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FREE QUASICONFORMALITY IN BANACH SPACES II

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1. Introduction

This paper is continuation to [Vä$_4$]. We consider homeomorphisms $f\colon G \to G'$ where G and G' are domains in Banach spaces E and E', respectively. In [Vä$_4$] we introduced the class of *freely* φ*-quasiconformal* (φ-FQC) maps, which in the case $E = R^n = E'$ essentially agrees with the class of K-quasiconformal (K-QC) maps. We also considered some related concepts, in particular, φ*-solid* maps.

In this paper, the boundary and distortion properties of these maps are studied. In Section 2 we show that an isolated boundary point is removable for solid and FQC maps. Since quasihyperbolic geodesics do not always exist, we prove in Section 3 existence theorems for a generalized concept called a *neargeodesic*. Section 4 deals with a new tool called the *coarse length* of an arc. We also introduce the class of *coarsely quasihyperbolic* (CQH) maps, which includes all solid and hence all FQC maps. In Section 5 we relativize the theory of quasisymmetric (QS) and quasimöbius (QM) maps. The theory of uniform domains in Banach spaces is developed in Section 6. The theory of Sections 3–6 is applied in Section 7 to prove various results on maps of uniform domains. For example, a CQH map $f\colon G \to G'$ between uniform domains extends to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$, which is QM rel ∂G. In particular, the induced boundary map $f_0\colon \partial G \to \partial G'$ is QM. If f is FQC, then f itself is QM. Many of the results are also new in the classical case $E = R^n = E'$. In Section 8 we apply the idea of relative quasisymmetry to reprove and generalize the recent interesting distortion theorem of D. Cooper on CQH maps of the n-ball.

We shall use the terminology and notation of [Vä$_4$]. In particular, X and Y will be metric spaces, E and E' will be real Banach spaces and $G \subset E$, $G' \subset E'$ domains. In the present paper we shall also assume that $\dim E \geq 2$, $\dim E' \geq 2$. The closure $\overline{A}$ and the boundary ∂A of a set $A \subset E$ are taken in the extended space $\dot{E} = E \cup \{\infty\}$. References to [Vä$_4$] will be given in the form I.2.5, which means the result 2.5 of [Vä$_4$].

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2. Isolated boundary points

2.1. *Introduction to Section* 2. Suppose that x_0 is an isolated boundary point of a domain $G \subset E$ and that $f\colon G \to G'$ is a homeomorphism. If $E = R^n = E'$ and f is K-QC, it is well known that f has a K-QC extension to $G \cup \{x_0\}$. We shall prove the corresponding result for solid and FQC maps.

For solid maps the result is new also in R^n. However, the proof for this case would be considerably shorter. For example, we know that the cluster set $\operatorname{clus}(f, x_0)$ is a component C of $\partial G'$. For small r the QH diameter of $S(x_0, r)$ is π. If C contains more than one point, then the QH diameter of $fS(x_0, r)$ tends to ∞ as $r \to 0$, which contradicts the solidity of f. This proves that f has a limit at x_0. In the infinite-dimensional case we can say very little of the cluster set by topological reasons, which makes the proof much longer.

We start with a basic result of Banach geometry.

2.2. Lemma. *Let $S = S(x_0, r)$ be a sphere in E. Then each pair of points in S can be joined by a 2-quasiconvex arc in S.*

Proof. Let $a, b \in S$, $a \neq b$. Let T be a 2-dimensional linear subspace of E containing a and b. Then $T \cap S$ is a topological circle, and the points a, b divide $T \cap S$ to arcs γ_1, γ_2. Assuming $l(\gamma_1) \le l(\gamma_2)$, the arc γ_1 is 2-quasiconvex by [Sc, 4.4]. □

2.3. *Remark.* In a Hilbert space we can replace the constant 2 of 2.2 by $\pi/2$. The bound 2 is sharp in the plane with the norm $|x| = |x_1| + |x_2|$.

2.4. *Notation.* For a set $A \subset G$ we let $k(A)$ denote the QH diameter of A, and $k(A, B)$ is the QH distance between two nonempty sets $A, B \subset G$. In G' we replace k by k'.

2.5. Lemma. *Suppose that x_0 is a finite isolated boundary point of G and that $B(x_0, 2r) \subset G \cup \{x_0\}$. Suppose also that $a, b \in G$ with $|a - x_0| \le |b - x_0| \le r$. Then*

$$k(a, b) \le \ln \frac{|b - x_0|}{|a - x_0|} + 4,$$

and thus $k\big(S(x_0, r)\big) \le 4$.

Proof. We may assume that $x_0 = 0$. Set $y = |b|a/|a|$. By 2.2 we can choose an arc $\alpha \subset S\big(x_0, |b|\big)$ joining y and b with $l(\alpha) \le 2|b - y| \le 4|b|$. Then $\gamma = [a, y] \cup \alpha$ joins a and b in G, and hence

$$k(a, b) \le l_k(\gamma) = \int_{|a|}^{|b|} \frac{dt}{t} + \frac{l(\alpha)}{|b|} \le \ln \frac{|b|}{|a|} + 4. \;\square$$

2.6. Lemma. *Suppose that $G \neq E$, $G' \neq E'$ and that $f\colon G \to G'$ is solid. Let $A \subset G$ with $k(A) < \infty$. Then fA is bounded, $d(fA, \partial G') > 0$, $\overline{fA} \subset G'$ and $\partial fA = f\partial A$.*

Proof. By solidity we have $k'(fA) < \infty$. By I.2.2(1) this implies that fA is bounded and that $d(fA, \partial G') > 0$. Hence $\overline{fA} \subset G'$, and ∂fA is the boundary of fA in the topology of G'. Since f is a homeomorphism, we have $\partial fA = f\partial A$. □

2.7. Theorem. *Suppose that x_0 is an isolated boundary point of $G \neq E$ and that $f\colon G \to G'$ is φ-solid. Then f has a limit $y_0 \in \dot{E}'$ at x_0, and y_0 is an isolated boundary point of $\partial G'$. Setting*

$$G_1 = G \cup \{x_0\}, \qquad G'_1 = G' \cup \{y_0\}, \qquad f_1(x_0) = y_0,$$

we obtain an extension of f to a homeomorphism $f_1\colon G_1 \to G'_1$. If $x_0 \neq \infty$, $y_0 \neq \infty$ and $G_1 \neq E$, then f_1 is φ_1-solid with $\varphi_1 = \varphi_1(\varphi)$.

Proof. Performing a preliminary inversion if necessary, we may assume that $x_0 \neq \infty$. By a translation we can normalize $x_0 = 0$. We break the proof to three lemmas 2.8, 2.15 and 2.16.

2.8. Lemma. *The map f has a limit y_0 at 0, possibly $y_0 = \infty$.*

Proof. Writing $U(r) = B(r) \setminus \{0\}$ we choose $r_0 > 0$ with $U(2r_0) \subset G$. We may assume that $0 \in fS(r_0)$. If $fx \to \infty$ as $x \to x_0$, there is nothing to prove. We may therefore assume that there are $R > 0$ and a sequence of points $x_j \in U(r_0)$ such that $r_j = |x_j| \to 0$ as $j \to \infty$, the sequence (r_j) is strictly decreasing, and $|fx_j| \leq R$ for all j. Since E' is complete, it suffices to show that $d(fU(r_j)) \to 0$ as $j \to \infty$. The proof consists of six steps.

Step 1. Writing $S_j = S(r_j)$ for $j \geq 0$ we infer from I.2.2(1) that

$$k(S_j, S_0) \geq \ln \frac{r_0}{r_j} \to \infty$$

as $j \to \infty$. By solidity this implies

$$\lim_{j\to\infty} k'(fS_j, fS_0) = \infty. \tag{2.9}$$

If $B(2R) \subset G'$, I.2.5 implies $k'(fx_j, 0) \leq 1$ for all j. By (2.9) this is impossible, and hence

$$B(2R) \cap \partial G' \neq \emptyset. \tag{2.10}$$

Step 2. For $j \geq 1$ let A_j be the annulus $B(r_0) \setminus \overline{B}(r_j)$. We next show that there is $R_1 = R_1(R, \varphi) > 0$ such that

$$fA_j \subset \overline{B}(R_1) \tag{2.11}$$

for all $j \geq 1$. If $x \in S_j$, then I.2.2(1), 2.5 and (2.10) imply

$$|fx - fx_j| \leq \delta'(fx_j)e^{k'(fx,fx_j)} \leq 3Re^{\varphi(4)}.$$

Hence $fS_j \subset \overline{B}(R_1)$ with $R_1 = 3Re^{\varphi(4)} + R$. Since $0 \in fS_0$, the same argument shows that $fS_0 \subset \overline{B}(R_1)$. From 2.5 we infer

$$k(\overline{A}_j) \leq \ln \frac{r_0}{r_j} + 4 < \infty. \tag{2.12}$$

By 2.6 this implies

$$\partial fA_j = fS_0 \cup fS_j. \tag{2.13}$$

Hence $fA_j \subset \overline{B}(R_1)$, since otherwise $\dot{E} \setminus \overline{B}(R_1)$ would be a connected set meeting fA_j and $\dot{E} \setminus fA_j$ but not ∂fA_j.

Step 3. We show that $\alpha_j = d(fS_j, fS_0) > 0$ for $j \geq 1$. By (2.12) and 2.6 we have $d(f\overline{A}_j, \partial G') = q_j > 0$. If $x \in S_0$, $y \in S_j$ and $|fx - fy| \leq q_j/2$, then I.2.5 yields

$$k'(fx, fy) \leq \frac{2|fx - fy|}{\delta'(fx)} \leq \frac{2|fx - fy|}{q_j}.$$

On the other hand, I.2.2(1) gives $k(x, y) \geq \ln\,(r_0/r_j)$. Hence

$$|fx - fy| \geq \frac{q_j}{2}\varphi^{-1}\left(\ln \frac{r_0}{r_j}\right) = \beta_j > 0,$$

which implies $\alpha_j \geq \beta_j \wedge (q_j/2) > 0$.

Step 4. We show that fS_j and fS_0 can be joined by an arc $\gamma_j \subset f\overline{A}_j$ which consists of two line segments. Since A_j is connected, there is $z \in A_j$ such that

$$d(fz, fS_0) = d(fz, fS_j) = \lambda.$$

Choose points $a_0 \in fS_0$ and $a_j \in fS_j$ such that

$$|fz - a_0| \wedge |fz - a_j| \leq \lambda + \alpha_j/2.$$

Replacing a_0 by a point in $[fz, a_0]$ we may assume that $[fz, a_0) \cap fS_0 = \emptyset$, and similarly $[fz, a_j) \cap fS_j = \emptyset$. If there is a point $u \in [fz, a_0] \cap fS_j$, then

$$|u - fz| \geq d(fz, fS_j) = \lambda,$$

and hence $|u - a_0| \leq \alpha_j/2$. Since $\alpha_j = d(fS_j, fS_0) \leq |u - a_0|$, we obtain a contradiction. Thus $[fz, a_0] \cap fS_j = \emptyset$. Similarly $[fz, a_j] \cap fS_0 = \emptyset$. By (2.13) the arc $\gamma_j = [fz, a_0] \cup [fz, a_j]$ lies in $\overline{A}_j$.

Step 5. We next show that

$$\lim_{j\to\infty} d(fS_j) = 0. \tag{2.14}$$

Set $\delta_j = d(\gamma_j, \delta G')$. By Step 4 and (2.11) we obtain

$$k'(fS_0, fS_j) \le l_k(\gamma_j) \le l(\gamma_j)/\delta_j \le 4R_1/\delta_j.$$

By (2.9) this implies that $\delta_j \to 0$ as $j \to \infty$. Hence $d(\partial fA_j, \partial G') \to 0$. In Step 3 we observed that $d(fS_0, \partial G') > 0$. In view of (2.13) we obtain $d(fS_j, \partial G') \to 0$. On the other hand, we have $k'(fS_j) \le \varphi(4)$, and hence by I.2.2(1),

$$d(fS_j) \le 2d(fS_j, \partial G')e^{\varphi(4)},$$

which gives (2.14).

Step 6. As the final step we prove the original claim $d\bigl(fU(r_j)\bigr) \to 0$. Let $\varepsilon > 0$. By (2.14) there is j such that $d(fS_i) \le \varepsilon$ for all $i \ge j$. It suffices to show that $d\bigl(fU(r_j)\bigr) \le 4\varepsilon$. Let A_{ij} be the annulus $B(r_j) \setminus \overline{B}(r_i)$ in G. It suffices to show that $d(fA_{ij}) \le 4\varepsilon$ for all $i > j$. Choose closed balls B_i and B_j of radius ε containing fS_i and fS_j, respectively. If $B_i \cap B_j = \emptyset$, then $\dot{E}' \setminus (B_1 \cup B_2)$ is a connected set meeting $\dot{E}' \setminus fA_{ij}$. As in (2.13) we obtain $\partial fA_{ij} = fS_i \cup fS_j \subset B_1 \cup B_2$. Hence $fA_{ij} \subset B_1 \cup B_2$. Since fA_{ij} is connected, this is impossible. Consequently, B_i meets B_j, and hence ∂fA_{ij} is contained in a ball B of radius 2ε. This implies that $fA_{ij} \subset B$, and hence $d(fA_{ij}) \le 4\varepsilon$. Lemma 2.8 is proved. □

2.15. Lemma. *Set $G_1 = G \cup \{0\}$, $G_1' = G' \cup \{y_0\}$. Then G_1 and G_1' are domains in E and $\dot{E}'$, respectively. The extension $f_1\colon G_1 \to G_1'$ with $f_1(0) = y_0$ is a homeomorphism.*

Proof. Clearly $y_0 \in \partial G'$, and G_1 is a domain. Performing a preliminary inversion and a translation, if necessary, we may assume that $y_0 = 0$. Write again $U(r) = B(r) \setminus \{0\}$ and choose $r_0 > 0$ with $U(2r_0) \subset G$. Since $k\bigl(S(r_0)\bigr) \le 4 < \infty$ by 2.5, Lemma 2.6 implies that $d\bigl(fS(r_0), 0\bigr) = t_0 > 0$. We show that $B(t_0) \cap \partial G' = \{0\}$. Assume that this is false. Since $fU(r_0)$ is an open set meeting $B(t_0)$, we can choose points z and z_1 in $B(t_0)$ such that $z \in fU(r_0)$, $z_1 \in \partial G'$, and $0 \notin [z, z_1]$. Replacing z_1 by a point in $[z, z_1]$ we may assume that $\beta = [z, z_1) \subset G'$. Then $\alpha = f^{-1}\beta$ does not meet $S(r_0)$, and hence $\alpha \subset B(r_0)$. Since f_1 is continuous at 0, $\overline{\alpha}$ does not contain 0. Hence $k(\alpha) < \infty$, which implies $k'(\beta) < \infty$. This is a contradiction, since $\overline{\beta}$ meets $\partial G'$.

We have proved that 0 is an isolated boundary point of G'. This implies that G_1' is a domain. Moreover, $f_1\colon G_1 \to G_1'$ is a continuous bijection. It remains to show that f_1^{-1} is continuous at the origin.

Let $0 < r < r_0$. As above, $d\bigl(fS(r), 0\bigr) = t > 0$. Then $f^{-1}U(t)$ is a connected set meeting $B(r)$ but not $S(r)$. Hence $f^{-1}U(t) \subset B(r)$, which implies the continuity of f_1^{-1} at 0. □

2.16. Lemma. *Suppose that $G_1 \neq E$ and $y_0 \neq \infty$. Then $G_1' \neq E'$ and $f_1\colon G_1 \to G_1'$ is φ_1-solid with $\varphi_1 = \varphi_1(\varphi)$.*

Proof. We may again assume that $y_0 = 0$. Assume that $G_1' = E'$. This means that $G' = E' \setminus \{0\}$. By Lemma 2.8, f_1^{-1} has a limit $x_1 \neq 0$ at ∞, possibly $x_1 = \infty$. Since $G_1 \neq E$, we can choose neighborhoods U of 0 and V of x_1 such that $\overline{U} \cap \overline{V} = \emptyset$ and $\dot{E} \setminus (U \cup V)$ meets ∂G. Then for the set $A = G \setminus (U \cup V)$ we have $k(A) = \infty$. However, $\overline{fA}$ does not meet $\{0, \infty\}$, and hence $k'(fA) < \infty$. This contradiction proves that $G_1' \neq E'$.

By auxiliary similarities we can normalize the situation so that $d(0, \partial G_1) = 1 = d(0, \partial G_1')$. We first show that there are a number $r_0 = r_0(\varphi)$, $0 < r_0 < 1/2$, and an increasing homeomorphism $\psi = \psi_\varphi\colon [0, r_0] \to [0, 1/2]$ such that

$$|f_1 x| \leq \psi(|x|) \tag{2.17}$$

whenever $|x| \leq r_0$.

We first observe that for each $r \in (0,1)$ we have $\overline{f_1 B(r)} \subset G_1'$ and hence $\partial f_1 B(r) = f S(r)$. Indeed, there is $r_1 < r$ with $f_1 B(r_1) \subset B(1/2)$. The annulus $A = B(r) \setminus \overline{B(r_1)}$ has a finite QH diameter in G. By 2.6, this implies $\overline{fA} \subset G'$, and hence $\overline{f_1 B(r)} \subset G_1'$.

Set $\varepsilon = \varepsilon(\varphi) = e^{-\varphi(4)}/2$, and choose a point $z \in \partial G_1'$ with $|z| \leq 1 + \varepsilon$. Let β be the ray from z through 0. We can choose points $y_1, y_2 \in \beta \cap f S(1/2)$ such that $0 \in [y_1, y_2] \subset f_1 \overline{B}(1/2)$ and $y_1 \in [z, 0]$. Writing $x_j = f^{-1} y_j$ we have $k(x_1, x_2) \leq 4$ by 2.5. Setting $q = |y_1 - z|$ and assuming $q \leq 1/2$ we have

$$d(y_1, \partial G') \leq q, \qquad |y_1 - y_2| \geq |y_1| \geq 1 - q.$$

By I.2.2(1) these yield

$$k'(y_1, y_2) \geq \ln \frac{1}{q}.$$

Since f is φ-solid, we obtain

$$q \geq e^{-\varphi(4)} = 2\varepsilon, \tag{2.18}$$

which is also valid if $q \geq 1/2$.

Next assume that $0 < r < 1/2$. Choose a point $x_3 \in S(r)$ such that $f x_3 = y_3 \in [y_1, 0]$. If $y \in [y_1, y_3]$ and $z_1 \in \partial G_1$, we have by (2.18)

$$1 \leq |z_1| \leq |y| + |y - z_1| \leq 1 + \varepsilon - q + |y - z_1| \leq 1 - \varepsilon + |y - z_1|,$$

which implies

$$d(y, \partial G') \geq \varepsilon \wedge |y_3| = t.$$

Hence

$$k'(y_1, y_3) \le |y_1 - y_3|/t < 2/t.$$

On the other hand, $k(x_3, x_1) \ge \ln(1/2r)$, and hence

$$t < \frac{2}{\varphi^{-1}\bigl(\ln(1/2r)\bigr)} = \psi_0(r) \to 0$$

as $r \to 0$. Setting $r_0 = e^{-\varphi(2/\varepsilon)}/2 = r_0(\varphi)$ we have $\psi_0(r_0) = \varepsilon$. Hence

$$|y_3| \le \psi_0(r) \tag{2.19}$$

for $0 < r \le r_0$.

Let $x \in S(r)$. Then $|fx| \le |y_3|$ or

$$k'(fx, y_3) \ge \ln\Bigl(1 + \frac{|fx - y_3|}{|y_3|}\Bigr) \ge \ln\frac{|fx|}{|y_3|}.$$

Since $k(x, x_3) \le 4$, this and (2.19) yield

$$|fx| \le e^{\varphi(4)}|y_3| \le e^{\varphi(4)}\psi_0(r) = \psi(r)$$

for $0 < r \le r_0$. Setting $\psi(0) = 0$ we obtain (2.17). Moreover, $\psi(r_0) = 1/2$.

