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GLEASON-TYPE DECOMPOSITIONS FOR $H^\infty(B_n)$ AND LUMER'S HARDY ALGEBRA OF THE BALL

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1. Introduction

We recall that a holomorphic function f on a bounded, balanced, simply connected domain Ω in $\mathbf{C}^n$ belongs to *Lumer's Hardy algebra* $LN_*(\Omega)$ if $\varphi(\log|f|) \le u$ for some pluriharmonic function u and some nondecreasing convex function $\varphi\colon \mathbf{R} \to \mathbf{R}_+$ satisfying $\varphi(t)/t \to \infty$ as $t \to \infty$. It is known that $LN_*(\mathbf{B}_n)$ endowed with the topology defined by the metric $d(f,g) = \|f-g\|$, where

$$\|f\| = \inf\{u(0) : u \text{ pluriharmonic}, \log(1+|f|) \le u\}$$

is an F-space (complete, metrizable topological vector space). We refer to [L, RU, R2, N2] for information on $LN_*(\Omega)$.

The topological vector space structure of Lumer's Hardy algebra (the Lumer–Smirnov class) $LN_*(\mathbf{U}^n)$ of the unit polydisc was extensively studied in [N2]. It turns out that many nice results obtained in the case $n = 1$ by N. Yanagihara [Y1, Y2] have very similar forms for $LN_*(\mathbf{U}^n)$, but in general there is no simple way of proceeding from one to several variables. The main trouble is that the space of polynomials on $\mathbf{U}^n$ *is not dense* in $LN_*(\mathbf{U}^n)$ if $n > 1$. Fortunately, as was proved in [N2], the Hartogs series of an arbitrary function $f \in LN_*(\mathbf{U}^n)$ is weakly convergent. Consequently, the *polynomials are weakly dense* in $LN_*(\mathbf{U}^n)$. This is a way to avoid the density problem at least while studying continuous linear functionals or general linear operators with values in locally convex spaces.

In the present paper we study Lumer's Hardy algebras $LN_*(\mathbf{B}_n)$ of the unit balls $\mathbf{B}_n$ in $\mathbf{C}^n$ for $n > 1$. In this setting, as in the case of polydiscs, the main difficulties are connected with the nonseparability of the spaces. Although the Hartogs series technique does not work for balls, we can find another tool to prove the weak density of polynomials in $LN_*(\mathbf{B}_n)$; namely, *Gleason-type decompositions* of functions in $H^\infty(\mathbf{B}_n)$.

We recall the well-known solution of Gleason's problem for $H^\infty(\mathbf{B}_n)$ to state that for each function $f \in H^\infty(\mathbf{B}_n)$ with $f(0) = 0$ there exist $g_1, \dots, g_n \in H^\infty(\mathbf{B}_n)$ such that $f(z) = \sum_{k=1}^n z_k g_k(z)$ and $\|g_k\|_\infty \le C\|f\|_\infty$, where C does not depend on individual f (see [AS, G, KN, R2]). Applying this result one can show by induction that if $f \in H^\infty(\mathbf{B}_n)$ is such that the Fourier coefficients of

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f with respect to z^α are equal to zero whenever the order $|\alpha| < k$, there are $g_\alpha \in H^\infty(\mathbf{B}_n)$, $|\alpha| = k$, such that

$$(*) \qquad f(z) = \sum_{|\alpha|=k} z^\alpha g_\alpha(z).$$

Unfortunately, the induction method gives rather poor exponential type estimates for norms of g_α ($\|g_\alpha\|_\infty = O(C^k)$). In any case it would be interesting to find the best possible estimates for $\|g_\alpha\|_\infty$ in the decomposition $(*)$.

In this paper, modifying the Ahern–Schneider solution [AS] of the classical Gleason's problem, we show that if "sufficiently many" first Fourier coefficients of $f \in H^\infty(\mathbf{B}_n)$ are equal to zero, we can express f in the form $(*)$ with a power-type estimate for $\|g_\alpha\|_\infty$ (see Theorem 3.1 for details). This result is crucial to our proof that the space of polynomials is weakly dense in $LN_*(\mathbf{B}_n)$.

The paper is organized as follows. Section 2 contains preliminary definitions and notation. In Section 3 we obtain Gleason-type decompositions of functions from $H^\infty(\mathbf{B}_n)$ mentioned above. Section 4 is devoted to proving the weak density of polynomials in $LN_*(\mathbf{B}_n)$.

