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EXPONENTIAL INTEGRABILITY OF THE QUASI-HYPERBOLIC METRIC ON HÖLDER DOMAINS

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Abstract. A proper subdomain D of R^n is called a Hölder domain if for a fixed y in D, the quasi-hyperbolic metric $k_D(x,y)$ is bounded by a constant plus a constant multiple of the logarithm of the Euclidean distance from x to the boundary of D. For simply connected planar domains D, it is known that these domains are characterized by the fact that the Riemann mapping function of the unit disk onto D satisfies a Hölder condition with some positive exponent.

For y in D fixed, we prove that $\exp(\tau k_D(x,y))$ is integrable over D for some $\tau > 0$. One corollary of this is that the boundaries of these domains have Hausdorff dimension less than n. Other applications pertain to Poincaré domains and to averaging domains. Our method involves extending some recent results of Carleson–Jones and Jones–Makarov on simply connected planar domains to multiply connected domains in R^n by using the quasi-hyperbolic metric.

1. Introduction

Consider an open, connected and proper subdomain D of Euclidean n-space R^n, $n \geq 2$. Following [GO] we define the quasi-hyperbolic metric k_D in D by

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\delta_D(x)}$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. Here we denote by $\delta_D(x)$ the Euclidean distance between x and ${}^cD$, the complement of D. As usual, we define

$$k_D(x_1, A) = \inf_{y \in A} k_D(x_1, y),$$

for $x_1 \in D$ and $A \subset D$.

If D is a simply connected planar domain, then the quasi-hyperbolic metric is comparable to the usual hyperbolic or Poincaré metric on D. See, for example, [BP]. For domains in R^n, the quasi-hyperbolic metric provides a useful substitute for the hyperbolic metric. Applications can be found, for example, in [GM], [GP], [GO], [H], [S] and [SS2].

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Definition. Let D be a proper subdomain of R^n and let $x_0 \in D$. The domain D is said to be a Hölder domain if there are constants c and C such that

$$k_D(x_0, x) \leq c \log \frac{\delta_D(x_0)}{\delta_D(x)} + C, \qquad x \in D.$$

The terminology is derived from the fact that in the plane, a simply connected domain D is a Hölder domain if and only if a Riemann mapping function from the unit disk onto D is Hölder continuous. (See [BP].) In general, Hölder domains are bounded and multiply connected domains.

Our motivation for this paper stems from our interest in the geometry of Hölder domains and the Poincaré inequality. In [SS2] Corollary 4 we showed that the volume of the boundary of a Hölder domain is 0, and we asked whether the Hausdorff dimension of the boundary must be less than n. This was known for John domains by the results in [MV]; see also Theorem 4 in [SS2]. It is a consequence of part (b) in the theorem below that this dimensionality result extends to Hölder domains.

Theorem A. *Let D be a proper subdomain of R^n and let $x_0 \in D$. The following are equivalent:*

(a) *D is a Hölder domain;*
(b) *There is a $\tau > 0$ such that*

$$\int_D \exp\left(\tau k_D(x_0, x)\right) dx < \infty;$$

(c) *D has finite volume, $m(D) < \infty$, and there is a $\tau > 0$ such that whenever u is integrable on D and satisfies*

$$\sup_B \frac{1}{m(B)} \int_B \exp\left(|u - u_B|\right) dx \leq 2$$

where the supremum is over all balls $B \subset D$ and u_B denotes the average of u over B, then

$$\frac{1}{m(D)} \int_D \exp\left(\tau |u - u_D|\right) dx \leq 2.$$

These equivalences are contained in Theorem 2 and Theorem 4 below. Our main result, Theorem 1, provides an estimate of the number of cubes, in a Whitney decomposition W of a general domain D, of a given size among those cubes in W that are at approximately the same quasi-hyperbolic distance from a fixed point $x_0 \in D$. We use this estimate together with geometric conditions on a domain to derive integrability properties of the quasi-hyperbolic metric. The applications to Hölder domains are given in Theorem 2 and Theorem 4. A precise statement of Theorem 1 and its proof can be found in Section 2. Theorem 1 extends some recent results of Carleson and Jones, see Section 10 of [CJ], and Jones and Makarov [M] who considered simply connected planar domains.

The next applications of Theorem 1 concern the Poincaré inequality on a domain in R^n. Let $D \subset R^n$ be a domain with finite volume and $1 \le p < \infty$. We denote by $W^{1,p}(D)$ the usual Sobolev space of functions on D that together with their first order weak partial derivatives are in $L^p(D)$. The norm for $W^{1,p}(D)$ is given by

$$\|u\|_{W^{1,p}(D)} = \Big(\int_D |u|^p\,dx + \int_D |\nabla u|^p\,dx\Big)^{1/p}.$$

Define

$$M_p(D) = \sup_u \frac{\left(\int_D |u - u_D|^p\,dx\right)^{1/p}}{\left(\int_D |\nabla u|^p\,dx\right)^{1/p}},$$

where the supremum is taken over all nonconstant $u \in W^{1,p}(D)$ and u_D is the average of u over D. D is said to be a p-Poincaré domain if $M_p(D) < \infty$.

