta_0 \pm \pi + o(1).$$

This is clearly a contradiction, as $\arg \lambda R(z)e^{Q(z)}$ varies continuously with z , and so there are no finite transcendental singularities of f^{-1} .

It now easily follows from Theorem 2.1 that, given a function f in the class F and a wandering domain U of f , the component U_n of $N(f)$ containing $f^n(U)$ contains no singularities of f^{-1} for large values of $n \in \mathbf{N}$.

3. Multiply-connected domains of normality

We now consider condition (ii) of Theorem B. The proof of Theorem B requires only that if a function f has a multiply-connected wandering domain U then the component U_n of $N(f)$ which contains $f^n(U)$ is simply-connected for large $n \in \mathbf{N}$ which, for entire functions, is equivalent to condition (ii). In this section we show that if f is a function in the class F then it has no wandering domain U such that U_n is multiply-connected for all $n \in \mathbf{N}$. In order to prove this we first consider the following result proved by Baker [2, Theorem 3.1].

Lemma 3.1. *If f is transcendental entire and U is a multiply-connected component of $N(f)$ then U is a wandering domain, and every component of $N(f)$ is bounded. Further, $f^n \rightarrow \infty$ in U as $n \rightarrow \infty$ and U contains a Jordan curve γ such that, for large n , the curve $\gamma_n = f^n(\gamma)$ in U_n is at a large distance from 0 and has a non-zero winding number about 0.*

We see from this that if f is an entire function in the class F and if U is a multiply-connected wandering domain of f then U contains a curve γ whose iterates nest around ∞ . We will show that all functions in F have a similar property. To do this we need the following result concerning meromorphic functions proved in [4, Lemma 2.1].

Lemma 3.2. *Suppose that f is a meromorphic function such that $O^-(\infty)$ contains infinitely many points and that (f^n) contains a subsequence with a non-constant limit function in a component N_0 of $N(f)$. Then N_0 is either periodic or pre-periodic.*

We are now able to show that a weaker form of Lemma 3.1 is true for all functions in the class F .

Lemma 3.3. *Suppose that f is a function in the class F and that U is a wandering domain of f such that U_n is multiply-connected for all $n \in \mathbf{N}$. Then, for some $k \in \mathbf{N}$, U_k is bounded and there exists a Jordan curve γ in U_k such that for any $M > 0$, $M \in \mathbf{R}$, there exists $L(M) \in \mathbf{N}$ and a continuum $\delta_{L(M)} \subset \gamma_{L(M)} = f^{L(M)}(\gamma)$ such that $\delta_{L(M)} \subset \{z : |z| > M\}$, with one of the components of the complement of $\delta_{L(M)}$ being bounded and containing $\{z : |z| < M\}$.*

Proof. As already observed, if f is entire then the result follows immediately from Lemma 3.1. Now suppose that f is a function in the class F with poles p_1, \dots, p_m and that U is a wandering domain of f such that U_n is multiply-connected for all $n \in \mathbf{N}$. From Section 2 we know that there exists $k \in \mathbf{N}$ such that, putting $V = U_k$, each V_n contains no singularities of f^{-1} .

We take a Jordan curve $\gamma(t)$ in V , $\gamma: [0, 1] \rightarrow V$, $\gamma(0) = p$, which is not null-homotopic in V . We claim that, for each $n \in \mathbf{N}$, γ_n is not null-homotopic in V_n . For suppose not then, for some $m \in \mathbf{N}$, $\gamma_m \sim f^m(p)$ in V_m where $\sigma \sim \psi$ if and only if σ is homotopic to ψ . We now take the branch g of f^{-m} such that $gf^m(p) = p$. As there are no singularities of f^{-1} in each V_n , $1 \leq n \leq m$, it follows that the continuation of g around γ_m gives γ . Hence g can be used to lift the homotopy $\gamma_m \sim f^m(p)$ in V_m to the homotopy $\gamma \sim p$ in V . This is clearly a contradiction.