We turn to the solidity of f_1. By I.3.7, f is θ-relative with $\theta = \theta_\varphi$. By symmetry and by I.3.7, it suffices to show that f_1 is (θ_1, q)-relative with (θ_1, q) depending only on φ. We show that one can choose $q = r_0/3$.

We write $\delta = \delta_G$, $\delta_1 = \delta_{G_1}$, $\delta' = \delta_{G'}$, $\delta'_1 = \delta_{G'_1}$. Then

$$\delta(x) = \delta_1(x) \wedge |x|, \qquad \delta'(y) = \delta'_1(y) \wedge |y|.$$

Set $q = r_0/3$, and let $a \in G_1$, $b \in B\bigl(a, q\delta_1(a)\bigr)$. It suffices to find an estimate

$$\frac{|f_1 a - f_1 b|}{\delta'_1(fa)} \le \theta_1\Bigl(\frac{|a-b|}{\delta_1(a)}\Bigr) \tag{2.20}$$

with some θ_1 with $\theta_1(t) \to 0$ as $t \to 0$. By continuity we may assume that $a \ne 0 \ne b$. We consider two cases.

Case 1. $|a| \ge r_0/2$. If $\delta(a) = |a|$,

$$\delta_1(a) \le 1 + |a| \le (2/r_0 + 1)|a| < \delta(a)/q.$$

If $\delta(a) = \delta_1(a)$, this is trivially true. Hence we have $|a - b| < \delta(a)$. Since f is θ-relative, this implies

$$\frac{|fa - fb|}{\delta'_1(fa)} \le \frac{|fa - fb|}{\delta'(fa)} \le \theta\Bigl(\frac{|a-b|}{\delta(a)}\Bigr) \le \theta\Bigl(\frac{|a-b|}{q\delta_1(a)}\Bigr).$$

Thus (2.20) holds with $\theta_1(t) = \theta(t/q)$.

Case 2. $|a| < r_0/2$. Now $\delta(a) = |a|$. We consider two subcases.
Subcase 2a. $|a-b| \le |a|^2$. We have

$$\frac{|fa-fb|}{\delta_1'(fa)} \le \frac{|fa-fb|}{\delta'(fa)} \le \theta\Big(\frac{|a-b|}{|a|}\Big).$$

Since $\delta_1(a) \le 1 + |a| < 2$, we obtain

$$\frac{|a-b|^2}{|a|^2} \le |a-b| < 2\frac{|a-b|}{\delta_1(a)}.$$

Hence (2.20) holds with $\theta_1(t) = \theta(\sqrt{2t})$.
Subcase 2b. $|a-b| \ge |a|^2$. Since

$$|a-b| \le q\delta_1(a) \le q(1+|a|) < r_0/2,$$

we have $|b| < r_0$ and

$$|b| \le |a| + |a-b| \le 2\sqrt{|a-b|}.$$

By (2.17) this yields

$$|fa-fb| \le |fa| + |fb| \le 2\psi(2\sqrt{|a-b|}).$$

Since

$$\delta_1'(fa) \ge 1 - |fa| \ge 1 - \psi(|a|) \ge 1/2, \qquad \delta_1(a) \le 1 + |a| \le 1 + r_0/2 \le 5/4,$$

we obtain (2.20) with $\theta_1(t) = 4\psi(\sqrt{5t})$. □

2.21. Theorem. *Suppose that x_0 is an isolated boundary point of G and that $f\colon G \to G'$ is φ-FQC. Then f has a limit $y_0 \in \dot{E}'$ at x_0, and y_0 is an isolated boundary point of $\partial G'$. Setting $f_1(x_0) = y_0$ we obtain an extension of f to a homeomorphism $f_1\colon G \cup \{x_0\} \to G' \cup \{y_0\}$. If $x_0 \ne \infty$ and $y_0 \ne \infty$, then f_1 is φ_1-FQC with $\varphi_1 = \varphi_1(\varphi)$.*

Proof. This is an easy corollary of 2.7. □

2.22. *Remark.* In the QC theory of R^n it is customary to allow the possibilities $\infty \in G$ and $\infty \in G'$. In the free theory this would involve technical complications, since we have not defined the QH metric of such domains. One can usually reduce the situation to the case $G \subset E$, $G' \subset E'$ by auxiliary inversions.

3. Neargeodesics

3.1. *Terminology.* Let $G \neq E$ and $c \geq 1$. An arc $\gamma \subset G$ is a c-*neargeodesic* in G, if γ is c-quasiconvex in the QH metric $k = k_G$. In other words, the inequality

$$l_k(\gamma[x,y]) \leq ck(x,y)$$

holds for each pair $x, y \in \gamma$. Thus γ is a QH geodesic if and only if it is a 1-neargeodesic. The arc γ may be closed, open or half open.

We showed in I.2.9 that geodesics do not always exist. In this section we shall prove two existence theorems for neargeodesics. In the first result we join two points in G, in the second one a point of G to ∂G.

3.2. Lemma. *Suppose that $a \in G \neq E$, that $0 < t \leq 1/2$, and that γ is a rectifiable arc in $\overline{B}(a, t\delta(a))$. Then*

$$\frac{1}{1+t} \leq \frac{l_k(\gamma)\delta(a)}{l(\gamma)} \leq \frac{1}{1-t}.$$

Moreover, every line segment in $\overline{B}(a, t\delta(a))$ is a c-neargeodesic in G with $c = (1+2t)^2$.

Proof. Since

$$\delta(a)(1-t) \leq \delta(x) \leq \delta(a)(1+t)$$

for all $x \in \overline{B}(a, t\delta(a))$, the inequalities follow directly by integration. Suppose that $\gamma = [x,y] \subset \overline{B}(a, t\delta(a))$. Then the second inequality and I.2.2(3) yield

$$l_k(\gamma) \leq (1+2t)|x-y|/\delta(a) \leq (1+2t)^2 k(x,y). \square$$

3.3. Theorem. *Let $a, b \in G \neq E$ and let $c > 1$. Then there is a c-neargeodesic joining a and b in G.*

Proof. For $q > 0$ we write $c_1 = c_1(q) = (1+2q)^2$. Choose $q_0 > 0$ such that

$$q_0 \leq k(a,b)/10, \qquad c_1(q_0) < 5/4. \tag{3.4}$$

Then $q_0 < (\sqrt{5}-2)/4 < 1/8$. We shall prove the theorem by constructing for every $q \leq q_0$ an arc β joining a and b such that β is a $c(q)$-neargeodesic, where $c(q) \to 1$ as $q \to 0$.

Let $0 < q \leq q_0$. Choose an arc γ joining a and b in G such that $l_k(\gamma) \leq k(a,b) + q^2$. Then

$$l_k(\gamma[x,y]) \leq k(x,y) + q^2 \tag{3.5}$$

for all $x, y \in \gamma$, since assuming $x \in \gamma[a,y]$ we have

$$k(a,x) + l_k(\gamma[x,y]) + k(y,b) \leq l_k(\gamma) \leq k(a,x) + k(x,y) + k(y,b) + q^2.$$

Since $q \leq k(a,b)/10$, we can choose a number $\lambda = \lambda(q)$ such that

(1) $q/4 \le \lambda \le q/2$,
(2) $l_k(\gamma) = n\lambda$

for some positive integer n. Divide γ by successive points $a = x_0, \ldots, x_n = b$ to subarcs $\gamma_j = \gamma[x_{j-1}, x_j]$ with $l_k(\gamma_j) = \lambda$. Setting

$$\beta_j = [x_{j-1}, x_j], \qquad \beta = \cup\{\beta_j : 1 \le j \le n\}$$

we show that β is the desired arc.

Let $x, y \in \beta$, $x \ne y$. Although we have not yet shown that β is an arc, the QH length $l_k(\beta[x,y])$ is defined in the obvious way, as soon as we fix i and j with $x \in \beta_i$, $y \in \beta_j$. Setting

$$p(x,y) = \frac{l_k(\beta[x,y])}{k(x,y)}$$

we must find an upper bound $p(x,y) \le c(q)$ with $c(q) \to 1$ as $q \to 0$. We consider four cases.

Case 1. For some i, β_i contains x and y. Since $k(x_{i-1}, x_i) \le \lambda \le q/2 < 1$, I.2.5 gives $|x_{i-1} - x_i| \le q\delta(x_i)$. Hence 3.2 implies that $p(x,y) \le (1+2q)^2 = c_1(q)$.

Case 2. x and y are vertices of β, say $x = x_i$, $y = x_{i+s}$, $s \ge 1$. Using Case 1 and (3.5) we obtain

$$\begin{aligned} l_k(\beta[x,y]) &= \sum_{j=1}^{s} l_k(\beta_{i+j}) \le c_1 \sum_{j=1}^{s} k(x_{i+j-1}, x_{i+j}) \le c_1 \sum_{j=1}^{s} l_k(\gamma_{i+j}) \\ &= c_1 l_k(\gamma[x,y]) \le c_1 k(x,y) + c_1 q^2. \end{aligned}$$

On the other hand, since $q < 1/8$, (3.5) implies

$$k(x,y) \ge l_k(\gamma[x,y]) - q^2 = s\lambda - q^2 \ge q/4 - q^2 > q/8.$$

Hence $p(x,y) \le c_1 + 8c_1 q = c_2(q)$.

Case 3. There are $i \ge 1$ and $s \ge i+2$ such that $x \in \beta_i$, $y \in \beta_s$. Using Case 2 we obtain

$$\begin{aligned} l_k(\beta[x,y]) &= l_k(\beta[x_{i-1}, x_s]) - l_k(\beta[x_{i-1}, x]) - l_k(\beta[y, x_s]) \\ &\le c_2 k(x_{i-1}, x_s) - k(x_{i-1}, x) - k(y, x_s) \\ &\le c_2 k(x,y) + (c_2 - 1)\big[k(x_{i-1}, x) + k(y, x_s)\big]. \end{aligned}$$

Here

$$k(x_{i-1}, x) \le l_k(\beta_i) \le c_1 k(x_{i-1}, x_i) \le c_1 \lambda \le c_1 q/2,$$

and a similar estimate is valid for $k(y, x_s)$. These and (3.5) yield

$$\begin{aligned} k(x,y) &\geq k(x_{i-1}, x_s) - k(x_{i-1}, x) - k(y, x_s) \\ &\geq l_k\big(\gamma[x_{i-1}, x_s]\big) - q^2 - 2c_1\lambda \geq (3-2c_1)\lambda - q^2 \geq (5-4c_1)q/8, \end{aligned}$$

where we also made use of the inequalities $\lambda \geq q/4$ and $q \leq 1/8$. By (3.4), the right-hand side is positive. Consequently,

$$p(x,y) \leq c_2 + \frac{8c_1(c_2-1)}{5-4c_1} = c_3(q) \to 1$$

as $q \to 0$. We have also proved that γ is an arc.

Case 4. There is i such that $x \in \beta_i$ and $y \in \beta_{i+1}$. If $x = x_{i-1}$ or if $y = x_{i+1}$, we are in Case 3. The general situation is reduced to this special case as follows:

We may assume that $x \neq x_i \neq y$. For $K \geq 1$ let $g\colon E \to E$ be the similarity defined by $gu = x_i + K(u - x_i)$. We can choose K such that $gx \in \beta_i$, $gy \in \beta_{i+1}$, and either $gx = x_{i-1}$ or $gy = x_{i+1}$. In Case 1 we showed that $\beta_i \cup \beta_{i+1} \subset \overline{B}\big(x_i, q\delta(x_i)\big)$. Applying 3.2 twice we obtain

$$l_k\big(\beta[x,y]\big) \leq \frac{l\big(\beta[x,y]\big)}{(1-q)\delta(x_i)} = \frac{l\big(\beta[gx,gy]\big)}{K(1-q)\delta(x_i)} \leq \frac{(1+q)l_k\big(\beta[gx,gy]\big)}{K(1-q)}.$$

By Case 3 we have $l_k\big(\beta[gx,gy]\big) \leq c_3 k(gx,gy)$. These estimates and I.2.2 yield

$$\begin{aligned} l_k\big(\beta[x,y]\big) &\leq \frac{(1+q)c_3|gx-gy|}{K(1-q)^2\delta(x_i)} = \frac{(1+q)c_3|x-y|}{(1-q)^2\delta(x_i)} \leq \frac{(1+q)(1+2q)c_3k(x,y)}{(1-q)^2} \\ &= c_4(q)k(x,y), \end{aligned}$$

where $c_4(q) \to 1$ as $q \to 0$. □

3.6. *Terminology.* A half open arc γ in a domain G is an *endcut* of G if $\overline{\gamma}$ is a closed arc with one endpoint in ∂G.

We want to show that each point x_0 in $G \neq E$ can be joined to ∂G by a neargeodesic endcut. If $\dim E < \infty$ this is easy: We choose $y_0 \in \partial G$ with $|y_0 - x_0| = \delta(x_0)$. Then $[x_0, y_0)$ is a 3-neargeodesic by Lemma 3.9 below. In the general case there is no nearest point y_0, and we must replace $[x_0, y_0)$ by a broken line consisting of a countable number of line segments.

We first prove some elementary inequalities in Banach spaces.

3.7. Lemma. *Suppose that $a, b \in G \neq E$ with $|a-b| \leq \delta(a)$ and that $x \in [a,b]$. Then $|x-b| \leq \delta(x)$ and $\delta(b) \leq 2\delta(x)$.*

Proof. Elementary estimates give

$$|x-a|+|x-b|=|a-b|\le\delta(a)\le\delta(x)+|x-a|,$$

$$\delta(b)\le\delta(x)+|x-b|\le 2\delta(x).\ \square$$

3.8. Lemma. *Let $x_0, x_1, x_2 \in E$ with $2|x_1-x_2|\le|x_0-x_1|\le|x_0-x_2|$, and let $x\in[x_0,x_1]$, $y\in[x_1,x_2]$. Then*

$$|x-x_1|\le 2|x-y|, \qquad |x_1-y|\le 2|x-y|.$$

Proof. We normalize $x_0=0$, $|x_1|=1$. Then $2|x_1-x_2|\le 1\le|x_2|$. Using a similarity of the form $f(x)=x_1+K(x-x_1)$ we see that it suffices to consider the cases where either $x=0$ or $y=x_2$.

Suppose first that $x=0$. Then

$$|x-y|=|y|\ge|x_1|-|x_1-y|\ge 1-|x_1-x_2|\ge 1/2,$$

and hence

$$|x-x_1|=1\le 2|x-y|, \qquad |x_1-y|\le|x_1-x_2|\le 1/2\le|x-y|.$$

Next assume that $y=x_2$. Now

$$|x-y|\ge|y|-|x|\ge 1-|x|=|x-x_1|,$$

which implies the first inequality. If $|x_1-y|\ge 2|x_1-x|$, then

$$|x_1-y|\le|x_1-x|+|x-y|\le|x_1-y|/2+|x-y|.$$

If $|x_1-y|\le 2|x_1-x|$, then

$$|x_1-y|/2\le|x_1-x|=1-|x|\le|y|-|x|\le|y-x|.$$

In both cases we obtain the second inequality. $\square$

3.9. Lemma. *Suppose that $a,b\in G\ne E$ and that $|a-b|\le\delta(a)$. Then*

$$l_k([a,b])\le 3\ln\Big(1+\frac{|a-b|}{\delta(b)}\Big),$$

and $[a,b]$ is a 3-neargeodesic.

Proof. . For each $x \in [a, b]$ and $y \in \partial G$ we have

$$|b - x| = |a - b| - |a - x| \leq \delta(a) - |a - x| \leq \delta(x), \quad \delta(b) \leq \delta(x) + |b - x| \leq 2\delta(x),$$

and hence

$$|x - b| + \delta(b) \leq 3\delta(x).$$

Consequently,

$$l_k([a, b]) \leq 3 \int_0^{|a-b|} \frac{dt}{\delta(b) + t} = 3 \ln \Big(1 + \frac{|a - b|}{\delta(b)}\Big).$$

The last statement follows now from I.2.2(1). □

3.10. Theorem. *Suppose that $x_0 \in G \neq E$ and that $\varepsilon > 0$. Then there is an endcut γ of G from x_0 such that*
(1) *γ is a c_0-neargeodesic with a universal c_0,*
(2) $\gamma \subset B(x_0, (1 + \varepsilon)\delta(x_0))$.

Proof. We may assume that $\varepsilon \leq 1/4$. For positive integers j set $\varepsilon_j = 2^{-j}\varepsilon$. We construct inductively a sequence of points $x_0, x_1, \ldots$ in $\overline{G}$ as follows: Suppose that $x_0, \ldots, x_i$ have been chosen. If $x_i \in \partial G$, the process stops. If $x_i \in G$, we choose a point $y_{i+1} \in \partial G$ with $|y_{i+1} - x_i| < (1 + \varepsilon_{i+1})\delta(x_i)$, and let x_{i+1} be the unique point in $[x_i, y_{i+1}] \cap S(x_i, \delta(x_i))$. Then

$$\delta(x_{i+1}) < \varepsilon_{i+1}\delta(x_i). \tag{3.11}$$

Writing $\gamma_i = [x_{i-1}, x_i)$ we claim that the union γ of all γ_i is the desired endcut.

As in the proof of 3.3, we can in the obvious way define the QH length of $\gamma[x, y]$ for $x \in \gamma_i$, $y \in \gamma_j$ as soon as i and j are fixed. We shall show that

$$l_k(\gamma[x, y]) \leq c_0 k(x, y) \tag{3.12}$$

for all $x, y \in \gamma$ with a universal constant c_0. This will imply that the arcs γ_j are disjoint. Moreover, since

$$\sum_j l(\gamma_j) = \sum_j \delta(x_{j-1}) < \delta(x_0)\Big(1 + \sum_{j=1}^{\infty} \varepsilon_j\Big) = \delta(x_0)(1 + \varepsilon),$$

γ lies in $B(x_0, (1+\varepsilon)\delta(x_0))$, and γ is rectifiable. Since E is complete, this implies that γ is an endcut of G. Thus it suffices to verify (3.12). We consider three cases.

Case 1. For some i, γ_i contains x and y. Now (3.12) follows from 3.9.

Case 2. For some i, $x_i \in \gamma_i$ and $y \in \gamma_{i+1}$. Now 3.9 gives

$$l_k(\gamma[x,y]) \le 3\ln\Big(1+\frac{|x-x_i|}{\delta(x_i)}\Big) + 3\ln\Big(1+\frac{|x_i-y|}{\delta(y)}\Big).$$

Applying 3.8 with the substitution

$$(x_0,x_1,x_2,x,y) \mapsto (x_{i-1},x_i,y_{i+1},x,y)$$

we obtain

$$|x-x_i| \vee |x_i-y| \le 2|x-y|.$$

Since

$$\delta(y) \le |y-y_{i+1}| < (1+\varepsilon_{i+1})\delta(x_i) \le 9\delta(x_i)/8,$$

and since $\ln(1+ta) \le t\ln(1+a)$ for $t \ge 1$ and $a \ge 0$, these inequalities and I.2.2 yield the estimate

$$l_k(\gamma[x,y]) \le \Big(\frac{27}{4}+6\Big)\ln\Big(1+\frac{|x-y|}{\delta(y)}\Big) \le 13k(x,y).$$

Case 3. For some $i \ge 1$ and $s \ge 2$, $x \in \gamma_i$ and $y \in \gamma_{i+s}$. By 3.9 we have

$$l_k(\gamma[x,y])/3 \le \ln\Big(1+\frac{|x_i-x|}{\delta(x_i)}\Big) + \sum_{j=1}^{s-1}\ln\Big(1+\frac{\delta(x_{i+j-1})}{\delta(x_{i+j})}\Big) + \ln\Big(1+\frac{|x_{i+s-1}-y|}{\delta(y)}\Big).$$

By (3.11) we obtain

$$\ln\Big(1+\frac{\delta(x_{i+j-1})}{\delta(x_{i+j})}\Big) \le \ln\frac{\delta(x_{i+j-1})}{\delta(x_{i+j})} + \ln(1+\varepsilon_{i+j}) \le \ln\frac{\delta(x_{i+j-1})}{\delta(x_{i+j})} + \varepsilon_{i+j}.$$

Writing

$$\alpha = \ln\frac{\delta(x_i)+|x_i-x|}{\delta(x)}, \qquad \beta = \ln\frac{\delta(y)+|y-x_{i+s-1}|}{\delta(x_{i+s-1})},$$

these inequalities yield

$$l_k(\gamma[x,y])/3 \le \alpha+\beta+\varepsilon+\ln\frac{\delta(x)}{\delta(y)}.$$

Furthermore, Lemma 3.7 implies that

$$\delta(x_i)+|x_i-x| \le 2\delta(x)+\delta(x) = 3\delta(x),$$

and hence $\alpha \le \ln 3$. Since $|y-x_{i+s-1}| \le \delta(x_{i+s-1})$, we have

$$\delta(y) \le \delta(x_{i+s-1}) + |y-x_{i+s-1}| \le 2\delta(x_{i+s-1}),$$

and thus $\beta \le \ln 3$. Since $\varepsilon < 1$, these estimates imply

$$l_k(\gamma[x,y]) \le M + 3\ln\frac{\delta(x)}{\delta(y)}, \tag{3.13}$$

where $M = 3(1+2\ln 3)$. We also see that

$$\delta(y) \le 2\delta(x_{i+s-1}) \le 2\varepsilon_1\delta(x_i) = \varepsilon\delta(x_i) \le \delta(x_i)/4 \le \delta(x)/2.$$

Hence (3.13) and I.2.2 give (3.12) with $c_0 = M/\ln 2 + 3 < 17$. □

4. Coarse length and CQH maps

4.1. *Introduction to Section* 4. We first introduce the concept of the *coarse length* of an arc, which will be our main tool when studying the boundary properties of FQC and more general maps. We also consider *coarsely bilipschitz* maps, which in the QH case will be called *coarsely quasihyperbolic* or CQH. The CQH maps include the solid maps, and hence the FQC maps.