In Theorem 1 [SS2] it was established that if D is a Hölder domain, then D is a p-Poincaré domain for $p \ge n$ and furthermore that this result is best possible for the class of all Hölder domains. Then Hurri [H] proved that a Hölder domain satisfying an additional geometric condition is a p-Poincaré domain for all $p > n - \varepsilon$, where ε depends on the domain. In Section 3, we use Theorem 1 to show that Hurri's geometric condition holds for all Hölder domains and hence the improved Poincaré inequality holds also. A special case of Theorem 1 [SS2] is that, if $D \subset \mathbf{C}$ is the image of the unit disk in the complex plane under a Hölder continuous Riemann mapping function, then $M_2(D) < \infty$. An extension of this result to other Riemann mapping functions is given in Section 3.

The final section of the paper is concerned with averaging domains. An averaging domain is a domain for which local BMO-type norm estimates imply a global BMO-type norm estimate. A precise definition can be found in Section 4. Our work here was motivated by that of Staples [S]. A characterization of averaging domains involving the quasi-hyperbolic metric is given, and then this and Theorem 2 are used to give the characterization of Hölder domains given in part (c) in the above theorem.

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2. Main result

Throughout this paper $W = \{Q\}$ will be a Whitney decomposition of D into closed dyadic cubes with disjoint interiors. This means that the coordinates of the vertices of each cube are dyadic rational numbers and that the diameter of each cube $Q \in W$, which we denote by $d(Q)$, is comparable to its Euclidean distance to ∂D. Further, the constants of comparability do not depend on D. See Chapter 6 of Stein's book [St] for the existence of such a decomposition.

The volume of a measurable subset E of R^n will be denoted by $m(E)$. The notation $a \approx b$ and $a \lesssim b$ will be used to mean that a and b are either comparable or satisfy an inequality with a constant depending only on the dimension.

The integral of Marcinkiewicz (associated with the domain D) is defined by

$$M(x) = \int_D \frac{\delta_D(y)}{|x-y|^{n+1}}\, dy.$$

See Chapter I of Stein's book [St]. We shall require the following fundamental result concerning these integrals; see Lemma 5 in [C1] or [Z]. We sketch a different proof which emphasizes the independence of the constants on the domain.

Lemma 1. *Let D be a domain in R^n and let Q_0 be a cube with $D \subset Q_0$. There are positive constants c_1 and c_2 depending only on the dimension n such that if $\lambda > 0$, then*

$$m\big(\{x \in Q_0 \setminus D \mid M(x) > \lambda\}\big) < c_1 m(Q_0) \exp(-c_2 \lambda).$$

Proof. We first use the Whitney decomposition of D to define a measure on $R^{n+1}_+ = \{(x,y) \mid x \in R^n, y > 0\}$. For $Q \in W$, let μ_Q be a point mass at $(x_Q, d(Q))$ of weight $m(Q)$, where x_Q is the center of Q, and let $\mu_D = \sum_{Q \in W} \mu_Q$. It is immediate that μ_D is a Carleson measure with norm bounded by a constant independent of D; that is, for an arbitrary cube $Q' \subset R^n$, $\mu(Q' \times d(Q')) \le c \cdot m(Q')$. It follows that the sweep of μ, $S_\mu(x) = \int P_y(x-t)\, d\mu(t,y)$ where $P_y(t)$ is the Poisson kernel on R^n, is a function of bounded mean oscillation with norm less than an absolute constant. See [C2] or Chapter 6 in [G]. Furthermore, an easy computation shows that for $x \in {}^cD$, $M(x) \approx S_\mu(x)$. The result is now a consequence of the John–Nirenberg Theorem [JN] (or see Chapter 6 in [G]) applied to the restriction of S_μ to Q_0.

For $r > 0$ and $x \in R^n$, let $T(r;x) = \{y \in D \mid |x-y| = r\}$, and define $\rho(r;x) = \max\{\delta_D(y) \mid y \in T(r;x)\}$.

