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HIGHER INTEGRABILITY AND THE BOUNDARY DIMENSION

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Abstract. We answer a question posed by K. Astala and P. Koskela in the negative by producing examples for each $n \geq 2$ and each $K > 1$ of bounded domains D, D' in $\mathbf{R}^n$ quasiconformally equivalent to the unit ball of $\mathbf{R}^n$ and a K-quasiconformal mapping f of D onto D' such that f lies in $\mathrm{locLip}_\alpha(D)$ for all $0 < \alpha < 1$ and $\dim_H(\partial D) = \dim_H(\partial D') = n-1$ but the derivative of of f does not belong to $L^p(D)$ for any exponent $p > n$.

Introduction

Let D, D' be domains in R^n with D' bounded and let f be a K-quasiconformal mapping of D onto D'. Then, as is well-known, f is differentiable a.e. on D and $\int_D |f'|^n \, dm < \infty$; in fact, $\int_D |f'|^n \, dm \leq K\big|f(D)\big|$ (where $|S|$ denotes the n-measure of $S \subset R^n$). On the other hand, it need not be true that

$$(1) \qquad \int_D |f'|^p \, dm < \infty \qquad \text{for some} \quad p > n.$$

Astala and Koskela [AK] recently showed that the higher integrability (1) is closely connected with the local Hölder continuity properties of f on D. They point out that (1) implies

$$(2) \qquad f \in \mathrm{locLip}_\alpha(D) \qquad \text{for some} \quad 0 < \alpha \leq 1.$$

Moreover, they show that the converse is true if one assumes that $\partial D'$ is not "too thick" in the sense of the Minkowski dimension. They also ask whether the converse remains true without the additional assumption on $\partial D'$. The main purpose of this note is to give an example which shows that some assumption of this type is necessary. Before stating our results, we define some of the terms used above and state Astala and Koskela's result as well.

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We first define the class $\operatorname{locLip}_\alpha(D)$, introduced by Gehring and Martio [GM]. If $D \subset R^n$ is a domain and g is a continuous function on D, we say that g belongs to the class $\operatorname{locLip}_\alpha(D)$, $0 < \alpha \le 1$, if there is a constant $M < \infty$ such that

$$\bigl|g(x) - g(y)\bigr| \le M|x-y|^\alpha$$

whenever x, y lie in a ball B contained in D. As usual, we write $g \in \operatorname{Lip}_\alpha(D)$ if the above inequality is valid for each pair x, y of points in D.

Next we define *Minkowski dimension* which gives a measure of the size of a set in R^n and which is analogous to Hausdorff dimension. If E is a compact set in R^n, $0 < \delta \le n$, and $r > 0$, set

$$M^\delta(E;r) = \inf\Bigl\{kr^\delta : E \subset \bigcup_1^k B(x,r)\Bigr\}.$$

The *Minkowski content* of E is now

$$M^\delta(E) = \limsup_{r\to 0} M^\delta(E,r)$$

and the *Minkowski dimension* of E is given by

$$\dim_M(E) = \inf\bigl\{\delta : M^\delta(E) < \infty\bigr\}.$$

Note that the Hausdorff dimension $\dim_H(E)$ of E satisfies $\dim_H(E) \le \dim_M(E)$; see [F] or [MV] for the relations between these two dimensions. We can now state Astala and Koskela's result [AK, Theorem 4.4].

Theorem A. *Let D' be a bounded domain with*

$$\dim_M(\partial D') < n. \tag{3}$$

If $f\colon D \to D'$ is K-quasiconformal, then (1) *and* (2) *are equivalent. Here α and p depend only on each other, n, K, and $\dim_M(\partial D')$.*

Our example shows that some condition like (3) is necessary in Theorem A, and, in particular, one can not replace the Minkowski dimension by the Hausdorff dimension:

Theorem B. *For each $n \ge 2$ and each $K > 1$ there are bounded domains D and D' in R^n quasiconformally equivalent to the unit ball of R^n and a K-quasiconformal mapping $f\colon D \to D'$ such that $\dim_H(\partial D) = \dim_H(\partial D') = n-1$ and* (2) *holds for all $0 < \alpha < 1$ but*

$$\int_D |f'|^p \, dm = \infty \qquad \textit{for all} \quad p > n. \tag{4}$$

Astala and Koskela [AK, 5.1] also showed that if in the above situation f satisfies (2) and $\alpha > \dim_M(\partial D)/n$, then (1) holds. Our second result shows that the dependence of α on $\dim_M(\partial D)$ in their theorem is essential.