The general idea in the coarse theory is that we forget what happens with small distances. Related concepts have been considered by M. Gromov [Gr, p. 186] and several others.

4.2. *Terminology.* Let γ be an arc in a metric space X. The arc may be closed, open or half open. Let $\overline{x} = (x_0, \dots, x_n)$, $n \geq 1$, be a finite sequence of successive points of γ. For $h \geq 0$ we say that $\overline{x}$ is h-*coarse* if $|x_{j-1} - x_j| \geq h$ for all $1 \leq j \leq n$. Let $\Phi(\gamma, h)$ be the family of all h-coarse sequences of γ. Set

$$s(\overline{x}) = \sum_{j=1}^{n} |x_{j-1} - x_j|,$$

$$l(\gamma, h) = \sup \left\{ s(\overline{x}) : \overline{x} \in \Phi(\gamma, h) \right\}$$

with the agreement that $l(\gamma, h) = 0$ if $\Phi(\gamma, h) = \emptyset$. The number $l(\gamma, h)$ is the h-*coarse length* of γ.

In this paper we shall use this concept in the case where X is a domain $G \neq E$ equipped with the QH metric k. We let $l_k(\gamma, h)$ denote the h-coarse QH length of γ.

This concept is useful in the theory of solid and FQC maps $f\colon G \to G'$, because we can compare suitable coarse lengths of an arc $\gamma \subset G$ and its image $f\gamma$. The ordinary length is useless, since f need not preserve the rectifiability of an arc.

We list some elementary properties of the coarse length:

4.3. Lemma. *Let γ be an arc in a metric space, and let $h \geq 0$.*

(1) *If γ is a closed arc and $h > 0$, then $l(\gamma, h) < \infty$.*
(2) *$l(\gamma, h)$ is decreasing in h.*
(3) *$l(\gamma, 0) = l(\gamma)$ is the ordinary length of γ.*
(4) *$l(\gamma, h) = 0$ for $h > d(\gamma)$.*
(5) *$d(\gamma) \leq h \vee l(\gamma, h)$.*
(6) *$\gamma' \subset \gamma$ implies $l(\gamma', h) \leq l(\gamma, h)$.*
(7) *$l(\gamma, h)$ is the supremum of $s(\overline{x})$ over all $\overline{x} = (x_0, \dots, x_n)$ which satisfy the condition $h \leq |x_{j-1} - x_j| < 2h$ for all $1 \leq j \leq n$.*

Proof. The property (1) follows easily from the compactness of γ, and the properties from (2) to (6) are direct consequences of the definition. Each h-coarse $\overline{x}$ has obviously a refinement $\overline{y}$ such that $h \leq |y_{j-1} - y_j| < 2h$ for all j. This implies (7). $\square$

4.4. *Remark.* The coarse length is not additive. If γ is divided to two subarcs γ_1, γ_2, we usually have $l(\gamma, h) \neq l(\gamma_1, h) + l(\gamma_2, h)$. One can easily prove the inequalities

$$l(\gamma_1, 2h) + l(\gamma_2, 2h) - h \le l(\gamma, h) \le l(\gamma_1, h) + l(\gamma_2, h) + 2h,$$

but they are not needed in this paper.

The following result will be needed in Section 6:

4.5. Lemma. *Let $G \neq E$ and let γ be an arc in $G \cap (\partial G + \overline{B}(r))$. If $0 \le h \le R$ and $l_k(\gamma, h) \le R$, then $d(\gamma) \le MRr$, where $M = M(h)$ is increasing in h. If $h = 0$, we have $l(\gamma) \le Rr$.*

Proof. The case $h = 0$ is easy, since

$$\frac{l(\gamma)}{r} \le \int_\gamma \frac{|dx|}{\delta(x)} = l_k(\gamma) \le R.$$

Assume that $h > 0$. We may assume that γ is a closed arc with endpoints a_0, a_1. Replacing γ by a subarc we may assume that $d(\gamma) = |a_0 - a_1|$. We show that the lemma is true with $M = 2(e^h - 1)/h$, which is easily seen to be increasing in h.

If $\gamma \subset B_k(a_0, h)$, then I.2.2 implies that

$$d(\gamma) = |a_0 - a_1| \le \delta(a_0)\big(e^{k(a_0,a_1)} - 1\big) \le r(e^h - 1) \le Mrh \le MrR.$$

Suppose that $\gamma \not\subset B_k(a_0, h)$. Choose a sequence $\overline{x} = (x_0, \dots, x_n)$ of successive points of γ such that $x_0 = a_0$, $k(x_{j-1}, x_j) = h$ for $1 \le j \le n$, and $k(x_n, a_1) < h$. Then $n \ge 1$ and

$$nh = s(\overline{x}) \le l_k(\gamma, h) \le R.$$

By I.2.2 this implies

$$d(\gamma) = |a_0 - a_1| \le s(\overline{x}) + |x_n - a_1| \le (n+1)r(e^h - 1) \le MRr. \ \square$$

4.6. *Terminology.* Let $M \ge 0$ and $C \ge 0$. A map $f\colon X \to Y$ is *C-coarsely M-Lipschitz* if

$$|fx - fy| \le M|x - y| + C$$

for all $x, y \in X$. If f is an embedding and if f and $f^{-1}\colon fX \to X$ are C-coarsely M-Lipschitz with $M \ge 1$, we say that f is *C-coarsely M-bilipschitz*, abbreviated (M, C)-CBL. This means that

$$(|x - y| - C)/M \le |fx - fy| \le M|x - y| + C \tag{4.7}$$

for all $x, y \in X$. In [Vä$_3$] the CBL maps were called roughly bilipschitz.

One could also consider maps satisfying (4.7) which are not injective or continuous. However, it is often helpful to be able to consider the inverse map f^{-1}. It seems to the author that one could develop an analogous theory for one-to-many "maps" (relations), but this would involve technical complications.

Recall from I.2.8 that a metric space X is c-quasiconvex if each pair of points $x, y \in X$ can be joined by an arc $\gamma \subset X$ with $l(\gamma) \le c|x - y|$.

4.8. Theorem. *Suppose that X is c-quasiconvex, that $d(X) = \infty$ and that $f\colon X \to Y$ is a map. Then the following conditions are quantitatively equivalent:*
(1) *f is C-coarsely M-Lipschitz.*
(2) *There are $t_1 \geq 0$ and $M_1 \geq 0$ such that $|fx - fy| \leq M_1|x - y|$ whenever $x, y \in X$ and $|x - y| \geq t_1$.*
(3) *There are $t_0 > 0$ and $M_0 \geq 0$ such that $|fx - fy| \leq M_0$ whenever $x, y \in X$ and $|x - y| \leq t_0$.*

Proof. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

If (1) holds and if $|x - y| \geq C$, then

$$|fx - fy| \leq M|x - y| + C \leq (M + 1)|x - y|.$$

Hence (2) is true with $t_1 = C$, $M_1 = M + 1$.

We next show that (2) implies (3) with $t_0 = t_1 + 1$ and $M_0 = 5M_1(t_1 + 1)$. Assume that $x, y \in X$ with $|x - y| \leq t_0$. Since X is connected and since $d(X) = \infty$, there is a point z in X such that $|z - x| = 2t_0$. Then $t_0 \leq |z - y| \leq 3t_0$, and we obtain

$$|fx - fy| \leq |fx - fz| + |fz - fy| \leq M_1|x - z| + M_1|z - y| \leq 5M_1(t_1 + 1).$$

Finally assume that (3) is true. Let $x, y \in X$. Choose an arc γ joining x and y with $l(\gamma) \leq c|x-y|$. Let $k \geq 0$ be the unique integer with $kt_0 < l(\gamma) \leq (k+1)t_0$. Choose successive points $x = x_0, \ldots, x_{k+1} = y$ such that each subarc $\gamma[x_{j-1}, x_j]$ has length at most t_0. Then $|x_{j-1} - x_j| \leq t_0$ and hence $|fx_{j-1} - fx_j| \leq M_0$. This implies

$$|fx - fy| \leq (k + 1)M_0 \leq M_0 l(\gamma)/t_0 + M_0 \leq cM_0|x - y|/t_0 + M_0,$$

which gives (1) with $M = cM_0/t_0$, $C = M_0$. □

4.9. Theorem. *Suppose that $f\colon X \to Y$ is a C-coarsely M-Lipschitz embedding, that γ is an arc in X and that $h \geq 0$. Then for $h_1 = M(h \vee C) + C$ we have*

$$l(f\gamma, h_1) \leq (M + 1)l(\gamma, h).$$

Proof. Let $\overline{y} = (y_0, \ldots, y_n)$ be a h_1-coarse sequence of $f\gamma$. Writing $x_j = f^{-1}(y_j)$ we obtain a sequence $\overline{x} = (x_0, \ldots, x_n)$. Since

$$|x_{j-1} - x_j| \geq \big(|y_{j-1} - y_j| - C\big)/M \geq (h_1 - C)/M = h \vee C,$$

$\overline{x}$ is h-coarse. Moreover,

$$s(\overline{y}) \leq \sum_{j=1}^{n} \big(M|x_{j-1} - x_j| + C\big),$$

where $|x_{j-1} - x_j| \geq h \vee C \geq C$, and hence

$$s(\overline{y}) \leq (M + 1)s(\overline{x}) \leq (M + 1)l(\gamma, h),$$

which implies the theorem. □

4.10. *Terminology.* Let $h \geq 0$ and $c \geq 1$. A metric space X is *h-coarsely c-quasiconvex* if each pair of points $x, y \in X$ can be joined by an arc γ with

$$l(\gamma, h) \leq c|x - y|.$$

In particular, an arc γ is h-coarsely c-quasiconvex if

$$l\big(\gamma[x, y], h\big) \leq c|x - y|$$

for all $x, y \in \gamma$. The case where γ is an arc in a domain $G \neq E$ with the QH metric plays an important role in the rest of the paper. We say briefly that an arc $\gamma \subset G$ is *(c, h)-solid* in G if it is h-coarsely c-quasiconvex in the QH metric of G. For $h = 0$ this means that γ is a c-neargeodesic.

4.11. Theorem. *Suppose that $f\colon X \to Y$ is (M, C)-CBL and that the arc $\gamma \subset X$ is h-coarsely c-quasiconvex. Then $f\gamma$ is h_1-coarsely c_1-quasiconvex with*

$$h_1 = M(h \vee 2cC) + C, \qquad c_1 = 2cM(M + 1).$$

Proof. Replacing γ by a subarc we see that it suffices to show that

$$l(f\gamma, h_1) \leq c_1|fx - fy|, \tag{4.12}$$

where x and y are the endpoints of γ. If $|x - y| \geq 2C$, we have

$$|fx - fy| \geq \big(|x - y| - C\big)/M \geq |x - y|/2M.$$

On the other hand, $l(\gamma, h) \leq c|x - y|$ implies by 4.9 that

$$l(f\gamma, h_1) \leq (M + 1)c|x - y|,$$

and (4.12) follows.

Next assume that $|x - y| < 2C$. It suffices to show that $|fu - fv| < h_1$ for all $u, v \in \gamma$, since then $l(f\gamma, h_1) = 0$, and (4.12) is trivially true. If $|u - v| < h$, then

$$|fu - fv| < Mh + C \leq h_1.$$

If $|u - v| \geq h$, then

$$|u - v| \leq l(\gamma, h) \leq c|x - y| < 2cC,$$

which implies

$$|fu - fv| < 2cCM + C \leq h_1. \qquad \square$$

4.13. *Terminology.* Recall from I.3.4 that a homeomorphism $f\colon G \to G'$ between domains $G \neq E$ and $G' \neq E'$ is M-quasihyperbolic or M-QH if f is M-bilipschitz in the QH metric. Similarly, we say that $f\colon G \to G'$ is *C-coarsely M-quasihyperbolic*, abbreviated (M,C)-CQH, if it is (M,C)-CBL in the QH metric. This means that f is a homeomorphism such that

$$(k(x,y) - C)/M \leq k'(fx, fy) \leq Mk(x,y) + C$$

for all $x, y \in G$.

We next give the relation between the CQH maps and some other classes considered in this paper. In 7.9 and in 7.22 we shall prove the close connection between the CQH maps and the maps which are quasimöbius relative to the boundary. More results on CQH maps will be given in [Vä$_5$]. For example, the properties φ-FQC and fully (M,C)-CQH are quantitatively equivalent.

4.14. Theorem. *For a homeomorphism $f\colon G \to G'$ with $G \neq E$, $G' \neq E'$, the following implications are quantitatively true:*

$$M\text{-QH} \Rightarrow \varphi\text{-FQC} \Rightarrow \varphi\text{-solid} \Rightarrow (M,C)\text{-CQH}.$$

In the last implication, one can choose an arbitrary $C > 0$ and then $M = C/\varphi^{-1}(C)$.

Proof. If f is M-QH, then f is φ-FQC with $\varphi(t) = 4M^2t$ by I.4.7. The second implication is trivial. Suppose that f is φ-solid and that $C > 0$, $c > 1$. Then $k'(fx, fy) \leq C$ whenever $k(x,y) \leq \varphi^{-1}(C)$. Since G is c-quasiconvex in the QH metric, the proof of 4.8 shows that f is C-coarsely M-Lipschitz in the QH metric with $M = cC/\varphi^{-1}(C)$. The same is true for f^{-1}. Since $c > 1$ is arbitrary, the theorem follows. □

4.15. Theorem. *Suppose that $G \neq E$, $G' \neq E'$, and that $f\colon G \to G'$ is (M,C)-CQH. If γ is a (c,h)-solid arc in G, then the arc $f\gamma$ is (c_1,h_1)-solid in G' with (c_1,h_1) depending only on (c,h,M,C). In particular, if f is φ-solid or φ-FQC, then (c_1,h_1) depends only on (c,h,φ).*

Proof. This follows from 4.11 and from 4.14. □

5. Relative quasisymmetry and quasimöbius

5.1. *Introduction to Section* 5. In this section we shall relativize the theory of quasisymmetric and quasimöbius maps. This theory will be applied in later sections to study the properties of CQH maps.

5.2. *Terminology.* By a *triple* in a space X we mean an ordered sequence $T = (x,a,b)$ of three distinct points in X. The *ratio* of T is the number

$$\rho(T) = \frac{|a-x|}{|b-x|}.$$

If $f\colon X \to Y$ is an injective map, the image of a triple $T = (x, a, b)$ is the triple $fT = (fx, fa, fb)$.

Suppose that $A \subset X$. A triple $T = (x, a, b)$ in X is said to be a *triple in the pair* (X, A) if $x \in A$ or if $\{a, b\} \subset A$. Equivalently, both $|a - x|$ and $|b - x|$ are distances from a point in A.

Let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f\colon X \to Y$ is said to be *η-quasisymmetric relative to* A, abbreviated η-QS rel A, if the condition

$$\rho(fT) \le \eta(\rho(T))$$

holds for every triple T in (X, A). Thus quasisymmetry rel X is equivalent to ordinary quasisymmetry.

Analogously, a *quadruple* in X is an ordered sequence $Q = (a, b, c, d)$ of four distinct points in X. The *cross ratio* of Q is the number

$$\tau(Q) = |a, b, c, d| = \frac{|a - b||c - d|}{|a - c||b - d|}.$$

Warning: The order of the points a, b, c, d varies in the literature. In particular, the cross ratio above is written as $|a, d, b, c|$ in [Vä$_1$]. The definition is extended in the well known manner to the case where one of the points is ∞. For example,

$$|a, b, c, \infty| = \frac{|a - b|}{|a - c|} = \rho(a, b, c).$$

If $X_0 \subset \dot{X}$ and if $f\colon X_0 \to \dot{Y}$ is an injective map, the image of a quadruple Q in X_0 is the quadruple $fQ = (fa, fb, fc, fd)$.

Suppose that $A \subset X_0 \subset \dot{X}$. We say that a quadruple $Q = (a, b, c, d)$ in X_0 is a *quadruple in the pair* (X_0, A) if $\{a, d\} \subset A$ or $\{b, c\} \subset A$. Equivalently, all four distances in the definition of $\tau(Q)$ are (at least formally) distances from a point of A.

Let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism and let $A \subset X_0 \subset \dot{X}$. An embedding $f\colon X_0 \to \dot{Y}$ is said to be *η-quasimöbius relative to* A, abbreviated η-QM rel A, if the inequality

$$\tau(fQ) \le \eta(\tau(Q)) \tag{5.3}$$

holds for each quadruple in (X_0, A). Thus η-QM rel X_0 is equivalent to ordinary quasimöbius.

5.4. *Remarks.* 1. Since $|a, b, c, d| = |b, a, d, c|$, an embedding $f\colon X_0 \to \dot{Y}$ is η-QM rel A as soon as (5.3) holds for each quadruple (a, b, c, d) with $\{a, d\} \subset A$.

2. It is possible to extend the relative concepts to some cases where the map is not everywhere injective. Let us say that a map $f\colon X \to Y$ is injective rel A if $f \mid A$ is injective and if $f^{-1}fA = A$. For such maps the definitions of QS and QM rel A still make sense. However, since such maps do not always have inverse maps, we have the difficulties mentioned in 4.6.

5.5. *Relative theory.* We shall next give a relative version of the basic quasimöbius theory of [Vä$_1$]. In most proofs it is sufficient to check that the corresponding proof in the absolute case makes only use of triples and quadruples in the given pair (X, A). In such cases the proof is omitted.

5.6. Theorem. *If $f\colon X \to Y$ is η-QS rel A, then f is θ-QM rel A with $\theta = \theta_\eta$.*

Proof. As the absolute case [Vä$_1$, 3.2]. □

5.7. Lemma. *Suppose that $f\colon X \to Y$ is an embedding, that $A \subset X$ and that $\eta\colon [0,\infty) \to [0,\infty)$ is an increasing function such that*
(1) *$\rho(fT) \le \eta(\rho(T))$ for each triple T in (X, A),*
(2) *$\rho(f^{-1}T') \le \eta(\rho(T'))$ for each triple T' in (fX, fA).*
Then f is η_1-QS rel A with η_1 depending only on η.