We now take a Jordan curve γ in V such that, for each $r \in \mathbf{N}$, γ_r is not null-homotopic in V_r . It is clear that, for each r , one of the bounded components of $\mathbf{C} \setminus \gamma_r$, say D_r , must contain a point in $J(f)$ and hence, as $J(f) = O^-(\infty)'$, a point in $O^-(\infty)$. We denote by $n(r)$ the least non-negative integer such that $f^{n(r)}$ maps a point in D_r to one of the poles $p_{i(r)}$ of f , where $i(r) \in \{1, \dots, m\}$. As $f^{n(r)}$ is analytic in D_r it follows that $f^{n(r)}(D_r)$ is a bounded domain containing $p_{i(r)}$. Without loss of generality we may assume that $i(r) = 1$ for infinitely many values of r .

From Lemma 3.2 we see that all convergent subsequences of $(f^{r+n(r)})$ in V must have constant limits and so the spherical diameter of $\gamma_{r+n(r)}$ must tend to zero as $r \rightarrow \infty$. So there must either be a subsequence of $(\gamma_{r+n(r)})$ which tends to ∞ or a subsequence which tends to p_1 .

We now take M to be a real value such that $M > \max(|p_1|, \dots, |p_m|)$. In the case where there is a sequence of curves $\gamma_{r(k)+n(r)}$ tending to ∞ , there exists $r(k)$ such that $\gamma_{r(k)+n(r)} \subset \{z : |z| > M\}$. As $p_1 \in f^{n(r)}(D_{r(k)})$, it follows that $f^{n(r)}(D_{r(k)}) \supset \{z : |z| > M\}$. In this case we take $\delta_{L(M)} = \partial f^{n(r)}(D_{r(k)}) \subset \gamma_{r(k)+n(r)}$.

In the case where there is a sequence of curves $\gamma_{r(k)+n(r)}$ tending to p_1 , as p_1 is a pole of f , it follows that there exists $r(k)$ such that $\gamma_{r(k)+n(r)+1} \subset \{z : |z| > M\}$, and also that $f^{n(r)+1}(D_{r(k)})$ is a full neighbourhood of ∞ contained in $\{z : |z| > M\}$. So we put $L(M) = n(r) + r(k) + 1$ and take $\delta_{L(M)}$ to be the boundary of the component of the complement of $\gamma_{r(k)+n(r)+1}$ containing p_1 .

Finally since $V \cap V_n = \emptyset$, $n > 0$, the above results clearly imply that V is bounded.

We are now in a position to prove the main result of this section.

Theorem 3.4. *A function f in the class F has no wandering component U of the set $N(f)$ such that U_n is multiply-connected for all $n \in \mathbb{N}$.*

Proof. The proof is by contradiction. Suppose that $f(z) = z + R(z)e^{Q(z)}$ is a function in F with a wandering domain U such that U_n is multiply-connected for all $n \in \mathbb{N}$. We may then take $k \in \mathbb{N}$ and a curve γ in $U_k = V$ satisfying the conditions of Lemma 3.3. Denoting hyperbolic distance in W by $[\cdot, \cdot]_W$ and recalling that V is bounded we see that there exists a constant L such that $[z_1, z_2]_V \leq L$ for all $z_1, z_2 \in \gamma$. Writing $f^n(z_i) = w_{n,i}$, $i = 1, 2$, it follows from Schwarz's lemma that $[w_{n,1}, w_{n,2}]_{f^n(V)} \leq L$ for all $w_{n,1}, w_{n,2} \in \gamma_n$. We also know that V is a wandering domain and so, for large n , $f^n(V) \subset \Omega$ where Ω is the plane punctured at the points $0, 1$. So, for large n , we have

$$L \geq [w_{n,1}, w_{n,2}]_{f^n(V)} \geq [w_{n,1}, w_{n,2}]_\Omega$$

for all $w_{n,1}, w_{n,2} \in \gamma_n$.