Theorem C. *For each $n \geq 2$, each $K > 1$, and each $0 < \alpha < 1$ there exist bounded domains D and D' in R^n quasiconformally equivalent to the unit ball of R^n and a K-quasiconformal mapping $f\colon D \to D'$ such that $\dim_M(\partial D) < n$ and $\dim_H(\partial D) = \dim_H(\partial D') = n-1$ and both (2) and (4) hold.*

Still another result of Astala and Koskela [AK] states that if ∂D is sufficiently smooth, say D satisfies a quasihyperbolic boundary condition [GM], then (2) implies (1). Hence neither of the domains D and D' in Theorem B (Theorem C respectively) can be too regular.

The basic idea behind our construction of f, D, and D' is the following: In order for (4) to be satisfied, we need to construct f and D so that $|f'|$ will be large on a significant portion of D. However, if B is a ball contained in D, then $f \in \mathrm{Lip}_\alpha(B)$ (we want f to satisfy (2)) and hence Hölder's inequality implies that $\int_B |f'|\,dm \leq C \mathrm{diam}(B)^{n-1+\alpha}$ where C is a constant independent of B. Consequently, the average of $|f'|$ on B is inversely related to the size of B. If for each $x \in D$ we let B_x be the largest ball satisfying $x \in B_x \subset D$, it follows that we must construct D so that B_x is small "most of the time". In order to accomplish this, we form D by adjoining an infinite sequence of narrower and narrower projections to a rectangular domain.

We first prove Theorem C. Theorem B will then be established by a similar construction.

Proof of Theorem C

To simplify our notation, we only consider the case $n = 3$ and leave the modifications needed for the general case to the reader. Let $K_0 > 1$, $0 < \alpha < 1$ and define $D_0 = \{(x,y,z) : 0 < x < a_0, 0 < y, z < 1\}$; the value of a_0 is determined by the formulas (5), (10), and (11) below. Then D (respectively D') is constructed by adjoining projections P_i (respectively P_i') to the bottom of D_0 as pictured in Figures 1 and 2.

The projections P_i (respectively P_i') consist of upper and lower chambers U_i and L_i (respectively U_i' and L_i'). Moreover, P_i and P_i' are attached to D_0 in an identical way so that $\overline{D}_0 \cap \overline{P}_i = \overline{D}_0 \cap \overline{P}_i'$. The location patterns of these attachments will be defined below.

The mapping f will be the identity on D_0 and map each U_i (respectively L_i) onto U_i' (respectively L_i'). On each L_i, f will be a similarity mapping with a streching factor s_i that increases to infinity as i approaches infinity. This will allow us to deduce (4). In order to make f K_0-quasiconformal we use the regions U_i as decompression chambers so that as one travels from L_i to D_0 through U_i the streching factor gradually reduces from s_i to 1. In order to define the subregions U_i, L_i, U_i', L_i' precisely, as well as give the mapping f on U_i and L_i, we fix a positive integer i and translate $\overline{U}_i \cup \overline{L}_i$ and $\overline{U}_i' \cup \overline{L}_i'$ to $\overline{U} \cup \overline{L}$ and $\overline{U}' \cup \overline{L}'$

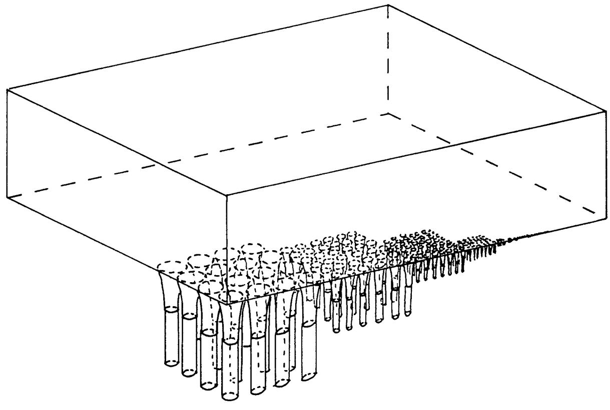

Figure 1.