Lemma 2. *Let $x_0 \in D$ and make the normalizing assumption that $\delta_D(x_0) = 1$. Suppose $x \in {}^cD$ is such that the distance from x to D satisfies $0 < \operatorname{dist}(x,D) = r_0 < 1$ and let c be a constant satisfying $1 < c < 1/r_0$. Then*

$$M(x) \gtrsim \left(\log \frac{1}{cr_0}\right)^{n+1} \cdot k_D^{-n}(x_0, T(cr_0;x)).$$

Proof. We first estimate $M(x)$ by using polar coordinates with center x. Let $d\sigma$ be surface measure on the unit sphere Ω in R^n.

$$
\begin{aligned}
M(x) &\geq \int_0^1 \int_\Omega \frac{\delta_D(rs)}{r^{n+1}} r^{n-1}\, d\sigma(s)\, dr \\
&\geq \int_{r_0}^1 \frac{1}{r^2} \int_{\{s\in\Omega \mid \delta_D(rs)\geq \rho(r;x)/2\}} \delta_D(rs)\, d\sigma(s)\, dr \qquad (2.1) \\
&\gtrsim \int_{r_0}^1 \frac{1}{r^2} \rho(r;x) \Big(\frac{\rho(r;x)}{r}\Big)^{n-1} dr \\
&= \int_{r_0}^1 \frac{\rho(r;x)^n}{r^{n+1}}\, dr.
\end{aligned}
$$

If γ is a rectifiable curve in D from x_0 to $x_1 \in T(cr_0;x)$, then

$$\int_{cr_0}^1 \frac{dr}{\rho(r;x)} \leq \int_\gamma \frac{ds}{\rho(|x-y|;x)} \leq \int_\gamma \frac{ds}{\delta_D(y)}.$$

Consequently,

$$\int_{cr_0}^1 \frac{dr}{\rho(r;x)} \leq k_D\big(x_0, T(cr_0;x)\big). \tag{2.2}$$

Thus by Hölder's inequality, (2.1) and (2.2), we have

$$
\begin{aligned}
\Big(\log \frac{1}{cr_0}\Big)^{n+1} &= \Big(\int_{cr_0}^1 \frac{\rho(r;x)^{n/(n+1)}}{r} \cdot \frac{1}{\rho(r;x)^{n/(n+1)}}\, dr\Big)^{n+1} \\
&\leq \int_{cr_0}^1 \frac{\rho(r;x)^n}{r^{n+1}}\, dr \cdot \Big(\int_{cr_0}^1 \frac{dr}{\rho(r;x)}\Big)^n \\
&\lesssim M(x) \cdot k_D^n\big(x_0, T(cr_0;x)\big),
\end{aligned}
$$

as required. This completes the proof of the lemma.

By refining the Whitney decomposition W of D, we may assume without loss of generality that if $Q \in W$ and $x_1, x_2 \in Q$, then $k_D(x_1,x_2) \leq 1/3$. That is, the quasi-hyperbolic diameter of every Whitney cube is less than $1/3$. Fix $x_0 \in D$, and for each $j \geq 0$, define

$$D_j = \cup\{Q \in W \mid k_D(x_0, Q) \leq j\}.$$

Let $x_1 \in D_{j-1}$ where $j \geq 1$. Then $k_D(x_0, x_1) \leq j - 2/3$ and it follows that $k_D(x_0, x) \leq j - 1/3$ for all points x within a ball centered at x_1 of radius $\varepsilon\delta_D(x_1)$, where ε is a positive numerical constant. Thus,

$$\delta_{D_j}(x) \approx \delta_D(x), \qquad x \in D_{j-1}. \tag{2.3}$$

Theorem 1. *Let D be bounded with $x_0 \in D$ and D_j as above, and put $\delta_0 = \delta_D(x_0)$. Suppose that $t > 0$ and $j \geq 1$ are given. There are positive constants c_3 and c_4 depending only on the dimension n such that*

$$\sum_{\substack{d(Q) \leq \delta_0 \exp(-tj) \\ Q \subset D_j \setminus D_{j-1}}} m(Q) \leq c_3 d(D)^n \exp(-c_4 t^{n+1} j). \tag{2.4}$$

Remark. We were led to this theorem by similar results, for simply connected planar regions, in [CJ] and [M].

Proof. By a dilation argument, we may assume that $\delta_0 = 1$. Fix $t > 0$ and $j \geq 1$, and suppose $Q \in W$ satisfies $d(Q) \leq \exp(-tj)$ and $Q \subset D_j \setminus D_{j-1}$. First assume that $1 \leq j \leq 4$. From Lemma 2.1 in [GP] we have that for some $x \in Q$,

$$\log \frac{\delta_0}{\delta_D(x)} \leq k_D(x_0, x) \leq j \leq 4 \tag{2.5}$$

and hence that

$$e^{-4} \leq \frac{\delta_D(x)}{\delta_0} \lesssim \frac{d(Q)}{\delta_0} \leq e^{-t}.$$

Thus, $t \lesssim 1$ and (2.4) follows.