Using the estimates for hyperbolic distance in Ω given in [1, p. 17, 18] we see that there exist real constants $M_0, \lambda > 0$ such that if $w_{n,1}, w_{n,2}$ are two points lying on γ_n such that $|w_{n,1}|, |w_{n,2}| > M_0$ then

$$L \geq [w_{n,1}, w_{n,2}]_\Omega \geq \lambda \int_\sigma \frac{|dz|}{|z| \log |z|} \geq \lambda \int_{|w_{n,1}|}^{|w_{n,2}|} \frac{d|z|}{|z| \log |z|}$$

where σ is a path joining $w_{n,1}$ to $w_{n,2}$. Hence for such points

$$\lambda (\log \log |w_{n,2}| - \log \log |w_{n,1}|) \leq L,$$

$$\Rightarrow \log \log |w_{n,2}| \leq \frac{L}{\lambda} + \log \log |w_{n,1}|,$$

$$\Rightarrow \log |w_{n,2}| \leq A \log |w_{n,1}|,$$

$$(3.1) \quad \Rightarrow |w_{n,2}| \leq |w_{n,1}|^A$$

where A is a positive constant dependent only on γ .

We now take a real number $M > 2M_0$ and a curve $\delta_{L(M)}$ which satisfies the conditions of Lemma 3.3. We see that there exist points $w_{L(M),1}, w_{L(M),2}$ on $\delta_{L(M)}$ such that $Q(w_{L(M),1})$ is real and positive and $Q(w_{L(M),2})$ is real and negative. Thus, for M sufficiently large,

$$\begin{aligned} |f(w_{L(M),1})| &= |w_{L(M),1} + R(w_{L(M),1})e^{Q(w_{L(M),1})}| \\ &> |e^{(1/2)Q(w_{L(M),1})}| > |w_{L(M),1}| > M_0 \end{aligned}$$

and

$$M_0 < (1/2)|w_{L(M),2}| < |f(w_{L(M),2})| = |w_{L(M),2} + R(w_{L(M),2})e^{Q(w_{L(M),2})}| < (3/2)|(w_{L(M),2})|.$$

It follows that $f(w_{L(M),1}), f(w_{L(M),2})$ also satisfy (3.1) and so

$$|e^{(1/2)Q(w_{L(M),1})}| < |f(w_{L(M),1})| \leq |f(w_{L(M),2})|^A < (3/2)^A |w_{L(M),2}|^A \leq (3/2)^A |w_{L(M),1}|^{A^2}.$$

From Lemma 3.3 we can take M and hence $|w_{L(M),1}|$ to be arbitrarily large which is clearly a contradiction. This completes the proof of Theorem 3.4.

4. Conjugation with quasiconformal maps

Finally, we consider conditions (iii) and (iv) of Theorem B. We show that all functions in the class F satisfy these conditions if the word ‘entire’ is replaced by ‘meromorphic’ in condition (iii). This is sufficient to allow the use of the proof of Theorem B modified to apply to meromorphic functions. It is clear that for each function f in the class F we must consider families of the form

$$G_K(f) = \left\{ f_\Phi = \Phi f \Phi^{-1} : \Phi \text{ is } K\text{-quasiconformal fixing } 0, 1, \infty, f_\Phi \text{ is meromorphic} \right\}.$$

Throughout the following section we will assume that if $f(z) = z + R(z)e^{Q(z)}$, where R is a rational function and Q is a polynomial of degree q , then $Kq < q+1$. The only result about quasiconformal maps that we use is the following (see, for example, [12] for a proof of this result and for more background material).

Lemma 4.1. *If f is a K -quasiconformal homeomorphism of the complex plane, fixing 0 and ∞ then there is a constant C such that for large z we have*

$$|f(z)| < C|z|^K \quad \text{and} \quad |f^{-1}(z)| < C|z|^K.$$

We also need the following result proved in [8].

Lemma 4.2. *Suppose that*

$$f(z) = z + a_{m+1}(z - z')^{m+1} + \dots$$

near $z = z'$, where $a_{m+1} \neq 0$ and $m \geq 1$. Then there exist m equally spaced rays at z' and m regions D_i , $1 \leq i \leq m$, each of which lies between two adjacent rays and is bounded by a smooth curve which is tangent to the rays at z' , such that $f(D_i) \subset D_i$ and, in D_i , $f^n(z) \rightarrow z'$ in the direction of the bisector, as $n \rightarrow \infty$. The regions D_i are known as petals.