so that $U \cup L$ and $U' \cup L'$ are symmetric with respect to the z-axis and $\overline{U} \cap \overline{L}$ and $\overline{U}' \cap \overline{L}'$ are contained in the plane $z = 0$. (see Figure 2).

Here U is bounded by the planes $z = 0$ and $z = Cr^{\gamma}$ and by the surface obtained by rotating the curve

$$z = C\Big(\frac{r^{\beta+1}}{x} - r^{\beta}\Big)\frac{r^{\gamma}}{r - r^{\beta}}$$

about the z-axis, $L = B(0,r) \times (-r^{1-\beta}, 0)$, $L' = B(0, r^{\beta}) \times (-1, 0)$ and $U' = B(0, r^{\beta}) \times \big(0, Cr^{\gamma-1}(r + r^{\beta})/2\big)$ where $B(0,s) = \{(x,y) : x^2 + y^2 < s^2\}$.

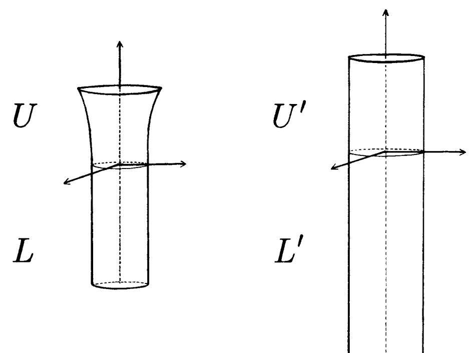

Figure 2.

Above $r = r_i$ is a number between 0 and 1 to be determined later, $C \geq 1$ is a constant to be determined later,

$$\beta = \max\Big\{\frac{2-\alpha}{3-2\alpha}, \frac{4}{5}\Big\} \tag{5}$$

and

$$\gamma = 2\beta - 1. \tag{6}$$

It follows from (5), and (6) that $\gamma > 0$.

We define g on $\overline{U} \cup \overline{L}$ as follows

$$g(x,y,z) = \begin{cases} (xw_1, yw_1, w_2), & (x,y,z) \in \overline{U} \\ (r^{\beta-1}x, r^{\beta-1}y, r^{\beta-1}z), & (x,y,z) \in \overline{L} \end{cases} \tag{7}$$

where

$$w_1 = \frac{r - r^\beta}{Cr^{\gamma+1}} z + r^{\beta-1}$$

and

$$w_2 = \frac{r - r^\beta}{2Cr^{\gamma+1}} z^2 + r^{\beta-1} z.$$

Then it is easy to verify that g is a homeomorphism of $\overline{U} \cup \overline{L}$ onto $\overline{U}' \cup \overline{L}'$ and g is C^∞ in $U \cup L$. We then define f on $U_i \cup L_i$ by composing g in an obvious way with translations so that f maps $U_i \cup L_i$ onto $U_i' \cup L_i'$.

We next show that if we take C large enough, then g is K_0-quasiconformal on $U \cup L \cup \{B(0,r) \times \{0\}\}$. It obviously suffices to show that g is K_0-quasiconformal on U. From (7) we have

$$Dg(x,y,z) = \begin{pmatrix} w_1 & 0 & xw_3 \\ 0 & w_1 & yw_3 \\ 0 & 0 & w_1 \end{pmatrix} \tag{8}$$

for all $(x,y,z) \in U$; here $w_3 = (r - r^\beta)/(Cr^{\gamma+1})$. Denote the operator norm of $Dg(x,y,z)$ by $\|Dg(x,y,z)\|$ and its determinant by $|Dg(x,y,z)|$.