Section 5 contains our main results, representations of the Fréchet envelope and the topological dual of $LN_*(\mathbf{B}_n)$. We recall that the *Fréchet envelope* of an F-space $X = (X, \tau)$ whose topological dual separates the points is the completion of the space (X, τ^c), where τ^c is the strongest locally convex topology on X weaker than τ. It is well-known that τ^c coincides with the Mackey topology of the dual pair (X, X'), where X' is the topological dual of (X, τ). Moreover, τ^c is metrizable and, in fact, defined by Minkowski's functionals of convex hulls of sets in an arbitrary τ-base at the origin (equivalently, by the family of all τ-continuous seminorms on X). The reader is referred to [S1, S2] for information on Mackey topologies and Fréchet envelopes.

In this paper we observe that the Fréchet envelope of $LN_*(\mathbf{B}_n)$ is isomorphic to a nuclear power series space, and apply this fact to obtain a representation of continuous linear functionals on $LN_*(\mathbf{B}_n)$. Finally, we show that the family of monomials $\{z^\alpha\}$ is a weak unconditional basis for $LN_*(\mathbf{B}_n)$ in spite of the fact that $LN_*(\mathbf{B}_n)$ is nonseparable in its own topology.

2. Preliminaries

Throughout the paper we use the standard notation of [R2]. Let $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$ $(z, w \in \mathbf{C}^n)$ denote the standard inner product on $\mathbf{C}^n$ and $|z| = \langle z, z\rangle^{1/2}$ $(z \in \mathbf{C}^n)$ the corresponding norm on $\mathbf{C}^n$. We denote the unit ball and the unit sphere in $\mathbf{C}^n$ by $\mathbf{B} = \mathbf{B}_n$ and $\mathbf{S} = \mathbf{S}_n = \partial\mathbf{B}_n$, respectively. Moreover, let $\mathbf{Z}_+$ denote the set of all nonnegative integers and $\mathbf{Z}_+^n$ its n-fold product. If $\mathbf{T} = \partial\mathbf{B}_1$ and dm is the normalized Lebesgue measure on $\mathbf{T}$, then $\mathbf{T}^n$ and dm_n

are the n-fold products of $\mathbf{T}$ and dm, respectively. By $d\sigma$ we denote the unique rotation invariant probability measure on $\mathbf{S}$.

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$, $z, \zeta \in \mathbf{C}^n$, and $r \in \mathbf{C}$, we denote $|\alpha| := \alpha_1 + \dots + \alpha_n$, $z\zeta := (z_1\zeta_1, \dots, z_n\zeta_n)$, $\alpha! := \alpha_1! \cdot \dots \cdot \alpha_n!$, $rz := (rz_1, \dots, rz_n)$, $z^\alpha := z_1^{\alpha_1} \cdot \dots \cdot z_n^{\alpha_n}$.

The class of all holomorphic functions on $\mathbf{B}_n$ will be denoted by $H(\mathbf{B}_n)$, while $H^\infty(\mathbf{B}_n)$ is the space of all bounded functions in $H(\mathbf{B}_n)$ endowed with the supnorm $\|\cdot\|_\infty$.

It is well-known that the analytic monomials z^α, $\alpha \in \mathbf{Z}_+^n$, are orthogonal on the sphere, i.e.,

$$\int_{\mathbf{S}} \zeta^\alpha \bar{\zeta}^\beta \, d\sigma(\zeta) = 0 \qquad \text{if } \alpha \neq \beta.$$

Moreover,

$$\int_{\mathbf{S}} |\zeta^\alpha|^2 d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} \tag{2.1}$$

(see [R2] 1.4.8, 1.4.9). Therefore,

$$\varphi_\alpha(z) = \Big(\frac{(n-1+|\alpha|)!}{(n-1)!\alpha!}\Big)^{1/2} z^\alpha, \qquad \alpha \in \mathbf{Z}_+^n,$$

is an orthogonal system of monomials on $\mathbf{B}_n$ which is normalized in $L_2(\mathbf{S}, \sigma)$.

Each function $f \in H(\mathbf{B}_n)$ has the Fourier expression

$$f(z) = \sum_{\alpha \in \mathbf{Z}_+^n} a_\alpha(f) \varphi_\alpha(z),$$

where the series is convergent uniformly and absolutely on each compact subset of $\mathbf{B}_n$ and

$$a_\alpha(f) = \lim_{r \to 1-} \int_{\mathbf{S}} f(r\zeta) \overline{\varphi_\alpha(\zeta)} \, d\sigma(\zeta).$$

If $k \in \mathbf{N}$, then $H^\infty(\mathbf{B}_n, k)$ will denote the subspace of $H^\infty(\mathbf{B}_n)$ consisting of functions f for which $a_\alpha(f) = 0$ for all $|\alpha| < k$.