Proof. Observe that we do not require $\eta(0) = 0$. Replacing η by a larger function we may assume that η is a homeomorphism onto $[r_0, \infty)$, $r_0 > 0$. Setting $t_0 = 1/r_0$ we define an increasing homeomorphism $\eta_0\colon (0, t_0) \to (0, \infty)$ by $\eta_0(t) = \eta^{-1}(t^{-1})^{-1}$. Suppose that $T = (x, a, b)$ is a triple in (X, A) with $\rho(T) < t_0$. Applying (2) to the ratio $T' = (fx, fb, fa)$ gives $1/\rho(T) \le \eta(\rho(T'))$, which implies $\rho(fT) \le \eta_0(\rho(T))$. Together with (1) this proves the lemma. □

5.8. Lemma. *Suppose that $A \subset X_0 \subset \dot{X}$, that $f\colon X_0 \to \dot{Y}$ is an embedding and that $\eta\colon [0,\infty) \to [0,\infty)$ is an increasing function such that*
(1) *$\tau(fQ) \le \eta(\tau(Q))$ for each quadruple $Q = (a, b, c, d)$ in X_0 with $\{a, d\} \subset A$.*
(2) *$\tau(f^{-1}Q') \le \eta(\tau(Q'))$ for each quadruple $Q' = (a', b', c', d')$ in fX_0 with $\{a', d'\} \subset fA$.*
Then f is η_1-QM rel A with η_1 depending only on η.

Proof. In view of 5.4.1, the proof is an obvious modification of the proof of 5.7. □

5.9. Theorem. *Suppose that X and Y are bounded spaces, that $A \subset X$, and that $f\colon X \to Y$ is θ-QM rel A. Suppose also that $\lambda > 0$, $z_1 \in X$ and $z_2, z_3 \in A$ are such that*

$$|z_i - z_j| \ge d(X)/\lambda, \qquad |fz_i - fz_j| \ge d(Y)/\lambda$$

for $i \ne j$. Then:
(1) *There is a homeomorphism $\mu = \mu_{\theta,\lambda}\colon [0,\infty) \to [0,\infty)$ such that*

$$\frac{|fx - fy|}{d(Y)} \le \mu\Big(\frac{|x-y|}{d(X)}\Big)$$

for all $x \in A$, $y \in X$.
(2) *f is η-QS rel A with $\eta = \eta_{\theta,\lambda}$.*

Proof. The part (1) is proved as the absolute case [Vä_1, 2.1]. In the part (2), we must replace the proof of the absolute case [Vä_1, 3.12] by the following argument:

We may assume that f is a homeomorphism and that $f^{-1}\colon Y \to X$ is θ-QM rel fA. We normalize the situation so that $d(X) = d(Y) = \lambda$ replacing the metric $|a-b|$ of X by $\lambda|a-b|/d(X)$ and similarly in Y. By (1), there is a homeomorphism $\varphi = \varphi_{\theta,\lambda}\colon [0,\infty) \to [0,\infty)$ such that

$$\varphi^{-1}\big(|x-y|\big) \le |fx - fy| \le \varphi\big(|x-y|\big)$$

for all $x \in A$, $y \in X$.

Suppose that $T = (x,a,b)$ is a triple in (X,A). Since f^{-1} is θ-QM rel fA, it follows from 5.7 that it suffices to find an estimate

$$\rho(fT) \le \eta\big(\rho(T)\big) \tag{5.10}$$

for some increasing $\eta = \eta_{\theta,\lambda}\colon [0,\infty) \to [0,\infty)$. Since $|z_2 - z_3| \ge 1$, we may assume that $|a - z_2| \ge 1/2$. We consider three cases.

Case 1. $|a-x| \ge 1/4$. Now $|b-x| \ge 1/4\rho(T)$. Since $b \in A$ or $x \in A$, we have $|fb - fx| \ge \varphi^{-1}\big(1/4\rho(T)\big)$, and hence (5.10) is true with $\eta(t) = \lambda/\varphi^{-1}(1/4t)$.

Case 2. $|b - z_2| \ge 1/8$. Now $|fb - fz_2| \ge \varphi^{-1}(1/8)$. The quadruple $Q = (x,a,b,z_2)$ is in (X,A), and $\tau(Q) \le 2\lambda\rho(T)$. Since $\tau(fQ) \ge \varphi^{-1}(1/8)\rho(fT)/\lambda$, we obtain (5.10) with

$$\eta(t) = \frac{\lambda\theta(2\lambda t)}{\varphi^{-1}(1/8)}.$$

Case 3. $|a - x| \le 1/4$ and $|b - z_2| \le 1/8$. Now

$$|b - x| \ge |a - z_2| - |a - x| - |z_2 - b| \ge 1/8.$$

Hence $|fb - fx| \ge \varphi^{-1}(1/8)$, which implies (5.10) with the constant function $\eta(t) = \lambda/\varphi^{-1}(1/8)$. □

5.11. Theorem. *Suppose that G and G' are bounded domains and that $c \ge 1$. Suppose also that $x_0 \in G$ and $x_0' \in G'$ are points with*

$$d(G) \le c\delta(x_0), \qquad d(G') \le c\delta'(x_0').$$

Let $f\colon \overline{G} \to \overline{G}'$ be a homeomorphism such that $fx_0 = x_0'$ and $fG = G'$. If f is θ-QM rel ∂G, then f is η-QS rel ∂G with $\eta = \eta_{\theta,c}$.

Proof. The proof of the absolute case in R^n, [Vä$_1$, 3.14] needs only slight modifications. Write $z_1 = x_0$, $M = d(G) = d(\partial G)$, $M' = d(G') = d(\partial G')$. Choose points $z_2, z_3 \in \partial G$ with $|z_2 - z_3| \geq M/2$. It suffices to show that the conditions of Theorem 5.9 hold with some $\lambda = \lambda(\theta, c)$. Since

$$|z_1 - z_2| \geq \delta(x_0) \geq M/c, \qquad |z_1 - z_3| \geq M/c, \qquad |z_2 - z_3| \geq M/2,$$

the first condition of 5.9 is true with $\lambda = c \vee 2$. For $j = 2, 3$ we have

$$|fz_j - fz_1| \geq \delta'(x_0') \geq M'/c,$$

and it remains to find an upper bound for $M'/|fz_2 - fz_3|$.

Choose $z_4 \in \partial G$ with $|fz_4 - fz_3| \geq M'/3$. The quadruple $Q = (z_2, z_1, z_3, z_4)$ is in $(\overline{G}, \partial G)$, and $\tau(Q) \leq 2c$. Since

$$\tau(fQ) \geq \frac{M'}{3c|fz_2 - fz_3|}$$

and since f is θ-QM rel ∂G, we obtain

$$\frac{M'}{|fz_2 - fz_3|} \leq 3c\theta(2c). \ \square$$

6. Uniform domains

6.1. *Introduction to Section* 6. Uniform domains in R^n were introduced by Martio and Sarvas [MS] in 1979. A related concept was independently studied by Jones [Jo], and the equivalence of these two approaches was proved in [GO]. In this section we shall consider uniform domains in a Banach space. The definition will be given in terms of length cigars, and alternative characterizations are given in terms of the QH metric.

6.2. *Cigars.* Let $\gamma \subset E$ be an arc with endpoints a, b. For $x \in \gamma$ we set

$$\varrho_d(x) = d(\gamma[a, x]) \wedge d(\gamma[x, b]).$$

If γ is rectifiable, we also define the function

$$\varrho_l(x) = l(\gamma[a, x]) \wedge l(\gamma[x, b]).$$

For $c \geq 1$, the sets

$$\operatorname{cig}_d(\gamma, c) = \cup\{B(x, \varrho_d(x)/c) : x \in \gamma \setminus \{a, b\}\},$$
$$\operatorname{cig}_l(\gamma, c) = \cup\{B(x, \varrho_l(x)/c) : x \in \gamma \setminus \{a, b\}\}$$

are the *diameter c-cigar* and the *length c-cigar*, respectively, with core γ. The length cigar is only defined for a rectifiable γ.

6.3. *Uniform domains.* Let $c \geq 1$. A domain $G \subset E$ is a *c-uniform domain* if each pair $a, b \in G$ can be joined by a rectifiable arc γ satisfying the following *uniformity conditions*:
(1) $\mathrm{cig}_l(\gamma, c) \subset G$,
(2) $l(\gamma) \leq c|a-b|$.
We call (1) the *cigar condition* and (2) the *turning condition*. Observe that (1) can be rewritten as
(1′) $\varrho_l(x) \leq c\delta(x)$
for all $x \in \gamma$.

In R^n one can also characterize the uniform domains by diameter cigars and the so-called distance cigars, cf. [MS] and [Vä$_2$]. This is no longer true in the general case. For example, the broken tube of I.4.12 is not a uniform domain although one can show that there is $c > 1$ such that each pair $a, b \in G$ can be joined by an arc γ such that $\mathrm{cig}_d(\gamma, c) \subset G$ and $d(\gamma) \leq c|a-b|$.

We first give examples of uniform domains. A simple lemma is needed:

6.4. Lemma. *If $y, z \in S(1)$, then $|y-z| \leq 2d(y, [0, z])$.*

Proof. Let $x \in [0, z]$ and set $\alpha = |y-z|/2$. If $|x| \leq 1-\alpha$, then $|y-x| \geq |y| - |x| \geq \alpha$. If $|x| \geq 1 - \alpha$, then

$$|y-x| \geq |y-z| - |z-x| = 2\alpha - (1-|x|) \geq \alpha. \ \square$$

6.5. Theorem. *For $x_0 \in E$ and $r > 0$, the domains $B(x_0, r), B(x_0, r) \setminus \{x_0\}$ and $E \setminus \{x_0\}$ are c-uniform with a universal c.*

Proof. We may assume that $x_0 = 0$ and $r = 1$. We first consider the domain $G = B(1) \setminus \{0\}$. Suppose that $a, b \in G$, $a \neq b$, $|a| \geq |b|$. We show that a and b can be joined by an arc γ satisfying the uniformity conditions in G. Setting

$$a_0 = a/|a|, \qquad b_0 = b/|b|, \qquad t = |a_0 - b_0|/4$$

we have $0 \leq t \leq 1/2$. We consider two cases.

Case 1. $|b| \leq 1-t$. Set $a_1 = |b|a_0$ and apply 2.2 to find an arc $\gamma_1 \subset S(|b|)$ joining a_1 and b with $l(\gamma_1) \leq 2|a_1 - b| = 8t|b|$. We show that $\gamma = \gamma_1 \cup [a_1, a]$ has the desired properties.

If $x \in \gamma_1$, then

$$\varrho_l(x) \leq l(\gamma_1) \leq 8t|b|, \qquad \delta(x) = |b| \wedge \big(1-|b|\big) \geq |b| \wedge t,$$

and hence $\varrho_l(x) \leq 8\big(t \vee |b|\big)\delta(x) < 8\delta(x)$. If $x \in [a_1, a] \cap B(1/2)$, then

$$\varrho_l(x) \leq l(\gamma_1) + |x - a_1| \leq 8t|b| + |x| \leq 5|x| = 5\delta(x).$$

If $x \in [a_1, a] \setminus B(1/2)$, then

$$\varrho_l(x) \le |a - x| \le 1 - |x| = \delta(x).$$

Hence the cigar condition holds with $c = 8$.

If $|a_1 - b| \le 2|a - a_1|$, then

$$l(\gamma) \le 5|a - a_1| = 5\big(|a| - |b|\big) \le 5|a - b|.$$

If $|a_1 - b| \ge 2|a - a_1|$, then

$$l(\gamma) \le 5|a_1 - b|/2, \qquad |a - b| \ge |a_1 - b| - |a - a_1| \ge |a_1 - b|/2,$$

and we obtain in both cases the turning condition with $c = 5$.

Case 2. $|b| \ge 1 - t$. Set $a_1 = (1 - t)a_0$, $b_1 = (1 - t)b_0$, and join the points a_1, b_1 with an arc $\gamma_1 \subset S(1 - t)$ with $l(\gamma_1) \le 2|a_1 - b_1| = 8t(1 - t) < 8t$. We show that $\gamma = [a, a_1] \cup \gamma_1 \cup [b_1, b]$ is the desired arc. If $x \in [a, a_1] \cup [b, b_1]$, then $\varrho_l(x) < \delta(x)$. If $x \in \gamma_1$, then

$$\varrho_l(x) \le |a - a_1| + l(\gamma_1) \le t + 8t = 9\delta(x),$$

and we obtain the cigar condition with $c = 9$.

By 6.4 we have

$$|a - b| = |a|\big|a_0 - b/|a|\big| \ge 2|a|t \ge t,$$

which yields the turning condition

$$l(\gamma) \le |a| - 1 + t + 8t + |b| - 1 + t < 10t \le 10|a - b|.$$

Hence the domain $B(1) \setminus \{0\}$ is 10-uniform. The case $G = E \setminus \{x_0\}$ follows immediately from this. For $G = B(1)$ it suffices to observe that the line segment $[0, x]$ satisfies for all $x \in G$ uniformity conditions with $c = 1$. □

6.6. *Other examples.* Suppose that T is a closed affine proper subspace of E. If codim $T \ge 2$, $E \setminus T$ is a domain. If codim $T = 1$, $E \setminus T$ consists of two domains called half spaces. All these domains are c-uniform with universal c. The case $T = \{x_0\}$ is contained in 6.5. The proof of the general case is contained in [Al]. In fact, c can be chosen to be any number greater than 2.

If G and D are c-uniform domains with $G \subset \overline{D}$, then $G \cap D$ is a c_1-uniform domain with $c_1 = c_1(c)$. In the case $E = R^n$ this is essentially Theorem 5.4 of [Vä$_2$]. In the general case, the proof needs some modification; for example, the distance cigars must be replaced by length cigars. A detailed proof is in [Al]. A direct proof for the case $D = E \setminus \{x_0\}$ is sketched in 6.7 below.

More examples can be obtained by auxiliary maps. We show in 6.26 that if G is c-uniform and $f\colon G \to G'$ η-QM, then G' is c_1-uniform with $c_1 = c_1(c, \eta)$.

6.7. Lemma. *Suppose that G is a c-uniform domain and that $x_0 \in G$. Then $G_0 = G \setminus \{x_0\}$ is c_0-uniform with $c_0 = c_0(c)$.*

Proof. We may assume that $x_0 = 0$ and $\delta(x_0) = 1$. Let $a, b \in G_0$, $a \neq b$. We describe how to construct an arc γ_0 from a to b satisfying the uniformity conditions in G_0. We consider 3 cases.

Case 1. $|a| < 1$ and $|b| < 1$. This case follows from 6.5.

Case 2. $|a| \geq 1/2$ and $|b| \geq 1/2$. Join a to b with an arc γ satisfying the uniformity conditions in G. If $0 \notin \mathrm{cig}_l(\gamma, 3c)$, choose $\gamma_0 = \gamma$. If $0 \in \mathrm{cig}_l(\gamma, 3c)$ it is easy to see that γ meets $S(1/2)$. Orient γ from a to b, and choose the first point a_1 and the last point b_1 of γ in $S(1/2)$. Apply 2.2 to choose an arc α joining a_1 and b_1 in $S(1/2)$ with $l(\alpha) \leq 2|a_1 - b_1|$. Then

$$\gamma_0 = \gamma[a, a_1] \cup \alpha \cup \gamma[b_1, b].$$

Case 3. $|a| < 1/2$ and $|b| \geq 1$. Let γ and b_1 be as in Case 2. We obtain γ_0 by replacing $\gamma[a, b_1]$ by the union of an arc in $S(|a|)$ and a radial segment from $S(|a|)$ to $S(1/2)$. □

6.8. *Other approaches to uniformity.* For $x, y \in G \neq E$, the numbers

$$r_G(x, y) = \frac{|x - y|}{\delta(x) \wedge \delta(y)}, \qquad j_G(x, y) = \ln\bigl(1 + r_G(x, y)\bigr)$$

are the *relative distance* and the *Jones distance* between x and y in G, respectively. We shall often abbreviate $j = j_G$, $j' = j_{G'}$. Slightly different but essentially equivalent versions of j_G have been considered by Jones [Jo] and by Gehring–Osgood [GO]; the present expression is due to Vuorinen [Vu_1]. By I.2.2 we always have

$$j_G(x, y) \leq k_G(x, y).$$

The uniform domains in R^n can be characterized by inequalities in the opposite direction. Indeed, either of the conditions

$$k_G \leq c j_G, \qquad k_G \leq c j_G + d$$

is quantitatively equivalent to c-uniformity; see [GO], [Vu_1, 2.50(2)] and [Ge, Theorem 6]. A free version of this result is given in 6.16.

We also consider a generalization of the inequality $k_G \leq c j_G$, suggested by Vuorinen [Vu_1, 2.49]. Let ψ: $[0, \infty) \to [0, \infty)$ be a homeomorphism. A domain $G \neq E$ is called *quasihyperbolically* ψ*-uniform*, or briefly QH ψ-uniform, if

$$k_G(x, y) \leq \psi\bigl(r_G(x, y)\bigr)$$

for all $x, y \in G$. For the function $\psi(t) = c\ln(1+t)$ this gives the inequality $k_G \le cj_G$. A somewhat surprising fact is that for a large class of functions ψ, this is no generalization at all. More precisely, let us call a homeomorphism $\psi\colon [0,\infty) \to [0,\infty)$ *slow* if $\psi(t)/t \to 0$ as $t \to \infty$. The function $\psi(t) = c\ln(1+t)$ is clearly slow. We shall prove that QH ψ-uniformity with a slow ψ quantitatively implies the condition $k_G \le cj_G$.

We start with results dealing with coarse length in uniform domains. They are needed in the proof of 6.22, which is useful in Section 7. To prove the equivalences described above, we only need the case $h = 0$ of these results. Recall that an arc is (c, h)-solid if it is h-coarsely c-quasiconvex in the QH metric of G. Roughly speaking, we show that a solid arc cannot travel long distances near the boundary of a uniform domain.

6.9. Lemma. *Suppose that $G \ne E$ and that γ is an arc in $G \cap (\partial G + \overline{B}(r))$ with endpoints a_0, a_1 such that $\delta(a_0) \wedge \delta(a_1) \ge r/c_1$. Suppose also that G is* QH *ψ-uniform with a slow ψ.*
(1) *If γ is (c, h)-solid, then $d(\gamma) \le M_1(c, h, c_1, \psi)r$.*
(2) *If γ is a c-neargeodesic, then $l(\gamma) \le M_1(c, c_1, \psi)r$.*

Proof. To prove (1) we set $t = d(\gamma)/r$ and look for an upper bound $t \le M_1$. The solidity and uniformity conditions give

$$l_k(\gamma, h) \le ck(a_0, a_1) \le c\psi\big(r_G(a_0, a_1)\big) \le c\psi(c_1 t).$$

If $c\psi(c_1 t) \le h$, we can choose $M_1 = \psi^{-1}(h/c)/c_1$. If $c\psi(c_1 t) \ge h$, then 4.5 gives $d(\gamma) \le M(h)c\psi(c_1 t)r$, and hence

$$1 \le M(h)cc_1 \frac{\psi(c_1 t)}{c_1 t}.$$

Since ψ is slow, this yields the desired bound $t \le M_1(c, h, c_1, \psi)$.