It follows that

$$\begin{aligned} \frac{\|Dg(x,y,z)\|^3}{|Dg(x,y,z)|} &= \sup_{|h|=1} \Big\{ \big[(h_1 w_1 + x h_3 w_3)^2 \\ &\qquad + (h_2 w_1 + y h_3 w_3)^2 + {h_3}^2 {w_1}^2\big]^{3/2} {w_1}^{-3} \Big\} \\ &\le \Big[1 + {w_3}^2 (x^2 + y^2) {w_1}^{-2} + 2\big(|xw_3| + |yw_3|\big)|w_1|^{-1}\Big]^{3/2}. \end{aligned}$$

This last quantity is maximized for $(x,y,z) \in U$ when $|x| = |y| = r^\beta$ and $z = Cr^\gamma$; so using (6) we get

$$\begin{aligned} \frac{\|Dg(x,y,z)\|^3}{|Dg(x,y,z)|} &\le [1 + 2r^{2\beta-2\gamma-2}(r - r^\beta)^2 C^{-2} + 4r^\beta (r^\beta - r) r^{-\gamma-1} C^{-1}]^{3/2} \\ &\le [1 + 6C^{-1}]^{3/2}. \end{aligned}$$

Hence, taking the constant C large enough, [V, 14.3] yields $K \le K_0$. It follows that f is K_0-quasiconformal on $U_i \cup L_i$ and hence K_0-quasiconformal on all of D.

We now describe how we choose the sequence $\{r_i\}$ and how each P_i (respectively P_i') is attached to D_0. For each i, let $E_i = \overline{P}_i \cap \overline{D}_0 = \overline{P}_i' \cap \overline{D}_0$. Then from the definition of P_i and P_i' it follows that E_i is a closed disk in the z-plane with radius $r_i{}^{\beta}$. We take the r_i's to be positive integer powers of $\frac{1}{2}$ and for each positive integer j we let $\psi(j)$ be the number of r_i's equal to 2^{-j}. We arrange the corresponding E_i's in a square pattern Q_j and then place the Q_j's so that Q_j and Q_{j+1} are adjacent and all the Q_j's lie along the line segment $y = 0$, $z = 0$, $0 < x < a_0$, (see Figure 1).

If follows easily from Theorem 10.3 in [GV] that D' is quasiconformally equivalent to the unit ball of R^3, and, consequently, the same holds for D.

We next show that (4) is satisfied as long as ψ is defined judiciously.

Let $p > 3$. Since $|f'| = r_i{}^{\beta-1}$ on L_i and L_i is a cylinder with dimensions $r_i \times r_i^{1-\beta}$, we have

$$\begin{aligned}\int_D |f'|^p dm &\ge \sum_1^\infty \int_{L_i} |f'|^p dm = \pi \sum_1^\infty \psi(i) 2^{-i(1-\beta)} 2^{-2i} 2^{-i(p(\beta-1))} \\ &= \pi \sum_1^\infty \psi(i) 2^{-i((p-1)(\beta-1)+2)}.\end{aligned} \tag{9}$$

On the other hand, by the definitions of D and D', D and D' will be bounded if (and only if)

$$a_0 = \sum_1^\infty \psi(i)^{1/2} 2^{-i\beta} < \infty. \tag{10}$$

From (5) it follows that $\beta < 1$ and hence

$$(p-1)(\beta-1) + 2 = 2\beta + (p-3)(\beta-1) < 2\beta.$$

Therefore, from (9) and (10) it follows that it is possible to choose ψ so that (4) is satisfied and both D and D' are bounded. For example, take

$$\psi(i) \approx 2^{i2\beta}/i^4. \tag{11}$$

Next we estimate the dimensions of ∂D and $\partial D'$.