3. Gleason-type decompositions

Theorem 3.1. *There is an increasing function* $\psi\colon \mathbf{N} \to \mathbf{N}$ *and a family* $\{T_\alpha : \alpha \in \mathbf{Z}_+^n\}$ *of bounded linear operators* $T_\alpha : H^\infty(\mathbf{B}, \psi(|\alpha|)) \to H^\infty(\mathbf{B})$ *such that*

(a) $f = \sum_{|\alpha|=k} \varphi_\alpha \cdot T_\alpha f$ *for each* $f \in H^\infty(\mathbf{B}, \psi(|k|))$, *and* $k \in \mathbf{Z}_+^n$,

(b) $\|T_\alpha\| = O(|\alpha|^{(n+1)/2})$.

Proof. For each $k = 0, 1, 2, \ldots$, $z \in \mathbf{B}$, and $\zeta \in \mathbf{S}$ we define

$$H_k(z,\zeta) = \sum_{|\alpha|=k} \varphi_\alpha(z)\overline{\varphi_\alpha(\zeta)} = \binom{n+k-1}{n-1}\langle z,\zeta\rangle^k,$$

$$C_k(z,\zeta) = \sum_{j=k}^{\infty} H_j(z,\zeta),$$

and

$$D_k(z,\zeta) = \sum_{j=0}^{\infty} \binom{n+k+j-1}{n-1}\langle z,\zeta\rangle^j.$$

It is well-known and easily seen that H_k is a reproducing kernel in the space of all homogeneous polynomials of degree k and that C_k is a reproducing kernel in $H^\infty(\mathbf{B}, k)$, i.e.

$$f(z) = \int_{\mathbf{S}} C_k(z,\zeta) f(\zeta)\, d\sigma(\zeta) \qquad \text{for each } f \in H^\infty(\mathbf{B}, k). \tag{3.1}$$

In particular, $C_0(z,\zeta) = C(z,\zeta)$ is the Cauchy kernel for $\mathbf{B}_n$. Moreover,

$$D_k(z,\zeta) = \sum_{j=0}^{\infty} c_{j,k} H_j(z,\zeta),$$

where

$$\begin{aligned} c_{j,k} &= \frac{(n+j)\cdots(n+j+k-1)}{(j+1)\cdots(j+k)} = \left(1+\frac{n-1}{j+1}\right)\cdots\left(1+\frac{n-1}{j+k}\right) \\ &= 1 + (n-1)\sum_{r=1}^{k} \frac{1}{j+r} + R_{j,k}, \end{aligned}$$

with $\sum_j R_{j,k} < \infty$ for each $k = 0, 1, \ldots$.

Now we can find and then fix an increasing function $\psi\colon \mathbf{N} \to \mathbf{N}$ so that

$$\sum_{j=\psi(k)}^{\infty} R_{j,k} \le 1 \qquad \text{for } k = 1, 2, \ldots. \tag{3.2}$$

Finally, we define operators U_α, W_α, $V_k\colon H^\infty(\mathbf{B}, k) \to H(\mathbf{B})$, where $k = |\alpha|$, by

$$\begin{aligned} U_\alpha f(z) &= \int_{\mathbf{S}} D_k(z,\zeta)\overline{\varphi_\alpha(z)} f(\zeta)\, d\sigma(\zeta), \\ W_\alpha f(z) &= \int_{\mathbf{S}} D_k(z,\zeta)\left[\overline{\varphi_\alpha(\zeta)} - \overline{\varphi_\alpha(\zeta)}\right] f(\zeta)\, d\sigma(\zeta), \\ V_k f(z) &= \int_{\mathbf{S}} D_k(z,\zeta) f(\zeta)\, d\sigma(\zeta). \end{aligned}$$

Since $\psi(k) \geq k$ for each k, we obtain by (3.1)

$$\begin{aligned} f(z) &= \int_{\mathbf{S}} C_k(z,\zeta) f(\zeta)\, d\sigma(\zeta) = \int_{\mathbf{S}} D_k(z,\zeta)\langle z,\zeta\rangle^k f(\zeta)\, d\sigma(\zeta) \\ &= \binom{n+k-1}{n-1}^{-1} \int_{\mathbf{S}} D_k(z,\zeta) H_k(z,\zeta) f(\zeta)\, d\sigma(\zeta) \\ &= \binom{n+k-1}{n-1}^{-1} \sum_{|\alpha|=k} \varphi_\alpha(z) \int_{\mathbf{S}} D_k(z,\zeta)\overline{\varphi_\alpha(\zeta)} f(\zeta)\, d\sigma(\zeta) \\ &= \binom{n+k-1}{n-1}^{-1} \sum_{|\alpha|=k} \varphi_\alpha(z) U_\alpha f(z). \end{aligned}$$