Now suppose that $j \geq 4$. We may assume that tj is large, for if $tj \lesssim 1$, then $t \lesssim 1$ and (2.4) is trivially true. Fix $x \in Q$ and let $r_0 = \operatorname{dist}(x, D_{j-2})$. Observe that if $Q_1 \in W$ shares a boundary point with Q, then Q_1 is disjoint from D_{j-2}. For otherwise, $k_D(x_0, Q) \leq k_D(x_0, Q_1) + 1/3 < j - 1$ which violates the condition that $Q \notin D_{j-1}$. Thus, $r_0 \gtrsim d(Q)$, since neighboring Whitney cubes are comparable in size. By [GO] Lemma 1, there is a geodesic γ for the quasi-hyperbolic metric on D from x to x_0. Let $x_1 \in \gamma$ satisfy $k_D(x_0, x_1) = j - 3$. Then, since $k_D(x_0, x) \leq j + 1/3$, we have

$$\begin{aligned} 3 + \frac{1}{3} \geq k_D(x, x_1) &= \int_{\gamma(x,x_1)} \frac{ds}{\delta_D(y)} \\ &\geq \int_{\gamma(x,x_1)} \frac{ds}{\delta_D(x) + s} = \log\Big(\frac{|\gamma(x, x_1)| + \delta_D(x)}{\delta_D(x)}\Big), \end{aligned}$$

where $\gamma(x, x_1)$ is the portion of γ from x to x_1 and $|\gamma(x, x_1)|$ is its length. Thus we have shown that

$$\operatorname{dist}(x, D_{j-3}) \leq |\gamma(x, x_1)| \lesssim \delta_D(x) \lesssim r_0, \tag{2.6}$$

and it follows that $r_0 \approx d(Q)$.

We now estimate $M_{j-2}(x)$ by using Lemma 2, where M_{j-2} is the integral of Marcinkiewicz associated with the domain D_{j-2}. Let $T_{j-2}(cr_0; x) = \{y \in D_{j-2} \mid |x-y| = cr_0\}$, where $c \lesssim 1$ has been chosen so that $T_{j-2}(cr_0; x) \cap D_{j-3}$ is not empty. This is possible by (2.6). Thus, by (2.3),

$$k_{D_{j-2}}\big(x_0, T_{j-2}(cr_0; x)\big) \approx k_D\big(x_0, T_{j-2}(cr_0; x)\big) \leq j - 3.$$

Since $r_0 \approx d(Q) \leq \exp(-tj)$ and tj is large we obtain from Lemma 2 that

$$M_{j-2}(x) \gtrsim \Big(\log \frac{1}{cr_0}\Big)^{n+1} \cdot k_{D_{j-2}}^{-n}\big(x_0, T_{j-2}(cr_0; x)\big) \gtrsim t^{n+1} j. \tag{2.7}$$

Applying Lemma 1, with Q_0 being the smallest cube containing D, along with the above estimate for M_{j-2} shows that

$$\begin{aligned} \sum_{\substack{d(Q) \leq \exp(-tj) \\ Q \subset D_j \setminus D_{j-1}}} m(Q) &\leq m\big(\{x \in Q_0 \setminus D_{j-2} \mid M_{j-2}(x) > c_0 t^{n+1} j\}\big) \\ &\lesssim d(D)^n \exp(-c_0 c_2 t^{n+1} j), \end{aligned}$$

where the constant $c_0 \approx 1$ comes from (2.7). Thus, the proof is complete.

3. Applications to Hölder domains

In this section several applications of Theorem 1 are given. We begin with applications to Hölder domains. If $0 < \alpha \leq 1$ we say that a Hölder domain $D \subset R^n$ is an α-Hölder domain if there is $x_0 \in D$ and $C < \infty$ such that

$$k_D(x_0, x) \leq \frac{1}{\alpha} \log \frac{1}{\delta_D(x)} + C, \qquad x \in D. \tag{3.1}$$

The restriction $\alpha \leq 1$ is needed because of the inequality (2.5). This terminology derives from the fact that, for a simply connected proper subdomain D of R^2, there is a Riemann mapping function from the unit disk onto D that is Hölder continuous with exponent α if and only if (3.1) holds when k_D is replaced by the comparable hyperbolic metric [BP].

Theorem 2. *Let D be a domain in R^n and let $x_0 \in D$. The following are equivalent:*

(a) *There exists $\alpha > 0$ such that D is an α-Hölder domain;*
(b) *There exists $\beta > 0$ such that $m(\{x \in D \mid k_D(x_0, x) > j\}) = O(\exp(-\beta j))$, as $j \to \infty$;*
(c) *There exists $\tau > 0$ such that $\int_D \exp(\tau k_D(x_0, x))\, dx < \infty$.*

Moreover, the constants α, β and τ are related as follows: If (b) *or* (c) *holds, then both hold with $\beta \approx \tau$; if D is an α-Hölder domain, then* (b) *holds with $\beta \gtrsim \alpha^{n+1}$; if* (b) *holds with $\beta > 0$, then D is a β/n-Hölder domain.*

Remark. It can be shown that if a proper subdomain D of R^n satisfies the integrability condition in part (c) above, then necessarily $\tau < 1$.