To prove (2) we set $t = l(\gamma)/r$. An easy modification of the argument above gives

$$1 \le cc_1 \frac{\psi(c_1 t)}{c_1 t},$$

and hence $t \le M_1(c, c_1, \psi)$. □

6.10. Lemma. *Suppose that G is a* QH *ψ-uniform domain with a slow ψ. Suppose also that γ is an arc in $G \cap (\partial G + \overline{B}(r))$.*
(1) *If γ is (c, h)-solid, then $d(\gamma) \le M_2(c, h, \psi)r$.*
(2) *If γ is a c-neargeodesic, then $l(\gamma) \le M_2(c, \psi)r$.*

Proof. Replacing r by a smaller number we may assume that $\delta(a_0) > r/2$ for some $a_0 \in \gamma$. Dividing γ to two subarcs we may further assume that a_0 is an endpoint of γ. Choose successive points $a_1, a_2, \dots$ of γ such that a_j is the last point of γ with $\delta(a_j) \ge 2^{-j} r$. The sequence (a_j) may be finite or infinite. Set $\gamma_j = \gamma[a_{j-1}, a_j]$. In the part (1) we obtain from 6.9(1)

$$d(\gamma_j) \le M_1(c, h, 2, \psi) r / 2^{j-1},$$

and hence $d(\gamma) \le 2M_1 r$. The part (2) follows similarly from 6.9(2). □

6.11. Lemma. *For every slow ψ and for all $c \ge 1$, $h \ge 0$ there is a number $q = q(c, h, \psi) \in (0,1)$ with the following property: Suppose that G is a QH ψ-uniform domain and that γ is a (c,h)-solid arc starting at x_0 and containing a point x with $\delta(x) \le q\delta(x_0)$. Then for $\gamma_x = \gamma \setminus \gamma[x_0, x)$ we have $d(\gamma_x) \le M_3(c, h, \psi)\delta(x)$. If $h = 0$, then $l(\gamma_x) \le M_3(c, \psi)\delta(x)$.*

Proof. Let $M_2 = M_2(c, h, \psi)$ be the constant given by 6.10. We show that one can choose

$$q = \exp\left[-2\big(h \vee c\psi(M_2)\big)\right].$$

Let γ, x_0, x satisfy the conditions of 6.11 with this q. Setting $r = \delta(x)/q$ we have $r \le \delta(x_0)$. It suffices to show that $\gamma_x \subset \partial G + \overline{B}(r)$, since then 6.10 gives the result with $M_3 = M_2/q$.

Assume that $\gamma_x \not\subset \partial G + \overline{B}(r)$. Let x_2 be the first point of γ_x with $\delta(x_2) = r$. Since $\delta(x_0) \ge r$, we can choose the last point x_1 of $\gamma[x_0, x]$ with $\delta(x_1) = r$. Then for $\alpha = \gamma[x_1, x_2]$ we have

$$l_k\big(\gamma[x_1, x], h\big) \le l_k(\alpha, h) \le ck(x_1, x_2) \le c\psi\big(|x_1 - x_2|/r\big) \le c\psi\big(d(\alpha)/r\big).$$

By 6.10 we have $d(\alpha) \le M_2 r$. Using I.2.2 and 4.3(5) we obtain

$$\ln\left(1 + \frac{|x_1 - x|}{qr}\right) \le k(x_1, x) \le k\big(\gamma[x_1, x]\big) \le h \vee c\psi(M_2) = \frac{1}{2}\ln\frac{1}{q}.$$

On the other hand, we have

$$|x_1 - x| \ge \delta(x_1) - \delta(x) = (1-q)r,$$

which gives the contradiction

$$\ln\left(1 + \frac{|x_1 - x|}{qr}\right) \ge \ln\frac{1}{q}. \ \square$$

6.12. Theorem. *Suppose that G is a QH ψ-uniform domain with a slow ψ and that $\gamma \subset G$ is a c-neargeodesic with endpoints a_0 and a_1. Then γ satisfies the uniformity conditions*

(1) $\operatorname{cig}_l(\gamma, c_1) \subset G$,

(2) $l(\gamma) \le c_1 |a_0 - a_1|$,

where c_1 depends only on c and ψ.

Proof. Choose $x_0 \in \gamma$ such that $\delta(x_0)$ is maximal. Let $q = q(c, 0, \psi)$ be the number given by 6.11. If $x \in \gamma[a_0, x_0]$ and $\delta(x) \le q\delta(x_0)$, then 6.11 implies

$$l(\gamma[a_0, x]) \le M_3(c, \psi)\delta(x).$$

If $x \in \gamma[a_0, x_0]$ and $\delta(x) \ge q\delta(x_0)$, then 6.10 with $r \mapsto \delta(x_0)$ yields

$$l(\gamma[a_0, x]) \le M_2(c, \psi)\delta(x_0) \le (M_2/q)\delta(x).$$

Considering similarly the arc $\gamma[a_1, x]$ we conclude that (1) is true with $c_1(c, \psi) = M_3 \vee (M_2/q)$.

To prove (2) write $t = |a_0 - a_1|$. We may assume that $\delta(a_0) \le \delta(a_1)$. We consider two cases.

Case 1. $\delta(a_0) \le t$. We may assume that $l(\gamma) > 2t$. Choose points b_0 and b_1 of γ such that

$$l(\gamma[a_0, b_0]) = t = l(\gamma[a_1, b_1]).$$

By (1) we have $t \le c_1\delta(b_0)$ and $t \le c_1\delta(b_1)$. Hence

$$r_G(b_0, b_1) \le \frac{|b_0 - a_0| + |a_0 - a_1| + |a_1 - b_1|}{t/c_1} \le 3c_1,$$

and hence

$$k(b_0, b_1) \le \psi(3c_1). \tag{6.13}$$

For each $x \in \gamma[b_0, b_1]$ we have

$$k(x, b_0) \le l_k(\gamma[b_0, x]) \le l_k(\gamma[b_0, b_1]) \le ck(b_0, b_1) \le c\psi(3c_1).$$

By I.2.2 this yields

$$|x - b_0| \le \delta(b_0)(e^{c\psi(3c_1)} - 1).$$

Since

$$\delta(b_0) \le \delta(a_0) + |a_0 - b_0| \le \delta(a_0) + t \le 2t,$$

we obtain

$$\delta(x) \le \delta(b_0) + |x - b_0| \le 2te^{c\psi(3c_1)} = M_4(c, \psi)t.$$

Integration along $\gamma[b_0, b_1]$ gives

$$ck(b_0, b_1) \ge l_k(\gamma[b_0, b_1]) \ge \frac{l(\gamma[b_0, b_1])}{M_4 t}.$$

By (6.13) this implies

$$l(\gamma[b_0, b_1]) \le M_5(c, \psi)t.$$

Hence (2) holds with c_1 replaced by $M_5 + 2$.

Case 2. $\delta(a_0) = r \ge t$. This case makes no use of the QH uniformity of G. Since $\delta(a_1) \ge r$, we have $\delta(x) \ge r/2$ for all $x \in [a_0, a_1]$. Integration along this line segment yields

$$k(a_0, a_1) \le 2t/r. \tag{6.14}$$

Let $\alpha\colon [0, \lambda] \to \gamma$ be the arc-length parametrization of γ with $\lambda = l(\gamma)$ and $\alpha(0) = a_0$. Since

$$\delta(\alpha(s)) \le \delta(a_0) + |a_0 - \alpha(s)| \le r + s,$$

we obtain

$$l_k(\gamma) = \int_0^\lambda \frac{ds}{\delta(\alpha(s))} \ge \int_0^\lambda \frac{ds}{r+s} = \ln\Big(1 + \frac{\lambda}{r}\Big).$$

Since γ is a c-neargeodesic, this and (6.14) imply

$$\ln\Big(1 + \frac{\lambda}{r}\Big) \le \frac{2ct}{r}.$$

Setting $u = r/t$ we obtain

$$\lambda/t = u(e^{2c/u} - 1).$$

Since $u \ge 1$ and since the right-hand side is bounded for $u \ge 1$, this implies (2). □

6.15. Lemma. *Suppose that* $a, b \in G \ne E$. *Then the following conditions are quantitatively equivalent:*
(1) $k(a,b) \le cj(a,b)$, $c \ge 1$.
(2) $k(a,b) \le cj(a,b) + d$, $c \ge 1$, $d \ge 0$.

Proof. Trivially (1) implies (2). Assume that (2) holds, and set $r = r_G(a,b)$. Suppose first that $r \le 1/2$. Since now $r \ln 2 \le \ln(1+r)$, I.2.5 implies

$$k(a,b) \le 2r \le \frac{2}{\ln 2} j(a,b).$$

Next assume that $r \ge 1/2$. Then $j(a,b) \ge \ln(3/2)$, and hence

$$\frac{k(a,b)}{j(a,b)} \le c + \frac{d}{\ln(3/2)}. \ \square$$

6.16. Theorem. *For a domain* $G \ne E$, *the following conditions are quantitatively equivalent:*
(1) G *is* c*-uniform,*
(2) $k_G \le cj_G$,
(3) $k_G \le cj_G + d$,
(4) G *is* QH ψ*-uniform with a slow* ψ.

Proof. By 6.15, (3) implies (2). Since the function $\psi(t) = c\ln(1+t)$ is slow, (2) clearly implies (4). The implication (4) $\Rightarrow$ (1) follows from 6.12 and 3.3. Finally, the implication (1) $\Rightarrow$ (3) can be proved with obvious modifications as in the case $E = R^n$ [GO, Theorem 1]. Observe that [GO] uses a slightly different version of the Jones distance, call it $j_G^*(a,b)$, but one has always $j_G^* \le j_G \le 2j_G^*$. □

6.17. *Remark.* Inspection of the proofs shows that one can replace the slowness condition in 6.16 by the weaker condition

$$\limsup_{t\to\infty} \frac{\psi(t)}{t} = v < 1. \tag{6.18}$$

Indeed, assume that (4) of 6.16 holds with such ψ. Set $c = c_1 = v^{-1/4}$. Thus c and c_1 depend only on ψ. Since $cc_1v = v^{1/2} < 1$, the proof of 6.9(2) is valid with these c and c_1, and we get $M_1 = M_1(\psi)$. In the proof of 6.10(2) we replace the conditions $\delta(a_j) \ge 2^{-j}r$ by $\delta(a_j) \ge c_1^{-j}r$ and obtain $l(\gamma) \le M_2 r$ with $M_2 = M_1(1-c_1^{-1})^{-1} = M_2(\psi)$. Then the case $h = 0$ of 6.11 is also true with these ψ and c giving q and M_3 depending on ψ. It follows that the proof of 6.12 is valid with these ψ and c giving a number $c_1 = c_1(\psi)$, which should not be confused with the number $c_1 = v^{-1/4}$ above. Since each pair of points in G can be joined by a c-neargeodesic by 3.3, G is c_1-uniform.

The condition (6.18) is sharp in the sense that it cannot be replaced by $v \le 1$. For example, each convex domain is QH ψ-uniform with $\psi(t) = t$, but a parallel strip in R^2 is not a uniform domain.

From 6.12 and 6.16 we immediately get the following result, which in the case $E = R^n = E'$, $c_1 = 1$ is given by [GO, Corollary 2, p. 59]:

6.19. Theorem. *Suppose that $G \ne E$ is a c-uniform domain and that γ is a c_1-neargeodesic in G with endpoints a_0, a_1. Then there is $c_2 = c_2(c, c_1) \ge 1$ such that*

(1) $\mathrm{cig}_l(\gamma, c_2) \subset G$,

(2) $l(\gamma) \le c_2|a_0 - a_1|$. □

6.20. *Remark.* Theorem 6.19 means that in a uniform domain, any neargeodesic is the core of a length cigar satisfying the uniformity conditions. We shall next prove a coarse version of this, replacing the neargeodesic by a solid arc and length by diameter. Both results will be needed in the proofs of 7.3 and 7.9. An auxiliary result is needed:

6.21. Lemma. *Suppose that $G \ne E$ and that γ is a (c,h)-solid arc in G with endpoints a_0, a_1 such that $\delta(a_0) \wedge \delta(a_1) = r \ge |a_0 - a_1|$. Then there is $c_2 = c_2(c) \ge 1$ such that*

$$d(\gamma) \le c_2|a_0 - a_1| \vee 2r(e^h - 1).$$

Proof. We may assume that $\delta(a_1) \geq \delta(a_0) = r$. Setting $t = |a_0 - a_1|$ and integrating along $[a_0, a_1]$ we get $k(a_0, a_1) \leq 2t/r$ as in the proof of (6.14). It suffices to find an estimate

$$|x - a_0| \leq c_2 t \vee r(e^h - 1)$$

for an arbitrary $x \in \gamma$.

If $k(x, a_0) \leq h$, then I.2.2 gives $|x - a_0| \leq r(e^h - 1)$. Suppose that $k(x, a_0) \geq h$. Choose successive points $a_0 = x_0, \ldots, x_n = x$ of γ, $n \geq 1$, such that

$$h \leq k(x_{j-1}, x_j) \leq 2h$$

for all j. Then

$$nh \leq l_k(\gamma, h) \leq ck(a_0, a_1) \leq 2ct/r.$$

This implies $k(x, a_0) \leq 2nh \leq 4ct/r$, and hence

$$|x - a_0| \leq \delta(a_0)\big(e^{k(x,a_0)} - 1\big) \leq r(r^{4ct/r} - 1).$$

Setting $u = r/t$ we have $u \geq 1$ and

$$|x - a_0|/t \leq u(e^{4c/u} - 1).$$

Since the right-hand side is bounded for $u \geq 1$, the lemma follows. □

6.22. Theorem. *Suppose that $\gamma \subset G \neq E$ is a (c, h)-solid arc with endpoints a_0, a_1 and that G is a c_1-uniform domain. Then there is $c_2 = c_2(c, h, c_1) \geq 1$ such that for $r = \delta(a_0) \wedge \delta(a_1)$ we have*
(1) $\mathrm{cig}_d(\gamma, c_2) \subset G$,
(2) $d(\gamma) \leq c_2\big(|a_0 - a_1| \vee 2r(e^h - 1)\big)$.

Proof. Choose $x_0 \in \gamma$ for which $\delta(x_0)$ is maximal. For (1) it suffices to find $c_2 = c_2(c, h, c_1)$ such that

$$d\big(\gamma[a_0, x]\big) \leq c_2 \delta(x) \tag{6.23}$$

for all $x \in \gamma[a_0, x_0]$.

By 6.16 G is QH ψ-uniform with a slow ψ depending only on c_1. Let $q = q(c, h, \psi) \in (0, 1)$ be the number given by 6.11. If $\delta(x) \leq q\delta(x_0)$, then (6.23) follows from 6.11. If $\delta(x) \geq q\delta(x_0)$, we apply 6.10 with the substitution $r \mapsto \delta(x_0)$, $\gamma \mapsto \gamma[a_0, x]$. We obtain

$$d\big(\gamma[a_0, x]\big) \leq M_2(c, h, \psi)\delta(x_0),$$

and hence (6.23) holds with $c_2 = M_2/q$.

To prove (2) set $t = |a_0 - a_1|$. We may assume that $\delta(a_1) \geq \delta(a_0) = r$ and that $d(\gamma) > 2t$. If $r \geq t$, then (2) follows from 6.21. Suppose that $r \leq t$. Choose points $y_0, y_1 \in \gamma$ such that $d(\gamma[a_i, y_i]) = t$ for $i = 1, 2$. By (1) we have

$$\delta(y_i) \geq d(\gamma[a_i, y_i])/c_2 = t/c_2.$$

Since

$$|y_0 - y_1| \leq |y_0 - a_0| + |a_0 - a_1| + |a_1 - y_1| \leq 3t,$$

and since G is QH ψ-uniform, this implies

$$k(y_0, y_1) \leq \psi(r_G(y_0, y_1)) \leq \psi(3c_2).$$

By 4.3(5) and by the (c,h)-solidity of γ we get

$$k(\gamma[y_0, y_1]) \leq h \vee l_k(\gamma[y_0, y_1], h) \leq h \vee ck(y_0, y_1) \leq h \vee c\psi(3c_2) = c_3(c, h, c_1).$$

Hence $\gamma[y_0, y_1]$ is contained in the QH ball $B_k(y_0, c_3)$. By I.2.2 this implies that $\gamma \subset B(y_0, R)$ with

$$R = \delta(y_0)(e^{c_3} - 1) + t \leq (r + |a_0 - y_0|)(e^{c_3} - 1) + t \leq 2te^{c_3} - t.$$

This implies (2) with c_2 replaced by $4e^{c_3} - 2$. □

6.24. *Quasimöbius invariance.* In the basic paper [MS, 2.15], Martio and Sarvas proved that QC maps $f\colon R^n \to R^n$ preserve the class of uniform domains. A different proof was given in [GO, Corollary 3, p. 65]. More generally, uniformity is preserved by QM maps $f\colon G \to G'$ of domains in R^n. This is obtained by modifying the proofs mentioned above or by using the characterization of uniform domains in terms of cross ratios, given by Martio [Ma]. The latter method was extended by the author [Vä$_1$, 4.11] to a large class of spaces including all Banach spaces. However, the definition of a uniform domain in [Vä$_1$] is not equivalent to the definition of the present paper in infinite-dimensional spaces. We shall next use the ideas of [GO] to prove the QM invariance of uniform domains in Banach spaces. For the notation of the following lemma, see 6.8.

6.25. Lemma. *Suppose that $G \neq E$, $G' \neq E'$ and that $f\colon G \to G'$ is an η-QM homeomorphism. Then*

$$j'(fa, fb) \leq Mj(a, b) + C$$

for all $a, b \in G$ with M and C depending only on η.

Proof. We first consider two special cases.

Case 1. $0 \notin G$ and f is the restriction of the inversion $u(x) = x/|x|^2$. Let $a, b \in G$ and set $r = r_G(a,b)$, $r' = r_{G'}(ua, ub)$. We may assume that $\delta'(ua) \le \delta'(ub)$. Choose $x \in \partial G$ such that $|ux - ua| \le 2\delta'(ua)$. By [Vä$_1$, (1.7)] we obtain

$$\frac{|ua - ub|}{\delta'(ua)} \le \frac{18|a-b||x|}{|x-a||b|}.$$

Together with the inequalities

$$|x| \le |x-a| + |a-b| + |b|, \qquad |x-a| \ge \delta(a), \qquad |b| \ge \delta(b)$$

this yields

$$r' \le 18|a-b|\Big(\frac{1}{\delta(b)} + \frac{|a-b|}{\delta(a)\delta(b)} + \frac{1}{\delta(a)}\Big) \le 18(2r + r^2).$$

Hence $1 + r' \le 18(1+r)^2$, and we obtain the lemma with $M = 2$, $C = \ln 18$.

Case 2. f is η-QS. Now f extends to an η-QS homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$. Let $a, b \in G$, and choose $x \in \partial G$ with $|fa - \overline{f}x| \le 2\delta'(fa)$. Setting $r = r_G(a,b)$ we obtain

$$\frac{|fa - fb|}{\delta'(fa)} \le 2\frac{|fa - fb|}{|fa - \overline{f}x|} \le 2\eta\Big(\frac{|a-b|}{|a-x|}\Big) \le 2\eta(r).$$

Since G is connected, [TV$_1$, 3.12] implies that one can choose η to be of the form $\eta(t) = C_1(t^\alpha \vee t^{1/\alpha})$, $C_1 \ge 1$, $\alpha \ge 1$. Since we may assume that $\delta'(fa) \le \delta'(fb)$, we obtain

$$r_{G'}(fa, fb) \le 2C_1(r^\alpha \vee r^{1/\alpha}).$$

If $r \le 1$, then the lemma holds with $M = 0$, $C = \ln(1 + 2C_1)$. If $r \ge 1$, then $1 + r^\alpha \le (1+r)^\alpha$, and thus we can chose $M = \alpha$, $C = \ln 2C_1$.

The general case is reduced to the special cases as follows: First extend f to an η-QM embedding $\overline{f}\colon \overline{G} \to \dot{E}'$ applying [Vä$_1$, 3.19], where the misprinted fA should be replaced by $\overline{fA}$. By auxiliary translations we may assume that $0 \in \partial G$ and that $\overline{f}(0)$ is either 0 or ∞. Furthermore, we can use auxiliary inversions and Case 1 to normalize the map so that $\infty \in \partial G$ and $\overline{f}(\infty) = \infty$. Then f is η-QS, and the result follows from Case 2. □

6.26. Theorem. *Suppose that* $f\colon G \to G'$ *is an* η-QM *homeomorphism and that* G *is a* c-*uniform domain. Then* G' *is* c_1-*uniform with* $c_1 = c_1(c, \eta)$.