Set $T_\alpha = \binom{n+k-1}{n-1}^{-1} U_\alpha$ for each $\alpha \in \mathbf{Z}_+^n$. We see that the family $\{T_\alpha : \alpha \in \mathbf{Z}_+^n\}$ satisfies the assertion (a). For the proof that it has the property (b) we need a few lemmas.

Lemma 3.2. *For each* $n \in \mathbf{N}$ *we have*
(a) $\|\varphi_\alpha\|_\infty = O\big(|\alpha|^{(n-1)/2}\big)$ *as* $|\alpha| \to \infty$;
(b) $\sup\big\{\big|\operatorname{grad}\varphi_\alpha(z)\big| : z \in \bar{\mathbf{B}}\big\} = O\big(|\alpha|^{(n+1)/2}\big)$ *as* $|\alpha| \to \infty$.

Proof. First we shall find the maximum of $|z^\alpha|$ on $\mathbf{S}$. It is enough to find the conditional maximum of the function of n real variables $f(x_1,\dots,x_n) = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ on the sphere $x_1^2 + \cdots + x_n^2 = 1$. The standard argument shows that f attains its maximum at the point $x = (x_1,\dots,x_n)$ with $x_j = \big(\alpha_j/|\alpha|\big)^{1/2}$ for $j = 1,\dots,n$, and so

$$\max\big\{|z^\alpha| : z \in S\big\} = \Big(\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}\Big)^{1/2}.$$

Consequently,

$$\|\varphi_\alpha\|_\infty^2 = \frac{(n+|\alpha|-1)!}{(n-1)!\alpha!}\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}.$$

Now, applying the Stirling formula ($k! \sim \sqrt{k}(k/e)^k$) to estimate factorials we obtain the desired result.

To prove (b) we have

$$\frac{\partial\varphi_\alpha}{\partial z_j} = \begin{cases} \big[(n+|\alpha|-1)\alpha_j\big]^{1/2}\varphi_\beta & \text{if } \alpha_j \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta = (\alpha_1,\dots,\alpha_{j-1},\alpha_j - 1,\alpha_{j+1},\dots,\alpha_n)$. Thus, (b) is an immediate consequence of (a).

Lemma 3.3. *Each operator V_k, $k \in \mathbf{Z}_+$, acts from $H^\infty(\mathbf{B}, \psi(k))$ into $H^\infty(\mathbf{B})$ and $\|V_k\| = O(k)$.*

Proof. Let $f(z) = \sum_{j=\psi(k)}^\infty F_j(z)$ be the homogeneous expansion of $f \in H^\infty(\mathbf{B}, \psi(k))$. Then

$$\begin{aligned}(V_k f)(z) &= \sum_{j=\psi(k)}^\infty c_{j,k} F_j(z) \\ &= \sum_{j=\psi(k)}^\infty \Big(1 + (n-1)\sum_{r=1}^k \frac{1}{j+r} + R_{j,k}\Big) F_j(z) \\ &= f(z) + (n-1) \sum_{j=\psi(k)}^\infty \sum_{r=1}^k \frac{1}{j+r} F_j(z) + \sum_{j=\psi(k)}^\infty R_{j,k} F_j(z).\end{aligned}$$

However, $F_j(z)/(j+r) = \int_0^1 t^{r-1} F_j(tz)\, dt$, so

$$(V_k f)(z) = f(z) + (n-1) \sum_{r=1}^k \int_0^1 t^{r-1} f(tz)\, dt + \sum_{j=\psi(k)}^\infty R_{j,k} F_j(z).$$

Finally,

$$\big|(V_k f)(z)\big| \le \|f\|_\infty + k(n-1)\|f\|_\infty + \|f\|_\infty \le (n+1)k \|f\|_\infty$$

for each $z \in \mathbf{B}$.

Lemma 3.4 [R2, Proposition 1.4.10]. *For $z \in \mathbf{B}_n$ and $c > 0$ define*

$$I_c(z) = \int_{\mathbf{S}} \big|1 - \langle z, \zeta\rangle\big|^{-n+c}\, d\sigma(\zeta).$$

Then I_c is bounded in $\mathbf{B}_n$.