Proof. We assume without loss of generality that $\delta_D(x_0) = 1$. Suppose first that D is an α-Hölder domain with Whitney decomposition $W = \{Q\}$, and let D_j be as in Section 2. By (3.1) and the properties of a Whitney decomposition, if j is sufficiently large, then

$$d(Q) \le \exp\Big(-\frac{\alpha}{2} j\Big), \qquad Q \subset D_j \setminus D_{j-1}.$$

Together with Theorem 1, this gives that if j is large, then

$$m(D_j \setminus D_{j-1}) = \sum_{\substack{d(Q) \le \exp(-\alpha j/2) \\ Q \subset D_j \setminus D_{j-1}}} m(Q) \le c_3 d(D)^n \exp\Big(-c_4\Big(\frac{\alpha}{2}\Big)^{n+1} j\Big).$$

Hence,

$$m(\{x \in D \mid k_D(x_0, x) > j\}) \le \sum_{i \ge j} m(D_i \setminus D_{i-1}) \le C \exp(-\beta j),$$

where $\beta = c_4(\alpha/2)^{n+1}$ and the constant C depends on the domain D. Thus (b) holds.

Next suppose that (b) holds with $\beta > 0$, and fix $Q \in W$. Then

$$d(Q)^n \le m(\{x \in D \mid k_D(x_0, x) > k_D(x_0, Q)\}) \le C \exp(-\beta k_D(x_0, Q)).$$

By taking logarithms and using that $d(Q) \approx \delta_D(x)$ for $x \in Q$ and that the quasi-hyperbolic diameter of a Whitney cube is comparable to 1, it follows that (3.1) holds for $x \in Q$ with $\alpha = \beta/n$ and $C \lesssim 1$. Thus, since $Q \in W$ was arbitrary, D is a β/n-Hölder domain.

The equivalence of (b) and (c) is elementary and is left to the reader. This completes the proof.

Corollary 1. *Let D be an α-Hölder domain with Whitney decomposition $W = \{Q\}$. There is a constant $\beta \gtrsim \alpha^{n+1}$ such that*

$$m\big(\cup\{Q \in W \mid d(Q) = 2^{-j}\}\big) = O(2^{-\beta j}), \qquad \text{as } j \to \infty. \tag{3.2}$$

Remark. A domain satisfying (3.2) is said to satisfy a Whitney-# condition by Martio and Vuorinen. In [MV], they prove that this condition holds for domains satisfying their c-covering condition. However, it is not hard to construct Hölder domains which do not satisfy this condition and hence this corollary does not follow from their results. In fact, it is unlikely that it follows from their techniques since Hölder domains are essentially more complicated domains.

Proof. By [GP] Lemma 2.1, we have that

$$\log\frac{\delta_D(x_0)}{\delta_D(x)} \le k_D(x_0, x),$$

so $\{Q \in W \mid d(Q) = 2^{-j}\} \subset \{x \in D \mid k_D(x_0, x) \gtrsim j\}$. The result now follows from Theorem 2.

The next corollary concerns the Minkowski dimension of the boundary of a Hölder domain. For the definition of the Minkowski dimension of a set in R^n, see [MV] where it is shown that the Hausdorff dimension of a set is less than or equal to its Minkowski dimension.

Corollary 2. *Let $D \subset R^n$ be an α-Hölder domain. There is a constant $C > 0$ depending only on the dimension n such that the Minkowski dimension of the boundary of D satisfies*

$$\dim_M(\partial D) \le n - C\alpha^{n+1}.$$

Proof. A Hölder domain is bounded, see [GM] or [SS2], and by Corollary 4 [SS2], the Euclidean measure of its boundary is 0. Thus we may apply Theorem 3.12 [MV] and Corollary 1 to conclude that $\dim_M(\partial D) \le n - \beta$, where $\beta \gtrsim \alpha^{n+1}$ is the constant from Corollary 1. This completes the proof.

Remark 1. This result can also be established using condition (c) of Theorem 2 and the method of proof of Corollary 2 [SS2].

Remark 2. Jones and Makarov have recently established this result for simply connected planar domains [M].

The next corollaries concern the Poincaré inequality on a domain in R^n; see the introduction for definitions and notation. In [SS2] it was established that if D is a Hölder domain, then D is a p-Poincaré domain for $p \ge n$. We now are able to improve this bound on p when D is an α-Hölder domain.

Corollary 3. *Let $D \subset R^n$ be an α-Hölder domain. There is a constant $C > 0$ depending only on the dimension n such that $M_p(D) < \infty$ for $p > n - C\alpha^{n+2}$.*

Proof. Since D is an α-Hölder domain, D is bounded and (3.2) holds with $\beta \gtrsim \alpha^{n+1}$. Thus we may apply Theorem 7.12 of [H] to conclude that $M_p(D) < \infty$ for $p > n - \alpha \cdot \beta$. This finishes the proof.

Remark. An alternate proof of Corollary 3 can be based on Theorem 2 (a) implies (b) and Theorem 9 of [SS2].