Proof. If $G = E$, then $G' = E'$ by [Vä$_1$, p. 226] or, alternatively, by I.5.13 and I.5.18. We may thus assume that $G \ne E$, $G' \ne E'$. By I.5.18 and by 4.14, there are $M \ge 1$ and $C \ge 0$ depending only on η such that

$$k'(fa, fb) \le Mk(a,b) + C$$

for all $a, b \in G$. Applying 6.25 to f^{-1} we can write

$$j(a,b) \le M_1 j'(fa, fb) + C_1$$

with M_1, C_1 depending only on η. By 6.16, we have $k \le c_0 j$ with $c_0 = c_0(c)$. These inequalities imply

$$k'(fa, fb) \le Mc_0 j(a,b) + C \le MM_1 c_0 j'(fa, fb) + Mc_0 C_1 + C,$$

and the theorem follows from 6.16. □

6.27. *Endcuts.* Recall from 3.6 that an endcut of a domain G is a half open arc $\gamma \subset G$ such that $\overline{\gamma}$ is a closed arc with one endpoint in ∂G. If $G \ne E$, the QH diameter $k(\gamma)$ of an endcut is always infinite. We next show that the converse is true for solid arcs in uniform domains:

6.28. Theorem. *Suppose that γ is a half open solid arc in a uniform domain $G \ne E$ and that $k(\gamma) = \infty$. Then γ is an endcut of G. If γ is also a neargeodesic, then either γ converges to ∞ or γ is rectifiable.*

Proof. Assume that G is QH ψ-uniform with a slow ψ and that γ is (c,h)-solid in G. Since $k(\gamma) = \infty$, we have $l_k(\gamma, h) = \infty$. We may assume that γ starts at the origin and that γ does not converge to ∞. For $x \in \gamma$ write $\gamma_x = \gamma \setminus \gamma[0,x)$. There is $R > 0$ such that $B(R)$ meets γ_x for every $x \in \gamma$. If $z \in \gamma_x \cap B(R)$, then

$$l_k(\gamma[0,z], h) \le ck(0,z) \le c\psi\Big(\frac{R}{\delta(0) \wedge \delta(z)}\Big).$$

Since $l_k(\gamma, h) = \infty$, this implies $d(\gamma, \partial G) = 0$.

Let $\varepsilon > 0$. To prove that γ is an endcut it suffices to find $x \in \gamma$ with $d(\gamma_x) \le \varepsilon$. Let $q = q(c,h,\psi)$ and $M_3 = M_3(c,h,\psi)$ be the numbers given by 6.11. Choose $x \in \gamma$ such that

$$\delta(x) \le q\delta(0) \wedge (\varepsilon/M_3).$$

Then 6.11 gives $d(\gamma_x) \le M_3\delta(x) \le \varepsilon$.

In the case $h = 0$, 6.11 yields $l(\gamma_x) \le M_3\delta(x) < \infty$ for any x satisfying the inequality above. Hence γ is rectifiable. □

7. Boundary behavior

7.1. *Introduction to Section* 7. This section deals mainly with the boundary properties of homeomorphisms $f\colon G \to G'$ between uniform domains. We show that if f is CQH, it can be extended to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$, which is QM rel ∂G in the norm metric. In the special case where f is FQC, $\overline{f}$ is QM in the whole $\overline{G}$. These results are quantitative.

Many of the results are new also in the classical case $E = E' = R^n$. In the case where G and G' are half spaces of R^n, V.A. Efremovich and E.S. Tihomirova [ET] proved in 1964 that a solid map $f\colon G \to G'$ extends to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$, and D.A. De-Spiller [DS] proved in 1970 that the induced map $\partial G \to \partial G'$ is quasiconformal. De-Spiller also proved that conversely, each quasiconformal map of R^{n-1} can be extended to a homeomorphism of $R^{n-1} \times [0,\infty)$ which is solid in the open half space. We shall construct the corresponding extension from a Banach space E to $E \times [0,\infty)$ in 7.26. A reflection principle is given in 7.35.

7.2. *Terminology.* Suppose that G is a domain in E and that $f\colon G \to E'$ is a map. The *cluster set* of f at a point $x_0 \in \partial G$ is defined as

$$\operatorname{clus}(f, x_0) = \cap \operatorname{cl} f[U \cap G]$$

over all neighborhoods U of x_0 in $\dot{E}$. Equivalently, a point $y \in \dot{E}'$ belongs to $\operatorname{clus}(f, x_0)$ if and only if there is a sequence of points $x_j \in G$ such that $x_j \to x_0$ and $fx_j \to y$.

In the general case, the cluster set may be empty even if f is an FQC map onto a domain G'; see I.4.12.

We next prove the crucial lemma of the paper. Its proof makes effective use of the theory of the preceding sections: the existence of neargeodesics, the CQH invariance of solid arcs, and the length cigar and diameter cigar theorems for uniform domains.

7.3. Fundamental lemma. *Suppose that $G \neq E$ and $G' \neq E'$ are unbounded c-uniform domains and that $f\colon G \to G'$ is (M,C)-CQH with $\infty \in \operatorname{clus}(f,\infty)$. Let x, a, b be points in G such that $|a - x| \le |b - x|$ and $k(b,x) \ge 2C \vee 1/2$. Then*

$$|fa - fx| \le H|fb - fx|$$

with $H = H(M,C,c)$.

Proof. By auxiliary similarities we normalize the situation so that $x = 0$, $fx = 0$, $|b| = 1 = |fb|$. Then $|a| \le 1$ and we must find an upper bound $|fa| \le H$ with H depending only on $v = (M,C,c)$.

We first show that

$$\delta'(0) \le 8M. \tag{7.4}$$

Since this is trivial if $\delta'(0) \le 2$, we may assume that $\delta'(0) \ge 2$. Then I.2.5 gives

$$k'(0, fb) \le 2|fb|/\delta'(0) = 2/\delta'(0). \tag{7.5}$$

If $C \le 1/4$, we have

$$k'(0, fb) \ge \big(k(0,b) - C\big)/M \ge 1/4M.$$

If $C \ge 1/4$, then $k(0,b) \ge 2C$, and hence

$$k'(0, fb) \ge C/M \ge 1/4M,$$

which thus holds in all cases. By (7.5) this gives (7.4).

In what follows, we let $c_1, c_2, \ldots$ denote constants depending only on v. Applying 6.19 we first choose $c_1 \ge 1$ such that the uniformity conditions hold with this c_1 for every 2-neargeodesic in G and in G'. Since $\infty \in \operatorname{clus}(f, \infty)$, we can choose $y \in G$ such that $|y| \ge 2$ and $|fy| \ge |fa|$. By 3.3 we can choose a 2-neargeodesic α' in G' joining fa and fy. Let z be a point of α' with minimal norm. Then

$$|z - fa| \wedge |z - fy| \ge |fa| - |z|.$$

Since $\operatorname{cig}_l(\alpha', c_1) \subset G'$, we obtain

$$c_1 \delta'(z) \ge |fa| - |z|.$$

Since (7.4) gives

$$\delta'(z) \le \delta'(0) + |z| \le 8M + |z|,$$

this implies

$$2c_1|z| \ge |fa| - 8c_1 M.$$

We may assume that $|fa| \ge 16 c_1 M$, since otherwise there is nothing to prove. Then $2c_1|z| \ge |fa|/2$, and thus

$$|fa| \le 4c_1 d(0, \alpha'). \tag{7.6}$$

Choose a 2-neargeodesic β' in G' joining 0 and fb. By 4.15, the arcs $\alpha = f^{-1}\alpha'$ and $\beta = f^{-1}\beta'$ are (c_2, h)-solid in G with $h = h(M, C)$. Hence, by 6.22, there is $c_3 \ge 1$ with

$$\operatorname{cig}_d(\alpha, c_3) \cup \operatorname{cig}_d(\beta, c_3) \subset G.$$

Choose points $a_0 \in \alpha$ and $b_0 \in \beta$ such that $|a_0| = 3/2$ and $|b_0| = 1/2$. Then

$$\delta(a_0) \wedge \delta(b_0) \ge 1/2c_3. \tag{7.7}$$

Applying once more 3.3 we join a_0 and b_0 with a 2-neargeodesic γ in G. Then $\mathrm{cig}_l(\gamma, c_1) \subset G$ and

$$l(\gamma) \le c_1 |a_0 - b_0| \le 2c_1.$$

For every $x \in \gamma$ we have $\delta(x) \ge 1/4c_1c_3$. Indeed, if $|x - a_0| \wedge |x - b_0| < 1/4c_3$, this follows from (7.7), and otherwise from the condition $\mathrm{cig}_l(\gamma, c_1) \subset G$. Hence

$$l_k(\gamma) \le 4c_1c_3 l(\gamma) \le 8c_1^2 c_3 = c_4.$$

This implies $k(a_0, b_0) \le c_4$ and hence

$$k'(fa_0, fb_0) \le Mc_4 + C = c_5.$$

Next observe that

$$|fb_0| \le l(\beta') \le c_1 |fb| = c_1. \tag{7.8}$$

By (7.4) this implies

$$\delta'(fb_0) \le \delta'(0) + |fb_0| \le 8M + c_1 = c_6.$$

By I.2.2 these estimates yield

$$|fa_0 - fb_0| \le \delta'(fb_0) e^{k'(fa_0, fb_0)} \le c_6 e^{c_5} = c_7.$$

Together with (7.6) and (7.8) this gives the desired bound

$$|fa| \le 4c_1 |fa_0| \le 4c_1(c_1 + c_7) = H(v). \ \square$$

7.9. Theorem. *Suppose that $G \ne E$ and $G' \ne E'$ are c-uniform domains and that $f\colon G \to G'$ is (M, C)-CQH. Then f extends to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$, and $\overline{f}$ is θ-QM rel ∂G with θ depending only on (M, C, c). In particular, $\overline{f} \mid \partial G$ is θ-QM.*

Proof. In the first part of the theorem, it suffices to show that f has a limit at every point $x_0 \in \partial G$. Indeed, then f has a continuous extension $\overline{f}\colon \overline{G} \to \overline{G}'$. By symmetry, $g = f^{-1}$ extends to a continuous map $\overline{g}\colon \overline{G}' \to \overline{G}$. Then clearly $\overline{g}\overline{f}$ and $\overline{f}\overline{g}$ are identity maps, and thus $\overline{f}$ is a homeomorphism.

Performing an auxiliary inversion and recalling 6.26 and I.4.9, we may assume that $x_0 \ne \infty$. Suppose that f has no limit at x_0. By auxiliary similarities we may assume that $0 \in G$, $f(0) = 0$, and $\delta(0) = 1 = \delta'(0)$. For $r > 0$ we set

$$D(r) = f\big[G \cap B(x_0, r)\big].$$

There is $\alpha > 0$ such that $d\big(D(r)\big) > \alpha$ for all $r > 0$, since otherwise f has a limit at x_0 by the completeness of E'. Since ∞ is not a limit, there is $R \geq \alpha$ such that the ball $B(R)$ meets $D(r)$ for all $r > 0$.

Let $r > 0$ and choose points $x, y \in G \cap B(x_0, r)$ such that $|fx| < R$ and $|fx - fy| > \alpha/2$. Join x and y by a 2-neargeodesic γ in G. By 4.15, $f\gamma$ is (c_1, h)-solid in G' with (c_1, h) depending only on (M, C). In what follows, we let $c_2, c_3, \ldots$ denote constants depending only on (M, C, c, α, R). We have $\mathrm{cig}_d(f\gamma, c_2) \subset G'$ by 6.22. Choose $z \in \gamma$ with $|fz - fx| = \alpha/4$. Then $|fz - fy| \geq \alpha/4$, and hence

$$\delta'(fz) \geq \alpha/4c_2. \tag{7.10}$$

Join 0 and fz by a 2-neargeodesic β' in G'. Then 6.19 gives c_3 with

$$\mathrm{cig}_l(\beta', c_3) \subset G', \qquad l(\beta') \leq c_3|fz|.$$

Since $\delta'(0) = 1$, this and (7.10) give a lower bound $\delta'(w) \geq 1/c_4$ for all $w \in \beta'$. Hence

$$k'(0, fz) \leq l_k(\beta') \leq c_4 l(\beta') \leq c_4 c_3 |fz|.$$

Here

$$|fz| \leq |fz - fx| + |fx| \leq \alpha/4 + R < 2R.$$

Since f is (M, C)-CQH, we obtain

$$k(0, z) \leq 2MRc_4c_3 + C = c_5. \tag{7.11}$$

On the other hand, 6.19 gives

$$l(\gamma) \leq c_3|x - y| \leq 2c_3 r.$$

Hence

$$\delta(z) \leq |z - x_0| \leq |z - x| + |x - x_0| \leq l(\gamma) + r \leq (2c_3 + 1)r = c_6 r.$$

By I.2.2 this implies

$$k(0, z) \geq \ln \frac{\delta(0)}{\delta(z)} \geq \ln \frac{1}{c_6 r}.$$

In view of (7.11), this gives a contradiction for small r. Hence f has a homeomorphic extension $\overline{f}\colon \overline{G} \to \overline{G}'$.

To prove the second part of the theorem, let $Q = (a, b, c, d)$ be a quadruple in $\overline{G}$ with $a, d \in \partial G$. Since f^{-1} satisfies the same conditions as f, it follows from 5.8 that it suffices to find an estimate

$$\tau(fQ) \leq \eta\big(\tau(Q)\big)$$

for some increasing $\eta\colon [0,\infty) \to [0,\infty)$ depending only on $v = (M, C, c)$. Performing auxiliary inversions we may assume that $d = \infty$ and $\overline{f}d = \infty$. Choose sequences $(a_n), (b_n), (c_n)$ in G converging to a, b, c, respectively, such that the points a_n, b_n, c_n are distinct for each n. Set $T_n = (a_n, b_n, c_n)$. Since $\rho(T_n) = |a_n - b_n|/|a_n - c_n| \to \tau(Q)$ and $\rho(fT_n) \to \tau(fQ)$, it suffices to find an estimate

$$\rho(fT_n) \le \eta\big(\rho(T_n)\big) \tag{7.12}$$

with $\eta = \eta_v$. Setting $\lambda = 2C \vee 1/2$ and observing that $\delta(a_n) \to 0$ we can assume that

$$|c_n - a_n| \ge e^{\lambda}\delta(a_n) \tag{7.13}$$

for each n.

Fix n and choose an arc γ joining a_n and b_n in G with $l(\gamma) \le c|a_n - b_n|$. Orient γ so that a_n is the first point. Set $y_0 = a_n$, and let y_1 be the last point of γ with $|y_1 - y_0| \le |c_n - y_0|$. Proceeding inductively, we let y_{j+1} be the last point of γ with $|y_{j+1} - y_j| \le |y_j - y_0|$, and we stop as soon as we obtain y_s with $y_s = b_n$. The process is finite, since γ is compact and since $|y_j - y_{j-1}| \ge |c_n - y_0|$ for all $j \le s - 1$. Assume that $s \ge 2$. For $1 \le j \le s-1$ we have

$$|y_j - a_n| \ge |y_1 - a_n| = |c_n - a_n|.$$

By (7.13) and by I.2.2, this implies that $k(y_j, a_n) \ge \lambda$ and $k(c_n, a_n) \ge \lambda$. By the Fundamental lemma 7.3, there is $H = H(v) \ge 1$ such that

$$|fy_1 - fa_n| \le H|fc_n - fa_n|$$

and

$$|fy_{j+1} - fy_j| \le H|fy_j - fa_n|$$

for $1 \le j \le s-1$. These inequalities imply

$$|fy_{j+1} - fa_n| \le (1+H)|fy_j - fa_n| \le (1+H)^j|fy_1 - fa_n| \le H(1+H)^j|fc_n - fa_n|$$

for $0 \le j \le s-1$, and hence

$$|fb_n - fa_n| \le H(1+H)^{s-1}|fc_n - fa_n|.$$

Clearly this is also true if $s = 1$. Since

$$c|a_n - b_n| \ge l(\gamma) \ge \sum_{j=1}^{s} |y_j - y_{j-1}| \ge (s-1)|c_n - a_n|,$$

we have $s - 1 \le c\rho(T_n)$. Hence (7.12) is true with $\eta(t) = H(1+H)^{ct}$. □

7.14. Theorem. *Suppose that G and G' are bounded c-uniform domains and that $c_0 \geq 1$, $x_0 \in G$, $x_0' \in G'$ with $d(G) \leq c_0\delta(x_0)$, $d(G') \leq c_0\delta'(x_0')$. Suppose also that $f\colon G \to G'$ is an (M,C)-CQH map with $fx_0 = x_0'$. Then f extends to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$, and $\overline{f}$ is η-QS rel ∂G with η depending only on (M,C,c,c_0).*

Proof. This follows directly from 7.9 and 5.11. □

7.15. Theorem. *Suppose that $f\colon B(1) \to B(1)$ is an (M,C)-CQH map between the unit balls of E and E', and that $f(0)=0$. Then f extends to a homeomorphism $\overline{f}\colon \overline{B}(1) \to \overline{B}(1)$, and $\overline{f}$ is η-QS rel $\partial B(1)$ with η depending only on (M,C).*

Proof. Since a ball is c-uniform with a universal c by 6.5, the theorem is a corollary of 7.14. □

7.16. Theorem. *Suppose that G and G' are c-uniform domains and that $f\colon G \to G'$ is φ-FQC. Then f is η-QM with η depending only on φ and c.*

Proof. If $G = E$ or $G' = E'$, the result follows from I.5.13. Suppose that $G \neq E$, $G' \neq E'$. By 7.9, f has a homeomorphic extension $\overline{f}\colon \overline{G} \to \overline{G}'$. In view of 6.26, we can use auxiliary inversions to normalize the situation so that $\infty \in \partial G$ and $\overline{f}(\infty) = \infty$. We show that f is η-QS with $\eta = \eta_{\varphi,c}$. Now G and G' are c-quasiconvex. By I.5.5, it suffices to show that f is weakly H-QS with $H = H(\varphi,c)$.

Let $x,a,b \in G$ with $|a-x| \leq |b-x|$. By 4.14, f is $(M,1/4)$-CQH with M depending only on φ. By 7.3, the desired inequality

$$|fa - fx| \leq H|fb - fx| \tag{7.17}$$

holds with $H = H(\varphi,c)$ provided that $k(b,x) \geq 1/2$. Suppose that $k(b,x) < 1/2$. Then I.2.2 gives

$$|b-x| \leq \lambda\delta(x), \qquad \lambda = e^{1/2} - 1 < 1.$$

Hence a and b are in the ball $B = B\big(x, \lambda\delta(x)\big)$. By I.5.10, $f \mid B$ is η-QS with $\eta = \eta_\varphi$. Thus (7.17) holds with $H = \eta(1)$. □

7.18. Theorem. *Suppose that G is a c-uniform domain and that $f\colon G \to G'$ is φ-FQC. Then the following conditions are quantitatively equivalent:*
(1) *G' is c_1-uniform,*
(2) *f is η-quasimöbius.*

Proof. This follows from 7.16 and 6.26. □

7.19. *Question.* Does 7.18 remain true if φ-FQC is replaced by (M,C)-CQH and η-QM by η-QM rel ∂G?

7.20. *Remarks.* In Theorem 7.9 we showed that for maps between uniform domains, CQH quantitatively implies QM rel ∂G. The uniformity condition can hardly be weakened. For example, conformal maps between non-uniform planar domains may have rather bad boundary behavior.

We next turn to the converse of 7.9 and show in 7.22 that for maps between uniform domains, QM rel ∂G quantitatively implies CQH. Here the uniformity plays a less important role. In fact, we shall prove the result for domains which are only QH ψ-uniform. Recall from 6.8 that G is QH ψ-uniform if $\psi\colon [0,\infty) \to [0,\infty)$ is a homeomorphism and if

$$k_G(a,b) \le \psi\big(r_G(a,b)\big)$$

for all $a,b \in G$; here $r_G(a,b) = |a-b|/\big(\delta(x) \wedge \delta(y)\big)$ is the relative distance between a and b in G. For example, all convex domains are QH ψ-uniform with $\psi(t) = t$.

In [Vä$_5$] we shall prove the result for a larger class of domains including all domains in R^n.