Lemma 3.5. *Each operator W_α, $\alpha \in \mathbf{Z}_+^n$, acts from $H^\infty(\mathbf{B}, \psi(|\alpha|))$ into $H^\infty(\mathbf{B})$ and $\|W_\alpha\| = O\big(|\alpha|^{(3n+1)/2}\big)$.*

Proof. Let us observe first that if $\big|\langle z, \zeta\rangle\big| \le k/(n+k) =: s$, then

$$\begin{aligned}\big|D_k(z,\zeta)\big| &\le \sum_{j=0}^\infty \binom{n+k+j-1}{n-1} s^j = s^{-k} \sum_{m=k}^\infty \binom{n+m-1}{n-1} s^m \\ &\le 1/s^k(1-s)^n \le Ck^n\end{aligned}$$

for some absolute positive constant C. Moreover, if $1 \geq |\langle z,\zeta\rangle| > k/(n+k)$, then

$$\begin{aligned} |D_k(z,\zeta)| &\leq \left(1+\frac{n}{k}\right)^k |C_k(z,\zeta)| \\ &= \left(1+\frac{n}{k}\right)^k \left| C(z,\zeta) - \sum_{j=0}^{k-1} \binom{n+j-1}{n-1} \langle z,\zeta\rangle^j \right| \\ &\leq e^n \left(|C(z,\zeta)| + \sum_{j=0}^{k-1} \binom{n+j-1}{n-1} \right) \leq C\left(|C(z,\zeta)| + k^n\right) \end{aligned}$$

for some $C > 0$. Consequently, we have

$$|D_k(z,\zeta)| \leq C\left(k^n + |C(z,\zeta)|\right) \tag{3.3}$$

for some $C > 0$ and each $z \in \mathbf{B}$, $\zeta \in \mathbf{S}$, and $k \in \mathbf{N}$. Since $|\zeta - z| \leq 2|1-\langle z,\zeta\rangle|^{1/2}$, we obtain by Lemma 3.2

$$\begin{aligned} |\varphi_\alpha(\zeta) - \varphi_\alpha(z)| &\leq \sup\left\{ |\operatorname{grad}\varphi_\alpha(z)| : z \in \bar{\mathbf{B}} \right\} |\zeta - z| \\ &\leq C|\alpha|^{(n+1)/2} |1 - \langle z,\zeta\rangle|^{1/2} \end{aligned}$$

for some $C > 0$ and all $\alpha \in \mathbf{Z}_+^n$, $z \in \mathbf{B}$, $\zeta \in \mathbf{S}$. Using this and (3.3) we can estimate $|W_\alpha f(z)|$ by

$$C|\alpha|^{(n+1)/2} \left(|\alpha|^n \int_{\mathbf{S}} |1 - \langle z,\zeta\rangle|^{1/2} d\sigma(\zeta) + \int_{\mathbf{S}} \frac{d\sigma(\zeta)}{|1 - \langle z,\zeta\rangle|^{n-1/2}} \right) \|f\|_\infty,$$

where $C > 0$ is independent of $z \in \mathbf{B}$, $\alpha \in \mathbf{Z}_+^n$, and $f \in H(\mathbf{B}, \psi(|\alpha|))$. Finally, by Lemma 3.4, $\|W_\alpha f\|_\infty \leq C|\alpha|^{(3n+1)/2} \|f\|_\infty$.

Proof of Theorem 3.1 (b). Since $U_\alpha = \varphi_\alpha V_{|\alpha|} + W_\alpha$, by Lemma 3.2 (a), Lemma 3.3, and Lemma 3.5 we obtain

$$\|U_\alpha f\|_\infty \leq \|\varphi_\alpha\|_\infty \left\|V_{|\alpha|} f\right\|_\infty + \|W_\alpha f\|_\infty \leq C\|f\|_\infty |\alpha|^{(3n+1)/2}.$$

However,

$$\binom{n+|\alpha|-1}{n-1} \geq |\alpha|^n/n!,$$

so $\|T_\alpha\| = O(|\alpha|^{(n+1)/2})$. The proof is complete.

4. Weak approximation by polynomials in $LN_*(B)$

Lemma 4.1. *For an arbitrary continuous seminorm* $\|\!|\cdot|\!\|$ *on* $LN_*(\mathbf{B}_n)$ *we have* $\|\!|\varphi_\alpha|\!\| = O(\exp(-c|\alpha|^{1/2})$ *for some* $c > 0$.

Proof. The general idea of the proof comes close to the spirit of [E, Proposition 3.3] and, to the author's knowledge, was suggested by N.J. Kalton.