Note first that ∂D and $\partial D'$ are both a union of a countable number of sets of finite 2-dimensional measure; hence $\dim_H(\partial D) = \dim_H(\partial D') = 2$. Now we sketch the estimate for the Minkowski dimension of the boundary of D. Let $2 < \delta \le 3$, $0 < r < 1$, and pick an integer i_0 with $2^{-i_0} < r \le 2^{-i_0+1}$. For each positive integer i there are $\psi(i)$ subdomains of D, say G_i^j, $j = 1, \dots, \psi(i)$, corresponding to the sets P_i constructed above with $r_i = 2^{-i}$. Since $M^2(\partial D_0) < \infty$, we have

$$M^\delta\Big(\partial D \setminus \bigcup_i \bigcup_j \partial G_i^j; r\Big) \le C_1 \tag{12}$$

where the constant C_1 is independent of r. Thus it suffices to consider the Minkowski content of $\cup_i \cup_j \partial G_i^j$. We estimate this content in two parts. A simple calculation which makes use of (5) and (6) shows that the 2-dimensional measure of the boundary of G_i^j for each i, j is comparable to $2^{i(\beta-2)}$.

By the geometry of the sets G_i^j one observes that if $i \le i_0$, then the surface area of the intersection of the boundary of G_i^j with any ball B of radius r centered at the boundary of G_i^j is comparable to 2^{-2i_0}. Hence a standard covering argument (e.g. using the Vitali theorem) shows that

$$M^\delta\Big(\bigcup_{i<i_0} \bigcup_j \partial G_i^j; r\Big) \le C_2 2^{-i_0\delta} \sum_{i<i_0} \psi(i) 2^{i(\beta-2)+2i_0} \le C_3 i_0 2^{i_0(3\beta-\delta)}, \tag{13}$$

where C_3 is a constant independent of i_0. Finally, we consider the Minkowski content of the rest of the boundary of D. By the definition of the sets G_i^j, $\bigcup_{i\ge i_0} \bigcup_j G_i^j$ is contained in a rectangular region G of dimensions comparable to $2^{-i_0(1-\beta)} \times {i_0}^{-2} \times {i_0}^{-1}$. Hence

$$M^\delta\Big(\bigcup_{i\ge i_0} \bigcup_j \partial G_i^j; r\Big) \le M^\delta(G; r) \le C_4 2^{-i_0\delta} 2^{i_0(\beta-1)} 2^{3i_0} {i_0}^{-3} \le C_4 2^{i_0(2+\beta-\delta)}; \tag{14}$$

here C_4 is independent of i_0. We conclude from (12), (13), and (14) that $M^\delta(\partial D)$ is finite provided $\delta > 2 + \beta$. Thus (5) yields

$$\dim_M(\partial D) \le 2 + \max\big\{(2-\alpha)/(3-2\alpha), 4/5\big\} < 3$$

as desired.

It remains to show that $f \in \mathrm{locLip}_\alpha(D)$. Clearly $f \in \mathrm{Lip}_1(D_0)$ and

$$\big|f(z) - f(z')\big| \le |z - z'|^\beta$$

whenever $z, z' \in L_i$ satisfy $|z - z'| \le r_i$. Since $\beta \ge \alpha$, it suffices to show that $f \in \mathrm{Lip}_\alpha(U_i)$ for each positive integer i with the $\mathrm{Lip}_\alpha(U_i)$ constant of f independent of i.

Fix a positive integer i. We again translate U_i and U_i' to U and U' and work with g as defined by (7). We need to show that $g \in \mathrm{Lip}_\alpha(U)$. Let

$$g_1(x,y,z) = \frac{x}{r}\Big(\frac{r - r^\beta}{Cr^\gamma} z + r^\beta\Big),$$

$$g_2(x,y,z) = \frac{y}{r}\Big(\frac{r - r^\beta}{Cr^\gamma} z + r^\beta\Big),$$

$$g_3(x,y,z) = \frac{1}{r}\Big(\frac{r - r^\beta}{2Cr^\gamma} z^2 + r^\beta z\Big).$$

It clearly suffices to show that g_1, g_2, and g_3 are each in $\mathrm{Lip}_\alpha(U)$.