In [SS2, Theorem 1] it was established that if $D \subset \mathbf{C}$ is the image of the unit disk in the complex plane under a Hölder continuous Riemann mapping function, then $M_2(D) < \infty$. Theorem 1 allows an extension of this to other Riemann mapping functions.

Corollary 4. *Let f be analytic and univalent in the unit disk. There is a positive constant c such that if*

$$(1-|z|)|f'(z)| \lesssim \exp\Big(-c\Big(\log\frac{1}{1-|z|}\Big)^{2/3}\Big(\log\log\frac{1}{1-|z|}\Big)^{1/3}\Big), \tag{3.3}$$

for $|z| < 1$, then $M_2(D) < \infty$, where $D = f(\{z \in \mathbf{C} \mid |z| < 1\})$.

Proof. Let $\psi(t) = \exp\big(-c(\log(1/t))^{2/3}(\log\log(1/t))^{1/3}\big)$, $0 < t < 1$. Notice that $\int_0^1(\psi(t)/t)\,dt < \infty$, and so (3.3) implies that D is bounded. Thus $m(D) < \infty$, as is necessary for D to be a Poincaré domain. By [SS2, Theorem 9], it is sufficient to show that

$$\int_D k_D(f(0),x)\,dx < \infty. \tag{3.4}$$

The Koebe Distortion Theorem implies that $(1-|z|)|f'(z)| \approx \delta_D(f(z))$, and furthermore that the hyperbolic metric, ρ_D, on D satisfies

$$\tfrac{1}{2}\rho_D(x_1,x_2) \le k_D(x_1,x_2) \le 2\rho_D(x_1,x_2).$$

See Corollary 1.4 in [P] or [SS1]. Thus (3.3) can be restated as

$$\begin{aligned}\delta_D(f(z)) &\lesssim \psi(1-|z|) \le \psi\Big(2\exp\Big\{-\frac{k_D(f(0),f(z))}{2}\Big\}\Big)\\ &\lesssim \exp\Big\{-a\Big(\frac{\log k_D(f(0),f(z))}{k_D(f(0),f(z))}\Big)^{1/3}\cdot k_D(f(0),f(z))\Big\},\end{aligned}$$

where $a \approx c$.

Now let $W = \{Q\}$ be a Whitney decomposition of D, and define D_j as in Section 2 with $x_0 = f(0)$. We assume without loss of generality that $\delta_D(x_0) = 1$. We have shown that, for sufficiently large j,

$$\delta_D(f(z)) \le \exp\Big(-\frac{a}{2}\Big(\frac{\log j}{j}\Big)^{1/3}\cdot j\Big), \qquad f(z) \in D_i \setminus D_{i-1}.$$

Using this and Theorem 1, we get that

$$\begin{aligned} m(D_j \setminus D_{j-1}) &= \sum_{\substack{d(Q)\le \exp(-(a/2)(\log j/j)^{1/3}\cdot j) \\ Q\subset D_j\setminus D_{j-1}}} m(Q) \\ &\le c_3 d(D)^n \exp\Big(-c_4\frac{a^3}{8}\frac{\log j}{j}\cdot j\Big) = O(j^{-b}), \end{aligned}$$

where $b \approx a^3 \approx c^3$. Thus if c is sufficiently large,

$$\int_D k_D(x_0, x)\,dx \lesssim \sum_{j=1}^{\infty} j\cdot m(D_j \setminus D_{j-1}) < \infty.$$

This establishes (3.4) and completes the proof.

4. Averaging domains

Let φ be a continuous increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$ and let D be a domain, with $m(D) < \infty$. Following the treatment of Orlicz spaces given in Chapter VIII of [A], we define the Orlicz norm

$$\|u\|_{L_\varphi(D)} = \inf\Big\{k > 0 \mid \frac{1}{m(D)}\int_D \varphi\Big(\frac{|u(x)|}{k}\Big)\,dx \le 1\Big\}$$

for any measurable function u on D. It follows that

$$\frac{1}{m(D)}\int_D \varphi\Big(\frac{|u(x)|}{\|u\|_{L_\varphi(D)}}\Big)\,dx \le 1 \tag{4.1}$$

whenever $\|u\|_{L_\varphi(D)}$ is finite.

Definition. A domain D, with $m(D) < \infty$, is a φ-averaging domain if there exists $M < \infty$ so that

$$\|u - u_D\|_{L_\varphi(D)} \le M \sup_{B\subset D} \|u - u_B\|_{L_\varphi(B)} \tag{4.2}$$

whenever u is an integrable function on D and the supremum ranges over all balls B contained in D.

Remark 1. The use of the family of balls is not essential in the above definition. The following theorem is also valid if balls are replaced by cubes or more generally by the dilates of a bounded open set.

Remark 2. This definition with $L_\varphi(D)$ replaced by the $L^p(D)$ spaces, with $1 \le p < \infty$, is given by Staples in [S], where the theorem below is proved in that setting. This is the special case $\varphi(t) = t^p$ in the following theorem.