In the case $n = 1$ pluriharmonic is the same as harmonic, so $LN_*(\mathbf{B}_1)$ coincides with the Smirnov class N^+. It is well-known (see [Y1]) that the Mackey topology of N^+ is defined by the sequence of norms

$$\|\!|f|\!\|_m = \sum_k |a_k(f)| \exp(-k^{1/2}/m),$$

$m = 0, 1, \ldots$. This immediately implies the lemma in the case $n = 1$.

In general, define an operator $\Gamma_\zeta \colon N^+ \to LN_*(\mathbf{B}_n)$ by $(\Gamma_\zeta f)(z) = f(\langle z, \zeta\rangle)$ for $z \in \mathbf{B}_n$ and $\zeta \in \mathbf{S}$. It is easily seen that the family $\{\Gamma_\zeta : \zeta \in \mathbf{S}\}$ is equicontinuous, so it remains equicontinuous if we equip N^+ and $LN_*(\mathbf{B}_n)$ with their Mackey topologies. In particular, there exists a continuous seminorm $\|\!|\cdot|\!\|_1$ on N^+ such that $\|\!|\Gamma_\zeta f|\!\| \le \|\!|f|\!\|_1$ for each $\zeta \in \mathbf{S}$. Using this and Yanagihara's result mentioned above we get

$$\|\!|\Gamma_\zeta \varphi_k|\!\| \le \|\!|\varphi_k|\!\|_1 \le C \exp(-ck^{1/2}) \tag{4.1}$$

for some C, $c > 0$ and each $k \in \mathbf{Z}_+$, where $\varphi_k(\lambda) = \lambda^k$ for each $\lambda \in \mathbf{U} = \mathbf{B}_1$. However,

$$(\Gamma_\zeta \varphi_k)(z) = \langle z, \zeta\rangle^k = \binom{n+k-1}{n-1} H_k(z, \zeta). \tag{4.2}$$

Moreover, since the vector-valued function $\mathbf{T}^n \ni \omega \to H_k(\cdot, \omega\zeta) \in (LN_*(\mathbf{B}), \|\!|\cdot|\!\|)$ is continuous, it is integrable and we have for $k = |\alpha|$

$$\int_{\mathbf{T}^n} H_k(\cdot, \omega\zeta)\omega^\alpha dm_n(\omega) = \varphi_\alpha(\cdot)\varphi_\alpha(\zeta). \tag{4.3}$$

Therefore, by (4.1), (4.2), and (4.3) we obtain

$$\begin{aligned} |\varphi_\alpha(\zeta)| \, \|\!|\varphi_\alpha|\!\| &\le \int_{\mathbf{T}^n} \|\!|H_k(\cdot, \omega\zeta)|\!\| \, dm_n(\omega) \\ &\le \binom{n+k-1}{n-1} \sup\Big\{ \|\!|\Gamma_\zeta \varphi_k|\!\| : \zeta \in \mathbf{S}\Big\} \le C' \exp(-c' k^{1/2}) \end{aligned}$$

for some C', $c' > 0$ and each $\zeta \in \mathbf{S}$, $\alpha \in \mathbf{Z}_+$, where $k = |\alpha|$. Now, it is enough to choose $\zeta = \zeta_\alpha \in \mathbf{S}$ such that $|\varphi_\alpha(\zeta)| = 1$. The proof is complete.

Theorem 4.2. *For each $f \in LN_*(\mathbf{B}_n)$ the functions $f_r(z) = f(rz)$, $0 < r < 1$, tend to f in the Mackey topology when r tends to 1.*

Proof. The same argument as in the proof of [N2, Theorem 6.1] shows that it suffices to prove the theorem for $f \in H^\infty(\mathbf{B}_n)$.

Fix $f \in H^\infty(\mathbf{B}_n)$ and an arbitrary continuous seminorm $|\!|\!|\cdot|\!|\!|$ on $LN_*(\mathbf{B}_n)$. Since $|\!|\!|\cdot|\!|\!|$ is dominated by the Minkowski functional of the convex hull of some neighbourhood of zero in $LN_*(\mathbf{B}_n)$, we can assume that $|\!|\!| gh |\!|\!| \le |\!|\!| g |\!|\!| \, \|h\|_\infty$ for each $g \in LN_*(\mathbf{B}_n)$ and $h \in H^\infty(\mathbf{B}_n)$.