We first note that (5) and (6) imply the following inequality

$$\gamma \geq \frac{1-\beta}{1-\alpha}. \tag{15}$$

Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in U$. Then

$$\begin{aligned}
|g_3(x_1,y_1,z_1) - g_3(x_2,y_2,z_2)| &= \left|\frac{1}{r}\left(\frac{r-r^\beta}{2Cr^\gamma}({z_1}^2 - {z_2}^2) + r^\beta(z_1 - z_2)\right)\right| \\
&\leq \frac{1}{r}\left(\left|\frac{r^\beta}{2Cr^\gamma}(z_1 - z_2)(z_1 + z_2)\right| + r^\beta|z_1 - z_2|\right) \leq 2r^{\beta-1}|z_1 - z_2| \\
&\leq 2C^{1-\alpha}r^{\beta-1}r^{\gamma(1-\alpha)}|z_1 - z_2|^\alpha \leq 2C|z_1 - z_2|^\alpha
\end{aligned}$$

where the last inequality follows from (15). Consequently $g_3 \in \mathrm{Lip}_\alpha(U)$.

We show that g_1 and g_2 belong to $\mathrm{Lip}_\alpha(U)$; by symmetry it suffices to verify that $g_1 \in \mathrm{Lip}_\alpha(U)$. Observe that

$$\begin{aligned}
&|g_1(x_1,y_1,z_1) - g_1(x_2,y_2,z_2)| \\
&\leq |g_1(x_1,y_1,z_1) - g_1(x_1,y_1,z_2)| + |g_1(x_1,y_1,z_2) - g_1(x_2,y_2,z_2)| \\
&= I_1 + I_2.
\end{aligned} \tag{16}$$

We estimate I_1 and I_2 separately. We have

$$\begin{aligned}
I_2 &= \left|\frac{1}{r}\left(\frac{r-r^\beta}{Cr^\gamma}z_2 + r^\beta\right)(x_1 - x_2)\right| \leq \frac{1}{r}(|r - r^\beta| + r^\beta)|x_1 - x_2| \\
&\leq 2r^{\beta-1}|x_1 - x_2| \leq 2r^{\beta-1}2r^{\beta(1-\alpha)}|x_1 - x_2|^\alpha \leq 4|x_1 - x_2|^\alpha
\end{aligned} \tag{17}$$

where the last inequality follows from (6) and (15). Estimating I_1 we get

$$\begin{aligned}
I_1 &= \left|x_1\frac{r-r^\beta}{Cr^{\gamma+1}}(z_1 - z_2)\right| \leq C^{-1}x_1 r^{\beta-\gamma-1}|z_1 - z_2| \\
&\leq C^{-1}r^{2\beta-\gamma-1}|z_1 - z_2| \leq |z_1 - z_2|.
\end{aligned} \tag{18}$$

From (16), (17), and (18) it follows that $g_1 \in \mathrm{Lip}_\alpha(U)$.

Hence $g \in \mathrm{Lip}_\alpha(U)$ and the above estimates show that the corresponding constant is independent of i. Therefore $f \in \mathrm{Lip}_\alpha(U_i)$ with the $\mathrm{Lip}_\alpha(U_i)$ constant independent of i and the proof is complete.

Proof of Theorem B

Again, we only consider the case $n = 3$. We modify the construction used to prove Theorem C as follows. Replace r^{β} with $r\log(1/r)$, r^{γ} with $r\big(\log(1/r)\big)^2$ and $r^{1-\beta}$ with $1/\log(1/r)$. Instead of taking the r_i's to be positive integer powers of $\frac{1}{2}$, the r_i's will now be of the form 2^{-2^i}, and we denote the number of r_i's equal to 2^{-2^j} by $\psi(j)$. Construct D, D' and f as above with these replacements.

Now

$$\int_D |f'|^p dm \geq \sum_1^\infty \int_{L_i} |f'|^p dm = \pi(\log 2)^{p-1} \sum_1^\infty \psi(i) 2^{i(p-1)} 2^{-2(2^i)},$$

and D and D' are bounded if (and only if)

$$\sum_1^\infty \psi(i)^{1/2} 2^i 2^{-2^i} < \infty.$$

Hence defining

$$\psi(i) \approx 2^{2(2^i)} 2^{-2i} i^{-4}$$

(4) is satisfied and both D and D' are bounded.

It is left to the reader to check that the mapping f satisfies all the remaining requiments.

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