Theorem 3. *Suppose that φ is as above and that $\varphi(t) \le e^{bt}$ for some $0 \le b < \infty$ and all $t \ge 1$. A domain D is a φ-averaging domain if and only if*

$$\int_D \varphi\big(\varepsilon k_D(x_0, x)\big)\, dx < \infty \tag{4.3}$$

for some $\varepsilon > 0$ and $x_0 \in D$.

Proof. Suppose that D is a domain which satisfies (4.3). Then by the dominated convergence theorem,

$$\lim_{\tau \to 0} \frac{1}{m(D)} \int_D \varphi\big(\tau k_D(x_0, x)\big)\, dx = 0. \tag{4.4}$$

Let u be an integrable function satisfying $\sup_B \|u - u_B\|_{L_\varphi(B)} < \infty$. Then by Jensen's inequality and (4.1) the $\mathrm{BMO}(D)$ norm of u satisfies

$$\|u\|_* = \sup_{B \subset D} \frac{1}{m(B)} \int_B |u - u_B|\, dx \le \varphi^{-1}(1) \sup_{B \subset D} \|u - u_B\|_{L_\varphi(B)}. \tag{4.5}$$

In addition, we see that the convexity of φ yields a variant of the familiar BMO-inequality, namely,

$$\frac{1}{m(D)} \int_D \varphi\big(a|u - u_D|\big)\, dx \le \frac{1}{m(D)} \int_D \varphi\big(2a|u - c|\big)\, dx \tag{4.6}$$

for any constant c and any positive constant a.

Let $W = \{Q\}$ be a Whitney decomposition of D into cubes. We may assume that each cube $Q \in W$ is contained in a ball B_Q, with $Q \subset B_Q \subset D$ and $m(B_Q) \lesssim m(Q)$. Using the techniques in Lemma 2.11 in [S] we see that

$$|u_{B_Q} - u_{B_0}| \lesssim k_D(x_0, x_Q)\|u\|_* \tag{4.7}$$

where we assume that x_0 is the center of $Q_0 \in W$, x_Q is the center of B_Q and $B_0 = B_{Q_0}$.

Define a by $a^{-1} = \sup_B \|u - u_B\|_{L_\varphi(B)}$ which we assume to be positive and let τ be a small positive number. By (4.6) we obtain the first inequality below:

$$\frac{1}{m(D)} \int_D \varphi\Big(\frac{\tau a}{2}|u - u_D|\Big)\, dx \le \frac{1}{m(D)} \int_D \varphi(\tau a|u - u_{B_0}|)\, dx$$
$$\le \sum_Q \frac{1}{m(D)} \int_Q \varphi(\tau a|u - u_{B_Q}| + \tau a|u_{B_Q} - u_{B_0}|)\, dx.$$

We assume that τ is small enough so that the convexity of φ, (4.5) and (4.7) imply that

$$\begin{aligned}\varphi(\tau a|u - u_{B_Q}| + \tau a|u_{B_Q} - u_{B_0}|) &\le \varphi(\tau a|u - u_{B_Q}| + \tau c_1 k_D(x_0, x_Q))\\ &\le \varphi(\tau a|u - u_{B_Q}| + (1-\tau)\tau c_2 \inf_{x\in Q} k_D(x_0, x))\\ &\le \tau\varphi(a|u - u_{B_Q}|) + (1-\tau) \inf_{x\in Q} \varphi(\tau_0 k_D(x_0, x))\end{aligned}$$

where $\tau_0 = \tau c_2$. Hence by (4.1), (4.4) and the properties of Whitney decompositions we get that

$$\begin{aligned}\frac{1}{m(D)} \int_D \varphi(\tfrac{1}{2}\tau a|u - u_D|)\, dx &\le \sum_Q \Big\{\frac{\tau}{m(D)} \int_{B_Q} \varphi\Big(\frac{|u - u_{B_Q}|}{\|u - u_{B_Q}\|_{L_\varphi(B_Q)}}\Big)\, dx\\ &\qquad + \frac{1}{m(D)} \int_Q \varphi(\tau_0 k_D(x_0, x))\, dx\Big\}\\ &\le \tau c + \frac{1}{m(D)} \int_D \varphi(\tau_0 k_D(x_0, x))\, dx \le 1\end{aligned}$$

provided τ is sufficiently small. Thus,

$$\|u - u_D\|_{L_\varphi(D)} \le (\tfrac{1}{2}\tau a)^{-1} = \frac{2}{\tau} \sup_{B\subset D} \|u - u_B\|_{L_\varphi(B)}$$

which proves that D is a φ-averaging domain.