Fix an $\varepsilon > 0$ and find a function $\psi\colon \mathbf{N} \to \mathbf{N}$, and a family $\{T_\alpha : \alpha \in \mathbf{Z}_+^n\}$ as in Theorem 3.1. By Lemma 4.1, there are C, $c > 0$ such that

$$\|T_\alpha\| \, |\!|\!| \varphi_\alpha |\!|\!| \le C|\alpha|^{(n+1)/2} \exp(-c|\alpha|^{1/2}) \tag{4.4}$$

for some $C > 0$ and all $\alpha \in \mathbf{Z}_+^n$. Choose $k \in \mathbf{N}$ so large that

$$C_k := C\binom{n+k-1}{n-1} k^{(n+1)/2} \exp(-ck^{1/2}) \, \|f\|_\infty < \varepsilon/6. \tag{4.5}$$

For $0 < r < 1$ define $h^{(r)} = \sum_{|\alpha|<\psi(k)} a_\alpha(f)(1 - r^{|\alpha|})\varphi_\alpha$ and $g^{(r)} = f - f_r - h^{(r)}$. There is an $r_0 \in (0,1)$ such that $|\!|\!| h^{(r)} |\!|\!| < \varepsilon/2$ and $\|h^{(r)}\|_\infty \le \|f\|_\infty$ for all $r_0 < r < 1$. Now, for an arbitrary $r \in (r_0, 1)$ we have $g^{(r)} \in H^\infty(B, \psi(k))$, $\|g^{(r)}\| \le \|f\|_\infty + \|f_r\|_\infty + \|h^{(r)}\|_\infty \le 3\|f\|_\infty$, and $g^{(r)} = \sum_{|\alpha|=k} \varphi_\alpha T_\alpha g^{(r)}$. Finally, by (4.4) and (4.5) we get

$$\begin{aligned}
|\!|\!| f - f_r |\!|\!| &\le |\!|\!| h^{(r)} |\!|\!| + \sum_{|\alpha|=k} \|T_\alpha g^{(r)}\|_\infty \, |\!|\!| \varphi_\alpha |\!|\!| \\
&\le \tfrac{1}{2}\varepsilon + \sum_{|\alpha|=k} \|T_\alpha\| \, |\!|\!| \varphi_\alpha |\!|\!| \, \|g^{(r)}\|_\infty \\
&\le \tfrac{1}{2}\varepsilon + \sum_{|\alpha|=k} 3C|\alpha|^{(n+1)/2} \exp(-c|\alpha|^{1/2}) \, \|f\|_\infty \\
&\le \tfrac{1}{2}\varepsilon + 3C_k < \varepsilon.
\end{aligned}$$

The proof is complete.

Corollary 4.6. *The ball algebra $A(\mathbf{B}_n)$ and the space of analytic polynomials are both Mackey and weakly dense in $LN_*(\mathbf{B}_n)$.*

Remark. The same argument as in [N1, Remark 6] shows that $LN_*(\mathbf{B}_n)$, $n > 1$, contains an isomorphic copy of the space of all bounded complex sequences l^∞. Consequently, $LN_*(\mathbf{B}_n)$ is nonseparable, and so the space of polynomials is not dense in $LN_*(\mathbf{B}_n)$ equipped with its original topology.

5. The Fréchet envelope and the dual of $LN_*(B)$

Let $LF_*(\mathbf{B}_n)$ be the space of all $f \in H(\mathbf{B}_n)$ for which

$$\| f \|_k = \sum_\alpha |a_\alpha(f)| \exp\left(-|\alpha|^{1/2}/k\right) < \infty$$

for all $k \in \mathbf{N}$. $LF_*(\mathbf{B}_n)$ equipped with the vector topology defined by the sequence of norms $\{ \| \cdot \|_k : k \in \mathbf{N}\}$ is a Fréchet space, which in fact is isomorphic to the nuclear power series space $\Lambda_1(\gamma)$, where $\gamma = (\gamma_\alpha)$ and $\gamma_\alpha = |\alpha|^{1/2}$ for $\alpha \in \mathbf{Z}_+^n$ (compare with [N2, 5]). We refer to [Ro, D] for general information on power series spaces.

Theorem 5.1. (a) *$LF_*(\mathbf{B}_n)$ contains $LN_*(\mathbf{B}_n)$ as a dense subspace;*

(b) *The topology induced on $LN_*(\mathbf{B}_n)$ from $LF_*(\mathbf{B}_n)$ coincides with the Mackey topology of $LN_*(\mathbf{B}_n)$.*

In particular, $LF_*(\mathbf{B}_n)$ is the Fréchet envelope of $LN_*(\mathbf{B}_n)$.