We are now in a position to show that functions in the class F satisfy the modified form of condition (iii).

Lemma 4.3. *For a specified $K > 1$, and a function f in the class F , the family $G_K(f)$ can be expressed uniquely in terms of a finite set of complex parameters $X_1, \dots, X_{n(K,f)}$.*

Proof. We take a function f in the class F and recall that f can be written in the form

$$f(z) = z + S(z)e^{Q(z)}/T(z)$$

where S, T, Q are polynomials. We may assume that S and T have leading coefficients equal to one and have no common roots. We denote the degrees of S, T, Q by s, t, q respectively. We write $S(z) = \prod_i (z - z_i)^{s(i)}$, where $\sum_i s(i) = s$, and note that the fixed points of f are precisely the points z_i . We now take a function f_Φ from the family $G_k(f)$. A point z is a fixed point of f_Φ if and only if $f_\Phi(z) = \Phi f \Phi^{-1}(z) = z$ implying $f \Phi^{-1}(z) = \Phi^{-1}(z)$. Thus $\Phi^{-1}(z) = z_i$ giving $z = \Phi(z_i)$, for some z_i .

We now write $T(z) = \prod_j (z - w_j)^{t(j)}$, and observe that the poles of f are precisely the points w_j . Using a similar argument to the above we see that a point z is a pole of f_Φ if and only if $f_\Phi(z) = \Phi f \Phi^{-1}(z) = \infty$ implying $f \Phi^{-1}(z) = \Phi^{-1}(\infty) = \infty$. Thus $\Phi^{-1}(z) = w_j$ giving $z = \Phi(w_j)$, for some w_j .

It follows from the above work that

$$f_\Phi(z) = z + S_\Phi(z)e^{g(z)}T_\Phi(z)$$

for some polynomials S_Φ, T_Φ and some entire function g . We now use Lemma 4.1 to see that the maximum modulus of f_Φ satisfies $M(f_\Phi, r) = O(\exp(r^{Kq}))$ and hence g must be a polynomial Q_Φ of degree $q' \leq Kq < q + 1$. We now use the same reasoning to deduce that Q has degree at most Kq' . As $K(q - 1) < q$, it follows that Q_Φ must have degree equal to q .

It is clear that T_Φ can be written as $T_\Phi(z) = \prod_j [z - \Phi(w_j)]^{t(j)}$ and so T_Φ has degree t .

We now consider the polynomial S_Φ . The roots this has are the points $\Phi(z_i)$ but it is not clear what orders these roots have. Suppose z_i is a multiple root of S , i.e. $s(i) > 1$. Then, near z_i , f satisfies the conditions of Lemma 4.2 and we see that there are $s(i) - 1$ petals at z_i . It is easy to see that these are mapped by Φ onto $s(i) - 1$ petals for f_Φ at $\Phi(z_i)$ and so $\Phi(z_i)$ is a root of S_Φ of order $s(i)$. We deduce that S_Φ can be written as $S_\Phi(z) = \prod_i [z - \Phi(z_i)]^{s(i)}$ and so S_Φ is of degree s .

It follows that the family $G_K(f)$ can be expressed in terms of a finite set of parameters $X_1, \dots, X_{n(K,f)}$ as required where $n(K, f) = s + t + q + 1$. The parameters are the zeros of S_Φ and T_Φ and the coefficients of Q_Φ .

We now consider the effect of letting Φ depend on a parameter t and find that the following result holds, and hence functions in the class F satisfy condition (iv).