Conversely, we now assume that D is a φ-averaging domain. Put $u(x) = k_D(x_0, x)$. If $B \subset D$ is a ball of radius r and center x_B, then we have by (4.6) that

$$\begin{aligned}\frac{1}{m(B)} \int_B \varphi(a|u - u_B|)\, dx &\le \frac{1}{m(B)} \int_B \varphi\left(2a\big|k_D(x_0, x) - k_D(x_0, x_B)\big|\right)\, dx\\ &\lesssim \frac{1}{r^n} \int_{|x|\le r} \exp\Big(2ab \log \frac{r}{r - |x|}\Big)\, dx\\ &= \int_{|x|\le 1} \exp\Big(2ab \log \frac{1}{1 - |x|}\Big)\, dx \le 1\end{aligned}$$

provided a is sufficiently small. Hence, $\sup_{B\subset D}\|u-u_B\|_{L_\varphi(B)}\le a^{-1}<\infty$.

Put $u_j(x)=\min\big(u(x),j\big)$. By a straightforward argument, we see from the above that

$$\sup_{B\subset D}\|u_j-(u_j)_B\|_{L_\varphi(B)}\le a^{-1}$$

holds for u_j, all $j>0$, provided we make a smaller. Since $k(x_0,x)\lesssim 1$ on B_0 we have by (4.2), that

$$\begin{aligned}\frac{1}{m(D)}\int_D u_j\,dx &\lesssim \frac{1}{m(B_0)}\int_{B_0}\big|u_j-(u_j)_D\big|\,dx+1\\ &\le \frac{m(D)}{m(B_0)}\frac{1}{m(D)}\int_D\big|u_j-(u_j)_D\big|\,dx+1\\ &\le \frac{m(D)}{m(B_0)}\varphi^{-1}(1)\|u_j-(u_j)_D\|_{L_\varphi(D)}+1\\ &\le \frac{m(D)}{m(B_0)}\varphi^{-1}(1)Ma^{-1}+1.\end{aligned}$$

Hence k_D is integrable on D, since the right hand side is independent of j. We may therefore apply (4.2) and the above estimate for $\sup_B\|u-u_B\|_{L_\varphi(B)}$ to conclude that $\|u-u_D\|_{L_\varphi(D)}\le Ma^{-1}$.

Finally, we have

$$\begin{aligned}\int_D\varphi\big(\varepsilon k_D(x_0,x)\big)\,dx &\le \int_D\varphi\big(\varepsilon|u-u_D|+\varepsilon|u_D|\big)\,dx\\ &\le \tfrac12\int_D\varphi\big(2\varepsilon|u-u_D|\big)\,dx+\tfrac12 m(D)\varphi\big(2\varepsilon|u_D|\big)<\infty\end{aligned}$$

provided ε is small enough. This completes the proof.

Theorem 4. *Let $\varphi(t)=e^t-1$. A domain D is a φ-averaging domain if and only if D is a Hölder domain.*

Proof. If D is a Hölder domain then k_D is exponentially integrable by Theorem 2 and hence (4.3) holds and D is a φ-averaging domain. Conversely, if D is a φ-averaging domain, then by Theorem 3 there is a $\tau>0$ such that

$$\int_B\exp\big(\tau k_D(x_0,x)\big)\,dx\le 1+m(B)$$

for all balls $B\subset D$. But this clearly implies that D is a Hölder domain.

Lastly, we have the following generalization of the well known John–Nirenberg Theorem for cubes; see [JN].

Corollary 5. *A necessary and sufficient condition that a domain D be a Hölder domain is that there exists a $\tau > 0$ such that the inequality*

$$\frac{1}{m(D)} \int\limits_D \exp\left(\frac{\tau|u-u_D|}{\|u\|_*}\right) dx \le 2 \tag{4.8}$$

holds for all $u \in \mathrm{BMO}(D)$.

Proof. Let D be a Hölder domain and $u \in \mathrm{BMO}(D)$. By [S], we can replace the family of balls used in our definition of $\mathrm{BMO}(D)$ by the family of cubes contained in D. Thus, by the John–Nirenberg Theorem, there is a $\tau > 0$ such that

$$\frac{1}{m(Q)} \int\limits_Q \exp\left(\frac{\tau|u-u_D|}{\|u\|_*}\right) dx \le 2$$

holds for any cube $Q \subset D$ and $u \in \mathrm{BMO}(D)$. Let φ be as in Theorem 4, so that we have

$$\sup_{Q \subset D} \|u-u_Q\|_{L_\varphi(Q)} \le \tau^{-1}\|u\|_*$$

and hence by Remark 1 (following (4.2)) and Theorem 4 we see that (4.8) holds for a smaller value of τ.

Conversely, if (4.8) holds, then the inequality $\|u-u_D\|_{L_\varphi(D)} \le \tau^{-1}\|u\|_*$ is valid for all $u \in \mathrm{BMO}(D)$. Trivially, $\|u\|_*$ is dominated by $\sup_{Q\subset D} \|u-u_Q\|_{L_\varphi(D)}$ and hence by Theorem 4 and Remark 1, D is a Hölder domain.

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