7.21. Lemma. *Suppose that $0 \notin G$ and that G is* QH *ψ-uniform. Let u be the inversion $u(x) = x/|x|^2$. Then uG is* QH *ψ_1-uniform with ψ_1 depending only on ψ.*

Proof. Let $a,b \in G$ and set $r = r_G(a,b)$, $r' = r_{G'}(ua,ub)$, where $G' = uG$. Applying Case 1 of the proof of 6.25 to the inverse map $G' \to G$ we obtain $r \le 18(2r' + r'^2)$. By I.4.9, u is fully 36-QH. Thus

$$k'(ua,ub) \le 36k(a,b) \le 36\psi(r) \le 36\psi\big(18(2r'+r'^2)\big) = \psi_1(r'). \ \square$$

7.22. Theorem. *Suppose that G and G' are* QH *ψ-uniform domains and that $f\colon \overline{G} \to \overline{G}'$ is a homeomorphism with $fG = G'$ such that f is η-*QM rel ∂G*. Then $f \mid G$ is (M,C)-*CQH *with (M,C) depending only on η and ψ.*

Proof. Performing auxiliary similarities we may assume that $0 \in \partial G$ and that $f(0)$ is either 0 or ∞. In view of 7.21, we can use auxiliary inversions to normalize the map so that $\infty \in \partial G$ and $f(\infty) = \infty$. Then f is η-QS rel ∂G.

Suppose that $a,b \in G$ with $k(a,b) \le 1$. By 4.8, it suffices to find an estimate

$$k'(fa,fb) \le M_0 \tag{7.23}$$

with $M_0 = M_0(\eta,\psi)$. We may assume that $\delta'(fb) \le \delta'(fa)$. Choose $x \in \partial G$ with $|fx - fb| \le 2\delta'(fb)$. Then

$$r_{G'}(fa,fb) = \frac{|fa-fb|}{\delta'(fb)} \le 2\frac{|fa-fx|}{|fb-fx|} + 2 \le 2\eta\Big(\frac{|a-x|}{|b-x|}\Big) + 2. \tag{7.24}$$

By I.2.2 we have

$$\frac{|a-b|}{\delta(b)} \le e^{k(a,b)} \le e.$$

Hence

$$\frac{|a-x|}{|b-x|} \le \frac{|a-b|+|b-x|}{|b-x|} \le \frac{|a-b|}{\delta(b)} + 1 \le e+1 < 4.$$

By (7.24) this implies

$$r_{G'}(fa, fb) \le 2\eta(4) + 2,$$

which gives (7.23) with $M_0 = \psi(2\eta(4)+2)$. □

7.25. *Extension to a half space.* For a Banach space E, we consider the space $E_1 = E \times R^1$ as a Banach space with the norm $|(x,t)| = |x| \vee |t|$, and we identify E with the subspace $E \times \{0\}$ of E_1. Let H be the half space $E \times (0,\infty)$. Then $\partial H = \dot{E} = E \cup \{\infty\}$. For another Banach space E' we similarly define E_1' and H'. The half spaces H and H' are c_0-uniform domains with a universal c_0 by 6.6. Suppose that $F\colon H \to H'$ is φ-solid. Then F extends to a homeomorphism $\overline{F}\colon \overline{H} \to \overline{H}'$ and induces an η-QM homeomorphism $f\colon \dot{E} \to \dot{E}'$ with $\eta = \eta_\varphi$ by 7.9.

We shall next show that conversely, every η-QM homeomorphism $f\colon \dot{E} \to \dot{E}'$ can be extended to a homeomorphism $\overline{F}\colon \overline{H} \to \overline{H}'$ such that the induced map $F\colon H \to H$ is φ-solid with $\varphi = \varphi_\eta$. In the case $E = R^n = E'$, this was proved by De-Spiller [DS] in 1970. In [TV$_2$] we proved the stronger result that F can be chosen to be QH and hence QC in $H^{n+1} = R^n \times (0,\infty)$. We do not know whether this is true in the general case.

The construction of F in the proof of 7.26 was used as a preliminary step in [TV$_2$], and the solidity of F in the euclidean case can be proved by compact families of embeddings; see [TV$_3$, 6.17] and [TV$_2$, 2.13]. We shall give an elementary but somewhat lengthy direct proof.

7.26. Theorem. *Suppose that $f\colon \dot{E} \to \dot{E}'$ is η-QM. Then, with the notation of 7.25, there is an extension of f to a homeomorphism $\overline{F}\colon \overline{H} \to \overline{H}'$ such that $F\colon H \to H'$ is φ-solid with $\varphi = \varphi_\eta$.*

Proof. By an auxiliary inversion we may assume that $f(\infty) = \infty$. Then f is η-QS in E. For $x \in E$ and $t \ge 0$ we set

$$\begin{aligned} \tau(x,t) &= \sup\{|fy - fx| : |y-x| \le t\}, \\ \overline{F}(x,t) &= (fx, \tau(x,t)). \end{aligned} \tag{7.27}$$

Setting $\overline{F}(\infty) = \infty$ we obtain an extension $\overline{F}\colon \overline{H} \to \overline{H}'$ of f and its restriction $F\colon H \to H'$. We show that $\overline{F}$ is the desired map.

We first show that there are $s_0 = s_0(\eta) \in (0, 1/2]$ and embeddings μ, θ: $[0, s_0] \to [0, \infty)$ with $\mu(0) = \theta(0) = 0$ depending only on η, with the following property: Let $a_0 = (x_0, t_0)$ and $a = (x, t)$ be points in H with $|a - a_0| = st_0$, $0 \le s \le s_0$. Then

$$\mu(s) \le \frac{|Fa - Fa_0|}{\tau(a_0)} \le \theta(s). \tag{7.28}$$

Suppose that $a_0, a \in H$ are as above, $|a - a_0| = st_0$, $0 < s \le 1/2$. Write $\tau_0 = \tau(a_0)$, $\tau = \tau(a)$. For $x, x_0, x_1, \dots$ we write $y = fx$, $y_0 = fx_0$, etc. Let x_1 be an arbitrary point of $S(x, t) \subset E$. Choose $x_2 \in S(x_0, t_0)$ such that x_2 lies on the ray from x_0 through x_1. Then either

$$|x_1 - x_2| = |x_2 - x_0| - |x_1 - x_0| \le t_0 - |x_1 - x| + |x - x_0| \le t_0 - t + |x - x_0|$$

or

$$|x_1 - x_2| = |x_1 - x_0| - |x_2 - x_0| \le |x_1 - x| + |x - x_0| - t_0 \le t - t_0 + |x - x_0|.$$

Hence in both cases

$$|x_1 - x_2| \le |t - t_0| + |x - x_0| \le 2st_0.$$

By quasisymmetry we obtain

$$\begin{aligned} |y_1 - y| &\le |y_1 - y_2| + |y_2 - y_0| + |y_0 - y| \\ &\le \eta\Big(\frac{|x_1 - x_2|}{|x_0 - x_2|}\Big)|y_0 - y_2| + \tau_0 + \eta\Big(\frac{|x_0 - x|}{|x_0 - x_2|}\Big)|y_0 - y_2| \\ &\le \eta(2s)\tau_0 + \tau_0 + \eta(s)\tau_0 \le \tau_0 + 2\eta(2s)\tau_0. \end{aligned}$$

Since $x_1 \in S(x, t)$ is arbitrary, this implies

$$\frac{\tau - \tau_0}{\tau_0} \le 2\eta(2s). \tag{7.29}$$

This inequality holds for all pairs a_0, a with $|a - a_0| = st_0$. Changing the roles of a and a_0 gives

$$\frac{\tau_0 - \tau}{\tau} \le 2\eta(2st_0/t). \tag{7.30}$$

Now $s \le 1/2$ implies $|t - t_0| \le t_0/2$, and hence $t_0 \le 2t$. From (7.29) and (7.30) we obtain

$$\tau_0 - \tau \le 2\eta(4s)\big(1 + 2\eta(2s)\big)\tau_0 = \theta(s)\tau_0,$$

where $\theta(s) = 2\eta(4s)\big(1 + 2\eta(2s)\big)$. Hence

$$\frac{|\tau - \tau_0|}{\tau_0} \le \theta(s) \tag{7.31}$$

whenever $s \le 1/2$.

On the other hand, since $|x - x_0| \le |a - a_0| = st_0$, we have

$$|fx - fx_0| \le \eta\Big(\frac{|x - x_0|}{|x_2 - x_0|}\Big)|fx_2 - fx_0| \le \eta(s)\tau_0 \le \theta(s)\tau_0.$$

Hence the second inequality of (7.28) is true with this θ and with $s_0 = 1/2$.

We turn to the first inequality of (7.28). In what follows, we let $\mu_j\colon [0,\infty) \to [0,\infty)$ denote homeomorphisms depending only on η. Using the same notation as above we assume that $a_0, a \in H$ with $|a - a_0| = st_0$, $0 < s \le 1/2$. We may assume that f and f^{-1} are η-QS, replacing $\eta(t)$ by $\eta(t) \vee \eta^{-1}(t^{-1})^{-1}$. Define μ_1 and μ_2 by

$$\mu_1(s) = \frac{1}{2\eta(2/s)}, \qquad \mu_1(0) = 0, \qquad \mu_2(s) = \frac{\eta^{-1}(\mu_1(s))}{2} \wedge \frac{s}{4}.$$

We consider two cases.

Case 1. $|x - x_0| \ge \mu_2(s)t_0$. Let x_4 be an arbitrary point of $S(x_0, t_0)$. Then

$$\frac{|y_4 - y_0|}{|y - y_0|} \le \eta\Big(\frac{|x_4 - x_0|}{|x - x_0|}\Big) \le \eta\big(1/\mu_2(s)\big),$$

and hence

$$\frac{|y - y_0|}{\tau_0} \ge \frac{1}{\eta\big(1/\mu_2(s)\big)} = \mu_3(s).$$

Since $|Fa - Fa_0| \ge |y - y_0|$, this gives the first inequality of (7.28).

Case 2. $|x - x_0| \le \mu_2(s)t_0$. Since

$$st_0 = |a - a_0| \le |x - x_0| + |t - t_0| \le \mu_2(s)t_0 + |t - t_0|,$$

one of the inequalities

$$t \ge t_0\big(1 + s - \mu_2(s)\big), \tag{7.32}$$

$$t \le t_0\big(1 - s + \mu_2(s)\big) \tag{7.33}$$

is true.

Suppose first that (7.32) holds. Since $\mu_2(s) \le s/4$, we have

$$|x - x_0| + t_0 \le \mu_2(s)t_0 + t - st_0 + \mu_2(s)t_0 < t.$$

Hence $\overline{B}(x_0, t_0) \subset B(x, t)$. Let $x_5 \in S(x_0, t_0)$ be arbitrary. Then y_5 and y belong to $fB(x, t)$. Hence there is $y_6 \in fS(x, t)$ such that $y_5 \in [y, y_6]$. Then

$$\frac{|y - y_0|}{|y_5 - y_0|} \le \eta\Big(\frac{|x - x_0|}{|x_5 - x_0|}\Big) \le \eta\big(\mu_2(s)\big) \le \mu_1(s).$$

Since

$$|x_5 - x_6| \ge |x - x_6| - |x - x_0| - |x_0 - x_5| \ge t - \mu_2(s)t_0 - t_0 \ge \big(s - 2\mu_2(s)\big)t_0 \ge st_0/2,$$

we have

$$\frac{|y_5 - y_0|}{|y_5 - y_6|} \le \eta\Big(\frac{|x_5 - x_0|}{|x_5 - x_6|}\Big) \le \eta\Big(\frac{2}{s}\Big) = \frac{1}{2\mu_1(s)}.$$

Since

$$\tau \ge |y - y_6| = |y - y_5| + |y_5 - y_6| \ge |y_5 - y_0| - |y - y_0| + |y_5 - y_6|,$$

these inequalities give

$$\frac{\tau - |y_5 - y_0|}{|y_5 - y_0|} \ge \frac{|y_5 - y_6|}{|y_5 - y_0|} - \frac{|y - y_0|}{|y_5 - y_0|} \ge \mu_1(s).$$

Since $|y_5 - y_0|$ is arbitrarily close to τ_0, we obtain the desired lower bound

$$\frac{|Fa - Fa_0|}{\tau_0} \ge \frac{\tau - \tau_0}{\tau_0} \ge \mu_1(s).$$

Finally assume that (7.33) is true. Since

$$|x - x_0| + t \le \mu_2(s)t_0 + t_0\big(1 - s + \mu_2(s)\big) \le t_0(1 - s + s/2) < t_0,$$

we have $\overline{B}(x, t) \subset B(x_0, t_0)$. Let $x_7 \in S(x, t)$ be arbitrary. Since $y_0, y_7 \in fB(x_0, t_0)$, there is $y_8 \in fS(x_0, t_0)$ such that $y_7 \in [y_0, y_8]$. Since

$$t \ge t_0 - |t - t_0| \ge t_0 - st_0 \ge t_0/2,$$

we have

$$\frac{|y - y_0|}{|y - y_7|} \le \eta\Big(\frac{|x - x_0|}{|x - x_7|}\Big) \le \eta\Big(\frac{\mu_2(s)t_0}{t}\Big) \le \eta\big(2\mu_2(s)\big) \le \mu_1(s).$$

By (7.33) and the inequality $\mu_2(s) \le s/4$ we have $t \le (1 - 3s/4)t_0 < t_0$. Hence

$$|x_8 - x_7| \ge |x_8 - x_0| - |x_0 - x| - |x - x_7| \ge t_0 - \mu_2(s)t_0 - t \ge st_0/2 > st/2.$$

Consequently,

$$\frac{|y - y_7|}{|y_8 - y_7|} \le \eta\Big(\frac{|x - x_7|}{|x_8 - x_7|}\Big) \le \eta\Big(\frac{2}{s}\Big) = \frac{1}{2\mu_1(s)}.$$

Since

$$\tau_0 \ge |y_8 - y_0| = |y_7 - y_0| + |y_8 - y_7| \ge |y_7 - y| - |y - y_0| + |y_8 - y_7|,$$

these inequalities yield

$$\frac{\tau_0 - |y_7 - y|}{|y_7 - y|} \ge \frac{|y_8 - y_7|}{|y_7 - y|} - \frac{|y - y_0|}{|y_7 - y|} \ge \mu_1(s),$$

and hence

$$\frac{\tau_0 - \tau}{\tau} \ge \mu_1(s).$$

Choose $s_0 \in (0, 1/2]$ such that $\theta(s_0) \le 1/2$. If $s \le s_0$, then (7.31) implies that $\tau \ge \tau_0/2$. Hence

$$\frac{|Fa - Fa_0|}{\tau_0} \ge \frac{\tau_0 - \tau}{2\tau} \ge \mu_1(s)/2.$$

We have now proved (7.28) for all $s \le s_0$.

From (7.28) it follows that F is continuous in H. If $|x - x_0| \le t$, then $\tau(x,t) \le \tau(x_0, 2t)$. Hence $\overline{F}$ is continuous in $\overline{H}$. Furthermore, (7.28) implies that $\overline{F}$ maps each ray $\{x_0\} \times [0, \infty)$ homeomorphically onto the ray $\{fx_0\} \times [0, \infty)$. From Lemma 7.34 below it follows that $\overline{F}$ is a homeomorphism.

By (7.28), the homeomorphism $F\colon H \to H'$ is (θ, s_0)-relative in the sense of I.3.6. If $a_0, a \in H$ and $|a - a_0| = st_0$ with $s \le s_0$, then

$$|Fa - Fa_0| \ge \mu(s)\tau_0.$$

Since F is a homeomorphism, this implies that $FB(a_0, st_0)$ contains the ball $B\big(Fa_0, \mu(s)\tau_0\big)$. Hence F^{-1} is (θ_1, s_1)-relative with $s_1 = \mu(s_0)$, $\theta_1 = \mu^{-1}$. From I.3.8 it follows that F is φ-solid with $\varphi = \varphi_\eta$. □

7.34. Lemma. *Suppose that X and Y are topological spaces and that $F\colon X \times [0, \infty) \to Y \times [0, \infty)$ is a continuous bijective map of the form $F(x,t) = \big(fx, \tau(x,t)\big)$ where $f\colon X \to Y$ is a homeomorphism. Then F is a homeomorphism.*

Proof. Let $a_0 = (x_0, t_0) \in X \times [0, \infty)$. We must show that F is open at x_0. Suppose first that $t_0 > 0$. Let $W = U \times V$ be a neighborhood of a_0, where U is open in X and $t_0 \in V = (t_1, t_2) \subset [0, \infty)$. The map $t \mapsto \tau(x_0, t)$ is a self homeomorphism of $[0, \infty)$. Choose numbers u_1, u_2 with

$$\tau(x_0, t_1) < u_1 < \tau(x_0, t_0) < u_2 < \tau(x_0, t_2).$$

Since τ is continuous, there is a neighborhood $U_1 \subset U$ of x_0 such that $x \in U_1$ implies $\tau(x, t_1) < u_1$ and $\tau(x, t_2) > u_2$. Since F maps the vertical segment $\{x\} \times (t_1, t_2)$ onto $\{fx\} \times \big(\tau(x, t_1), \tau(x, t_2)\big)$, the set FW contains the neighborhood $fU_1 \times (u_1, u_2)$ of Fa_0.

The case $t_0 = 0$ is proved by an obvious modification of the argument above. □

7.35. *Reflection.* The reflection principle in the QS theory of R^n enables us to extend a K-QC map between balls or half spaces to a K-QC map of the whole space $\dot{R}^n$. We shall next present a free version of this idea. In fact, it turns out that the principle follows easily from a general result on unions of QS maps.

Suppose first that G and G' are unit balls of E and E', respectively, and that $f\colon G \to G'$ is a φ-FQC map with $f(0) = 0$. Then f is η-QM with $\eta = \eta(\varphi)$ by 6.5 and 7.16. In particular, f extends to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$. Let u and u' be the inversions $x \mapsto x/|x|^2$ of $\dot{E}$ and $\dot{E}'$, respectively. Then we can extend $\overline{f}$ to a homeomorphism $f^*\colon \dot{E} \to \dot{E}'$ by setting $f^*x = u'fux$ for $x \in \dot{E} \setminus \overline{G}$. We say that f^* is obtained from f by *reflection.*

Next assume that G is a half space in E with $0 \in \partial G$. This means that $\partial G \setminus \{\infty\}$ is a closed linear subspace T of E of codimension 1. Let $e \in G$ be a unit vector. Then E is spanned by $T \cup \{e\}$, and there is a unique linear map $v\colon E \to E$ such that $v \mid T = \mathrm{id}$ and $ve = -e$. We say that v is a *reflection* in T. If E is Hilbert space, we can choose e to be orthogonal to T, and then v is an isometry. In a Banach space, an isometric reflection does not always exist, but for every $M > 3$, there is an M-bilipschitz reflection; see Lemma 7.37 below. In what follows, we assume that the reflections are 4-bilipschitz.

Suppose that G' is another half space with $0 \in T' = \partial G' \setminus \{\infty\}$. Let $f\colon G \to G'$ be φ-FQC. Again by uniformity, f is η-QM, $\eta = \eta(\varphi)$, and extends to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}'$. Suppose that $\overline{f}(\infty) = \infty$. Using reflections v and v' in T and in T', respectively, we extend $\overline{f}$ to a homeomorphism $f^*\colon \dot{E} \to \dot{E}'$ with $f^*x = v'fvx$ for $x \in E \setminus \overline{G}$. We again say that f^* is obtained from f by reflection.

In both cases we have $f^*(\infty) = \infty$. Hence f^* defines a homeomorphism $f_*\colon E \to E'$. We want to show that this map is φ_1-FQC with $\varphi_1 = \varphi_1(\varphi)$. Since the inversions u, u' and the reflections v, v' are η_0-QM with a universal η_0, the maps $f^* \mid \overline{G}$ and $f^* \mid E \setminus G$ are η_1-QM with $\eta_1 = \eta_1(\varphi)$. Since $f^*(\infty) = \infty$, $f^* \mid \overline{G} \setminus \{\infty\}$ is η_1-QS in the second case, and $f^* \mid E \setminus G$ is η_1-QS in both cases.