Lemma 4.4. *If, for a given family $G_K(f)$, where $f \in F$, each homeomorphism $\Phi = \Phi(t, z)$ depends on a system of parameters t which vary in an open neighbourhood \mathcal{I} of the origin of \mathbf{R}^M for some M , and $\Phi(0, z) = z$, $\Phi(t, z) \in C^1(\mathcal{I})$ for fixed z , then for some open subset \mathcal{I}' of \mathcal{I} , the parameters X_i are in $C^1(\mathcal{I}')$.*

Proof. From Lemma 4.3 we know that if $f(z) = z + S(z)e^{Q(z)}/T(z)$ where S, T, Q are polynomials of degree s, t, q respectively then each function $f_\Phi = \Phi f \Phi^{-1}$ in $G_K(f)$ is of the form

$$f_\Phi(z) = z + S_\Phi(z)e^{Q_\Phi(z)}/T_\Phi(z)$$

where S_Φ, T_Φ, Q_Φ are polynomials of degree s, t, q respectively.

Writing $S(z) = \prod_i (z - z_i)^{s(i)}$ we also know that S_Φ can be written as $S_\Phi(z) = \prod_i [z - \Phi(z_i)]^{s(i)}$. Similarly, if $T(z) = \prod_j (z - w_j)^{t(j)}$, then $T_\Phi(z) = \prod_j [z - \Phi(w_j)]^{t(j)}$. Thus S_Φ and T_Φ are uniquely determined by the functions $\Phi(z_i)$ and $\Phi(w_j)$ all of which are in $C^1(\mathcal{I})$.

We now write $Q_\Phi(z) = a_q z^q + \dots + a_0$ where the $a_k, 0 \leq k \leq q$, depend only on t . The result will clearly be proved if we show that each a_k is in $C^1(\mathcal{I}')$ for some open subset \mathcal{I}' of \mathcal{I} .

As shown in the proof of Theorem 2.1, f has infinitely many critical points $\alpha_c, c \in \mathbf{N}$. The critical points of f_Φ are precisely $\alpha_c(t) = \Phi(t, \alpha_c)$. Each such $\alpha_c(t)$ satisfies

$$\begin{aligned} F_c[\Phi(t, z_i), \Phi(t, w_j), a_k(t), t] &= T_\Phi^2(\alpha_c) \frac{\partial}{\partial z} f_\Phi(\alpha_c(t)) \\ &= T_\Phi^2(\alpha_c(t)) + [-S_\Phi(\alpha_c(t))T'_\Phi(\alpha_c(t)) + S'_\Phi(\alpha_c(t))T_\Phi(\alpha_c(t))] e^{Q_\Phi(\alpha_c(t))} \\ &\quad + S_\Phi(\alpha_c(t))T_\Phi(\alpha_c(t))Q'_\Phi(\alpha_c(t))e^{Q_\Phi(\alpha_c(t))} = 0. \end{aligned}$$

We see that each F_c depends analytically on each of the parameters of f_Φ and belongs to $C^1(\mathcal{I})$.

We have $\alpha_c(0) = \alpha_c$ and so, at $t = 0$,

$$\begin{aligned} (4.1) \quad \frac{\partial F_c}{\partial a_k} &= [-S(\alpha_c)T'(\alpha_c) + S'(\alpha_c)T(\alpha_c) + S(\alpha_c)T(\alpha_c)Q'(\alpha_c)] \alpha_c^k e^{Q(\alpha_c)} \\ &\quad + kS(\alpha_c)T(\alpha_c)\alpha_c^{k-1} e^{Q(\alpha_c)}. \end{aligned}$$

We claim that there are $q + 1$ choices of c for which the vectors

$$\underline{u}_c = \left(\frac{\partial F_c}{\partial a_0}, \dots, \frac{\partial F_c}{\partial a_q} \right)$$

are linearly independent at $t = 0$. If not, then all of the \underline{u}_c span a space in \mathbb{C}^{q+1} of dimension at most q . This implies that there exists a vector $\underline{v} = (v_0, \dots, v_q)$ such that $\underline{v} \neq \underline{0}$, and for each c at $t = 0$

$$(4.2) \quad v_0 \frac{\partial F_c}{\partial a_0} + \dots + v_q \frac{\partial F_c}{\partial a_q} = 0.$$

Combining (4.1) and (4.2) we see that the polynomial P where

$$P(z) = \sum_{k=0}^q v_k \{ [-S(z)T'(z) + S'(z)T(z) + S(z)T(z)Q'(z)]z^k + kS(z)T(z)z^{k-1} \}$$

then vanishes for each α_c and hence for all z . The highest power of z in this polynomial is $s + t + 2q - 1$ and this comes from the term $v_k S(z)T(z)Q'(z)z^q$ and so $v_q = 0$. Using the same line of argument it follows successively that $v_{q-1} = 0, v_{q-2} = 0, \dots, v_0 = 0$ and so $\underline{v} \equiv \underline{0}$ which is a contradiction.