Proof. (a) Since for each positive pluriharmonic function u on $\mathbf{B}_n$ and $\zeta \in \mathbf{S}$ the slice function u_ζ defined by $u_\zeta(\lambda) = u(\zeta\lambda)$, $\lambda \in \mathbf{U}$, is a positive harmonic function on $\mathbf{U}$, we have $u_\zeta(\lambda) \le 2u(0)/(1-|\lambda|)$. Consequently, $|f(z)| < \exp\left(2\|f\|/(1-|z|\right)$ for each $f \in LN_*(\mathbf{B}_n)$ and $z \in \mathbf{B}$. Arguments similar to those in [Y3, Theorem 1] and [ST] show that if $f \in LN_*(\mathbf{B}_n)$, then $|a_\alpha(f)| = O\left(\exp\left(c|\alpha|^{1/2}\right)\right)$ for each $c > 0$. This implies that $LN_*(\mathbf{B}_n)$ is contained in $LF_*(\mathbf{B}_n)$. It is easily seen that the Taylor series of an arbitrary function $f \in LF_*(\mathbf{B}_n)$ is convergent in the topology of $LF_*(\mathbf{B}_n)$ to f. Thus, the space of all polynomials $\mathcal{P}(\mathbf{B}_n) \subset LN_*(\mathbf{B}_n)$ is dense in $LF_*(\mathbf{B}_n)$.

(b) We know that $LN_*(\mathbf{B}_n) \subset LF_*(\mathbf{B}_n)$ and that, by the closed graph theorem, the inclusion mapping is continuous. Thus, the inclusion remains continuous if we equip $LN_*(\mathbf{B}_n)$ with the Mackey topology, and the topology ν induced on $LN_*(\mathbf{B}_n)$ from $LF_*(\mathbf{B}_n)$ is weaker than the Mackey topology μ of $LN_*(\mathbf{B}_n)$.

Since $\mathcal{P}(\mathbf{B})$ is Mackey-dense both in $LN_*(\mathbf{B}_n)$ and $LF_*(\mathbf{B}_n)$ (see Corollary 4.6), in order to prove $m \le \nu$ it is enough to show that each μ-continuous seminorm $\| \cdot \|$ on $\mathcal{P}(\mathbf{B})$ is dominated by some seminorm $\| \cdot \|_k$, $k \in \mathbf{N}$.

Fix $\| \cdot \|$. By Lemma 4.1, there exist $C > 0$ and $k \in \mathbf{N}$ such that

$$\| \varphi_\alpha \| \le C \exp\left(-|\alpha|^{1/2}/k\right)$$

for all $\alpha \in \mathbf{Z}_+^n$. Therefore, for an arbitrary polynomial $f = \sum a_\alpha \varphi_\alpha$ we have $\| f \| \le \sum |a_\alpha| \, \| \varphi_\alpha \| \le C \, \| f \|_k$. The proof is complete.

Corollary 5.2. *The family $\{\varphi_\alpha : \alpha \in \mathbf{Z}_+^n\}$ is an unconditional Mackey and weak basis of $LN_*(\mathbf{B}_n)$.*

Proof. It is easily seen that $\{\varphi_\alpha : \alpha \in \mathbf{Z}_+^n\}$ is an absolute basis in $LF_*(\mathbf{B}_n)$ $(\simeq \Lambda_1(\gamma))$.

Theorem 5.2. *Each continuous linear functional T on $LN_*(\mathbf{B}_n)$ is of the form*

$$Tf = \sum a_\alpha(f)\lambda(\alpha),$$

where $\{\lambda(\alpha) : \alpha \in \mathbf{Z}_+^n\}$ is a family of complex numbers such that

$$\sup_\alpha |\lambda(\alpha)| \exp\left(|\alpha|^{1/2}/k\right) < \infty$$

for some $k \in \mathbf{N}$.

Proof. $LN_*(\mathbf{B}_n)$ has the same dual as its Fréchet envelope $LF_*(\mathbf{B}_n)$ (more precisely, the restriction mapping $\left(LF_*(\mathbf{B}_n)\right)' \ni T \to T_{|LN_*(\beta)} \in \left(LN_*(\mathbf{B}_n)\right)'$ is an algebraic isomorphism). Since the mapping $f \to \left(a_\alpha(f)\right)$ is a linear topological isomorphism of $LF_*(\mathbf{B}_n)$ onto the space $\Lambda_1(\gamma)$ described at the beginning of this section, we obtain the theorem applying the well-known representation of continuous linear functionals on nuclear power series spaces (see for example [RO] Proposition 7.4.8 or 4.4.5).

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