From Lemma 7.38 below it follows that in the first case, $f_* \mid \overline{G}$ is η_2-QS with $\eta_2 = \eta_2(\varphi)$. From a general result of QS theory, given as Theorem 7.39 below, it follows that f_* is η_3-QS with $\eta_3 = \eta_3(\varphi)$. This implies that f_* is φ_1-FQC with $\varphi_1 = \varphi_1(\varphi)$ by I.5.18, and we obtain the following result:

7.36. Theorem. (Reflection principle). *Suppose that $f_*\colon E \to E'$ is obtained from a φ-FQC map $f\colon G \to G'$ by reflection. Then f_* is φ_1-FQC with $\varphi_1 = \varphi_1(\varphi)$.*

7.37. Lemma. *Suppose that $H \subset E$ is a half space and that $M > 3$. Then there is an M-bilipschitz reflection of E in ∂H.*

Proof. Setting $r = 2/(M-1)$ we have $0 < r < 1$. Fix an arbitrary $h_0 \in H$. Write $T = \partial H$ and choose $a \in T$ such that $|h_0 - a| < d(h_0, T)/r$. Write $h_1 = h_0 - a$ and $e = h_1/|h_1|$. Then $e \in H$, $|e| = 1$, and

$$d(e,T) = d(h_1,T)/|h_1| = d(h_0,T)/|h_1| > r.$$

Let v be the reflection in T with $ve = -e$. We show that the norm $|v|$ of v is at most M.

Let $x \in E$ be a unit vector. We can write $x = y + te$ with $y \in T$, $t \in R^1$. Then

$$1 = |x| \ge d(x,T) = d(te,T) = |t|d(e,T) \ge |t|r.$$

Hence

$$|vx| = |x - 2te| \le |x| + 2|t| \le 1 + 2/r = M,$$

which implies that $|v| \le M$. Since $v^{-1} = v$, the map v is M-bilipschitz. □

7.38. Lemma. *Suppose that G and G' are the unit balls of E and E', respectively, and that $f\colon \overline{G} \to \overline{G}'$ is an η-QM homeomorphism with $f(0) = 0$ and $f\partial G = \partial G'$. Then f is η_1-QS with $\eta_1 = \eta_1(\eta)$.*

Proof. Fix $z_1 \in \partial G$ and set $z_2 = -z_1$ and $z_3 = 0$. By [Vä$_1$, 3.12] or by 5.9, it suffices to show that $|fz_1 - fz_2| \ge \lambda$ for some $\lambda = \lambda(\eta) > 0$. Setting $z_4 = f^{-1}(-fz_1)$ we have

$$\frac{|z_1 - z_4||z_2 - 0|}{|z_1 - z_2||z_4 - 0|} \le 1, \qquad \frac{|fz_1 - fz_4||fz_2 - 0|}{|fz_1 - fz_2||fz_4 - 0|} = \frac{2}{|fz_1 - fz_2|},$$

and hence $|fz_1 - fz_2| \ge 2/\eta(1)$. □

7.39. Theorem. *Suppose that $f\colon E \to E'$ is a homeomorphism and that $E = A_1 \cup A_2$ such that $f \mid A_1$ and $f \mid A_2$ are η-QS. Then f is η_1-QS with $\eta_1 = \eta_1(\eta)$.*

Proof. Replacing the sets A_1, A_2 with their closures, we may assume that they are closed in E. By I.5.5, it suffices to show that f is weakly H-QS with $H = H(\eta)$. Suppose that x, a, b are distinct points in E with $|a-x| = |b-x|$. In the present situation, it suffices to show that $|fa - fx| \le H(\eta)|fb - fx|$. We may normalize the situation so that $x = 0$, $fx = 0$, and $|a| = |b| = 1 = |fb|$. Then we must find an upper bound $|fa| \le H(\eta)$. We may assume that $0 \in A_1$ and that $\{a, b\} \not\subset A_1$. We consider three cases.

Case 1. $a \in A_1$, $b \in A_2$. Choose $x_1 \in A_1 \cap A_2$ with $fx_1 \in [0, fb]$. If $|x_1| \ge 1/2$, we have

$$|fa| \le \eta\Big(\frac{|a|}{|x_1|}\Big)|fx_1| \le \eta(2).$$

Suppose that $|x_1| \le 1/2$. Since $S(1)$ is connected, there is $x_2 \in S(1) \cap A_1 \cap A_2$. Then

$$|fx_2 - fx_1| \le \frac{|fx_2 - fx_1|}{|fb - fx_1|} \le \eta\Big(\frac{|x_2 - x_1|}{|b - x_1|}\Big) \le \eta(3),$$

and hence

$$|fa| \le \eta\Big(\frac{|a|}{|x_2|}\Big)|fx_2| \le \eta(1)\big(1 + \eta(3)\big).$$

Case 2. $a \in A_2$, $b \in A_1$. Choose $x_2 \in S(1) \cap A_1 \cap A_2$ as above and a point $x_3 \in [0, a] \cap A_1 \cap A_2$. Then

$$|fa - fx_3| \le \eta\Big(\frac{|a - x_3|}{|x_2 - x_3|}\Big)\eta\Big(\frac{|x_2 - x_3|}{|x_2|}\Big)\eta\Big(\frac{|x_2|}{|b|}\Big)|fb| \le \eta(1)\eta(2)\eta(1),$$

$$|fx_3| \le \eta\Big(\frac{|x_3|}{|b|}\Big)|fb| \le \eta(1),$$

and hence $|fa| \le \eta(1) + \eta(1)^2\eta(2)$.

Case 3. $a \in A_2$, $b \in A_2$. We again choose $x_1 \in A_1 \cap A_2$ with $fx_1 \in [0, fb]$. If $|x_1| \le 1/2$, then

$$|fa - fx_1| \le \eta\Big(\frac{|a - x_1|}{|b - x_1|}\Big)|fb - fx_1| \le \eta(3),$$

and hence $|fa| \le 1 + \eta(3)$. Suppose that $|x_1| \ge 1/2$. Choose again $x_3 \in [0, a] \cap A_1 \cap A_2$. Then

$$|fx_3| \le \eta\Big(\frac{|x_3|}{|x_1|}\Big)|fx_1| \le \eta(2),$$

$$|fa - fx_3| \le \eta\Big(\frac{|a - x_3|}{|b - x_3|}\Big)|fb - fx_3| \le \eta(1)\big(1 + |fx_3|\big),$$

and hence $|fa| \le \eta(2) + \eta(1)\big(1 + \eta(2)\big)$. □

7.40. *Remark.* The one-dimensional version of 7.39 is false. For example, define $f\colon R^1 \to R^1$ by $fx = x$ for $x \le 0$ and $fx = x^2$ for $x \ge 0$. Then f is not QS although it is QS in $(-\infty, 0]$ and in $[0, \infty)$.

7.41. *Hyperbolic geometry.* Suppose that E is a Hilbert space and that G is a half space of E. Then the QH metric k of G is the *hyperbolic metric* of G. Each pair $a, b \in G$ can be joined by a unique hyperbolic segment, which is a subarc of a semicircle or a ray which joins two points of ∂G and is orthogonal to ∂G. This semicircle or ray is called a *hyperbolic line.* The Möbius maps preserving G are the hyperbolic isometries of G.

Maps related to CQH maps of G have been studied by several people. For example, Thurston [Th, 5.9] considers maps $f\colon G \to G$ called pseudo-isometries. They are defined by the condition

$$\big(k(x,y) - C\big)/M \le k(fx, fy) \le Mk(x,y).$$

Essential use has been made by the fact that the image of a hyperbolic line lies in a hyperbolic neighborhood of another hyperbolic line. We next show that for CQH maps, this result follows easily from the fact (Theorem 7.9) that f extends to a map which is QM rel ∂G. However, I feel that in the applications it is usually easier to make direct use of Theorem 7.9 or the related results 7.14 and 7.15. Moreover, these results also apply to many other domains, in which we do not have well-defined hyperbolic lines.

The hyperbolic metric h of the unit ball $B(1)$ is obtained with the aid of any Möbius map of G onto $B(1)$. Alternatively, h is defined by the density $2/\big(1 - |x|^2\big)$. Then h and the QH metric k of $B(1)$ satisfy the inequalities $k \le h \le 2k$.

7.42. Theorem. *Suppose that G is a half space of a Hilbert space and that $f\colon G \to G$ is (M,C)-CQH. Let γ be a hyperbolic line in G. Then there is a unique hyperbolic line γ' such that $f\gamma$ lies in the hyperbolic neighborhood*

$$N(\gamma', r) = \big\{y \in G : k(y, \gamma') \le r\big\},$$

where r can be chosen to depend only on (M,C).

Proof. Let a and b be the endpoints of γ. By 7.9, f extends to a homeomorphism $\overline{f}\colon \overline{G} \to \overline{G}$. By auxiliary Möbius maps we may assume that $a = 0 = \overline{f}a$ and $b = \infty = \overline{f}b$. Then γ is the ray from 0 to ∞, orthogonal to ∂G. The uniqueness of γ' is clear: we must have $\gamma' = \gamma$. The neighborhood $N(\gamma, r)$ is a cone with axis γ. Let $x \in \gamma$, and let $y \in \partial G$ be the point for which $\overline{f}y$ is the orthogonal projection of fx on ∂G. Let α be the angle between the vector fx and the ray γ. It suffices to obtain an upper bound for $\tan\alpha$ in terms of (M,C).

By 7.9, $\overline{f}$ is η-QM rel ∂G with η depending only on (M, C). Applied to the quadruple $(y, 0, x, \infty)$ this gives

$$\tan \alpha = \frac{|\overline{f}y|}{|\overline{f}y - fx|} \leq \eta\Big(\frac{|y|}{|y - x|}\Big) \leq \eta(1).$$

This proves the theorem. From the formulas in [Vu$_2$, p. 22] we can obtain the explicit bound $r = \ln\big(H + \sqrt{1 + H^2}\big)$ with $H = \eta(1)$. □

8. Distortion

8.1. *Introduction to Section* 8. This section was inspired by a manuscript [Co] of D. Cooper. Let B^n be the open unit ball of R^n, and let $f\colon B^n \to B^n$ be (M, C)-CQH. Cooper considered the distortion of hyperbolic spheres under f. By auxiliary Möbius homeomorphisms of B^n we reduce the problem to the case where $f(0) = 0$ and the center of the sphere is the origin. We can then consider the QH sphere

$$S_k(r) = \partial B_k(r) = \big\{x \in B^n : k(x, 0) = r\big\},$$

where k is the QH metric of B^n. This sphere is also a euclidean sphere $S(s)$ with radius $s = 1 - e^{-r}$. If $r \geq 2C$, then

$$r/2M \leq k(fx, 0) \leq 2Mr \tag{8.2}$$

for all $x \in S_k(r)$. Examples show that for any $r > 0$, $k(fx, 0)$ may have the order of magnitude of Mr or r/M at certain points of $S_k(r)$. Cooper made the important new observation that for large r, this can happen only in a set of small area, provided that $n \geq 3$. Indeed, the *multiplicative* bounds in (8.2) can be replaced by the *additive* bounds

$$r - \alpha \leq k(fx, 0) \leq r + \alpha \tag{8.3}$$

for $x \in S_k(r)$ except for a subset whose area has an upper bound $\varepsilon(\alpha, M, C, n)$ which tends to zero as $\alpha \to \infty$. Observe that ε does not depend on r.

I have not been able to follow all details of Cooper's proof. The purpose of this section is to give a new proof for this result. In fact, we prove in Theorem 8.7 that the first inequality of (8.3) also holds for $n = 2$. To prove the second inequality we need the absolute continuity of the boundary map $S^{n-1} \to S^{n-1}$. This requires $n \geq 3$; the result is given as Theorem 8.9.

Since we are working in R^n, this section differs from the preceding sections. However, our proof is based on Theorem 7.15 on relative quasisymmetry. On the other hand, the result is also new for QC maps of B^n. A reader interested only in this special case can skip the preceding sections, since a K-QC map $f\colon B^n \to B^n$ with $f(0) = 0$ is well known to be η-QS with $\eta = \eta_K$. For example, reflect f to R^n and use [AVV, 5.23].

8.4. *Notation.* Suppose that $f\colon X \to Y$ is a map. For $x \in X$ and $r > 0$ we set $S(x,r) = \partial B(x,r)$ and

$$L(x,f,r) = \sup\{|fy - fx| : y \in S(x,r)\},$$
$$l(x,f,r) = \inf\{|fy - fx| : y \in S(x,r)\}.$$

The n-dimensional outer measure of a set $A \subset S^n = \partial B^{n+1}$ is written as $m_n(A)$. For $x \in S^n$, $0 < r \le 2$, we let $U(x,r)$ be the cap $S^n \cap B(x,r)$. There are positive numbers $a_n < b_n$ such that

$$a_n r^n \le m_n\big(U(x,r)\big) \le b_n r^n$$

for all $x \in S^n$ and $0 < r \le 2$. We also write $\omega_n = m_n(S^n)$.

8.5. Lemma. *Suppose that $n \ge 1$ and that $f\colon S^n \to S^n$ is a homeomorphism. Suppose also that $0 < r \le 2$, that $t > 0$ and that $A \subset S^n$ is such that*

$$l(x,f,r) \ge tr$$

for all $x \in A$. Then $m_n(A) \le \mu(t,n)$ where $\mu(t,n) \to 0$ as $t \to \infty$.

Proof. The proof is based on a simple packing argument. Choose a maximal set $F \subset A$ with the property that $|x-y| \ge 2r$ whenever $x,y \in F$ with $x \ne y$. Write $k = \operatorname{card} F$. Since A is covered by the caps $U(a,2r)$, $a \in F$, we have $m_n(A) \le kb_n(2r)^n$. Hence it suffices to find an estimate

$$kr^n \le \mu(t,n). \tag{8.6}$$

Since

$$fU(a,r) \supset U\big(fa, l(a,f,r)\big) \supset U(fa,tr)$$

for $a \in F$ and since the sets $fU(a,r)$ are disjoint, we have

$$\omega_n = m_n(S^n) \ge \sum_{a\in F} m_n\big(fU(a,r)\big) \ge ka_n(tr)^n.$$

This gives (8.6) with $\mu(t,n) = \omega_n/a_n t^n$. □

8.7. Theorem. *Suppose that $n \ge 2$, that $f\colon B^n \to B^n$ is (M,C)-CQH, that $f(0) = 0$, and that $0 < \alpha < r$. Suppose also that A is a subset of the* QH *sphere $S_k(r)$ such that $k(fx,0) \le r - \alpha$ for all $x \in A$. Then*

$$m_{n-1}(A) \le \varepsilon_1(\alpha,M,C,n)$$

where $\varepsilon_1(\alpha,M,C,n) \to 0$ as $\alpha \to \infty$.

Proof. By 7.15, f extends to a homeomorphism $\overline{f}\colon \overline{B}^n \to \overline{B}^n$ which is η-QS rel S^{n-1} with $\eta = \eta_{M,C}$. Let $f_0\colon S^{n-1} \to S^{n-1}$ be the restriction of $\overline{f}$. Writing as usual $\delta(x) = d(x, \partial B^n) = 1 - |x|$ we have $\delta(a) = e^{-r}$ and $\delta(fa) \geq e^{\alpha - r}$ for all $a \in A$. Let $p\colon R^n \setminus \{0\} \to S^{n-1}$ be the radial projection $px = x/|x|$. We first show that

$$l(pa, f_0, e^{-r}) \geq e^{\alpha - r}/\eta(1) \tag{8.8}$$

for each $a \in A$. Let $y \in S^{n-1}$ with $|y - pa| = e^{-r}$. Then $|a - pa| = |y - pa|$, and hence

$$e^{\alpha - r} \leq \delta(fa) \leq |fa - f_0 pa| \leq \eta(1)|f_0 y - f_0 pa|.$$

This proves (8.8). By Lemma 8.5 we obtain

$$m_{n-1}(A) \leq m_{n-1}(pA) \leq \mu\big(e^{\alpha}/\eta(1), n-1\big) = \varepsilon_1(\alpha, M, C, n). \ \square$$

8.9. Theorem. *Suppose that $n \geq 3$, that $f\colon B^n \to B^n$ is (M,C)-CQH, that $f(0) = 0$ and that $r > 0$, $\alpha > 0$. Suppose also that A is a subset of the QH sphere $S_k(r)$ such that $k(fx, 0) \geq r + \alpha$ for all $x \in A$. Then*

$$m_{n-1}(A) \leq \varepsilon_2(\alpha, M, C, n)$$

where $\varepsilon_2(\alpha, M, C, n) \to 0$ as $\alpha \to \infty$.

Proof. Let $a \in A$. Using the notation of the proof of 8.7 we first show that

$$L(pa, f_0, e^{-r}) \leq ce^{-r-\alpha} \tag{8.10}$$

with $c = c(M, C)$. Assume that $y \in S^{n-1}$ with $|y - pa| = e^{-r}$. Set $z = f_0^{-1} pfa$. Since

$$|z - pa| \leq |z - a| + |a - pa| \leq 2|z - a|,$$

the relative η-quasisymmetry of $\overline{f}$ gives

$$|f_0 z - f_0 pa| \leq \eta(2)|f_0 z - fa|.$$

This implies

$$|fa - f_0 pa| \leq \big(1 + \eta(2)\big)|f_0 z - fa| \leq \big(1 + \eta(2)\big)e^{-r-\alpha}.$$

Since $|y - pa| = |a - pa|$, we have

$$|f_0 y - f_0 pa| \leq \eta(1)|fa - f_0 pa|.$$

These estimates yield (8.10) with $c = \eta(1)\big(1 + \eta(2)\big)$.

From (8.10) it follows that

$$l(b, f_0^{-1}, ce^{-r-\alpha}) \geq e^{-r}$$

for each $b \in f_0 pA$. Hence Lemma 8.5 gives

$$m_{n-1}(f_0 pA) \leq \mu(e^{\alpha}/c, n-1) = \varepsilon_3(\alpha, M, C, n),$$

and $\varepsilon_3 \to 0$ as $\alpha \to \infty$. The theorem now follows easily from the absolute continuity of QC maps. This is the point where the condition $n \geq 3$ is needed. We give the required result as Lemma 8.11 below. $\square$

8.11. Lemma. *Suppose that $n \geq 2$ and that $f\colon S^n \to S^n$ is an η-QS homeomorphism. Then for each $A \subset S^n$ we have $m_n(fA) \leq \varepsilon\big(m_n(A), \eta, n\big)$, where $\varepsilon(t, \eta, n) \to 0$ as $t \to 0$.*

Proof. Let $\varphi\colon S^n \to \dot{R}^n$ be the stereographic projection with $\varphi(e_{n+1}) = \infty$. Then φ maps the lower hemisphere H onto B^n. Let $Q \subset \overline{B}^n$ be a closed n-cube with vertices on S^{n-1}. Since there is an integer $k = k(n)$ such that S^n can be covered by k rotations of $\varphi^{-1}Q$, we may assume that $A \subset \varphi^{-1}Q$. We may also assume that $fe_{n+1} = e_{n+1}$. The map $g = \varphi f \varphi^{-1}\colon \dot{R}^n \to \dot{R}^n$ is a K-QC map with $g(\infty) = \infty$ and $K = \eta(1)^{n-1}$. From the η-quasisymmetry of f we obtain a lower bound $|fx - e_{n+1}| \geq q = q(\eta) > 0$ for each $x \in H$. Indeed, setting $y = f^{-1}(-e_{n+1})$ we have

$$|y - e_{n+1}| \leq 2 \leq \sqrt{2}\,|x - e_{n+1}|,$$

and hence

$$2 = |fy - fe_{n+1}| \leq \eta\big(\sqrt{2}\big)|fx - e_{n+1}|.$$

It follows that $m(gQ)$ has an upper bound depending only on η and n. From [Re, Corollary, p. 262] we obtain an estimate

$$m(g\varphi A) \leq \varepsilon_0\big(m(\varphi A), \eta, n\big)$$

where $\varepsilon_0(t, \eta, n) \to 0$ as $t \to 0$. This proves the lemma, since φ is L-bilipschitz in $S^n \setminus B(e_{n+1}, q)$ with $L = L(\eta, n)$. □

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