We are now able to take $q + 1$ choices of c , which we label as $c(d)$ where $d = 0, 1, \dots, q$, such that the vectors $\underline{u}_{c(d)}$ are linearly independent at $t = 0$. It follows that the matrix

$$\left(\frac{\partial F_{c(d)}}{\partial a_k} \right), \quad 0 \leq d, k \leq q,$$

is non-singular at $t = 0$. Letting $\underline{b}(t)$ be the vector with components $\Phi(t, z_1), \Phi(t, z_j)$, and $\underline{a}(t) = (a_0(t), \dots, a_q(t))$ it now follows from the implicit function theorem that there is a C^1 function $g: \underline{b}(t) \rightarrow \underline{a}(t)$ in a neighbourhood of $\underline{a}(0)$. We know that all the components of \underline{b} are in $C^1(\mathcal{I})$ and hence all the $a_k, 0 \leq k \leq q$, are in $C^1(\mathcal{I}')$ for some open subset \mathcal{I}' of \mathcal{I} as required.

Using the results of the last three sections we are now able to prove Theorem C using essentially the same argument as that used to prove Theorem B in [2, Section 7].

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References

- [1] AHLFORS, L.: Conformal invariants. - McGraw-Hill, New York, 1973.
- [2] BAKER, I.N.: Wandering domains in the iteration of entire functions. - Proc. London Math. Soc. (3) 49, 1984, 563-576.
- [3] BAKER, I.N., J. KOTUS, and LÜ YINIAN: Iterates of meromorphic functions I. - Ergodic Theory Dynamical Systems 11, 1991, 241-248.
- [4] BAKER, I.N., J. KOTUS, and LÜ YINIAN: Iterates of meromorphic functions III. - Ergod. Theory Dynam. Systems (submitted).
- [5] BAKER, I.N., J. KOTUS, and LÜ YINIAN: Iterates of meromorphic functions IV. - Manuscript.

- [6] BERS, L.: On Sullivan's proof of the finiteness theorem and the eventual periodicity theorem. - *Amer. J. of Math.* 109, 1987, 833-852.
- [7] EREMENKO, A., and M.YU. LYUBICH: Iterates of entire functions. - Preprint 6 of the Physico-technical Institute of Low Temperatures, Ukrainian S.S.R. Academy of Sciences, Kharkov, 1984 (in Russian).
- [8] FATOU, P.: Sur les équations fonctionnelles. - *Bull. Soc. Math. France* 47, 1919, 161-271; 48, 1920, 33-94, 208-314.
- [9] FATOU, P.: Sur l'itération des fonctions transcendentes entières. - *Acta Math.* 47, 1926, 337-370.
- [10] GOLDBERG, L.R., and L. KEEN: A finiteness theorem for entire functions. - *Ergod. Theory Dynam. Systems* 6, 1986, 183-192.
- [11] HAYMAN, W.K.: *Meromorphic functions*. - The Clarendon Press, Oxford, 1964.
- [12] LEHTO, O., and K.I. VIRTANEN: *Quasiconformal mappings in the plane*. - Springer-Verlag, 1973.
- [13] SULLIVAN, D.: Itération des fonctions analytiques complexes. - *C. R. Acad. Sci. Paris Sér. A-B* 294, 1982, 301-303.
- [14] SULLIVAN, D.: Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains. - *Ann. of Math.* 122, 1985, 401-418.

Imperial College of Science, Technology and Medicine
Department of Mathematics
London SW7 2AZ
England

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