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EXTENSIONS WITH BOUNDED $\overline{\partial}$-DERIVATIVE

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1. Introduction

In what follows D denotes the unit disk $\{|z|<1\}$ of the complex plane, Γ is the unit circle $\{|z|=1\}$, and $\overline{\partial}=\partial_x+i\partial_y$ is the usual complex derivative with respect to $\bar z$. For $q\in L^\infty(D)$, let

$$(Pq)(z)=-\frac{1}{\pi}\iint_D\frac{q(\zeta)\,d\xi\,d\eta}{\zeta-z},\qquad z\in\mathbf{C}.$$

Then [10] $(Pq)(z)$ is continuous with modulus of continuity $O(-\delta\log\delta)$ and has the generalized derivative $\overline{\partial}(Pq)=q$. Of course, since $\overline{\partial}h=0$ for any function h, holomorphic in D, $\overline{\partial}\big[(Pq)+h\big]=q$, also.

If, conversely, $F(z)$ is a complex-valued function, continuous in $D\cup\Gamma$, and if F has the generalized derivative $\overline{\partial}F=q$, $q\in L^\infty(D)$, then $F(z)=(Pq)(z)+h(z)$, where $h(z)$ is holomorphic in D and continuous in $D\cup\Gamma$. In the terminology of Ahlfors [3] F is a "quasiconformal deformation". (This is connected with the fact that F approximates the deviation from the identity mapping of a close-to-conformal quasiconformal mapping.) For given $\overline{\partial}F$, the additive holomorphic function h can always be determined, and uniquely, in fact, if we *normalize* F by requiring

$$\text{(1.1)}\qquad \operatorname{Re}\big[\bar zF(z)\big]=0\quad(z\in\Gamma),\qquad\text{and}\qquad F(1)=F(i)=F(-1)=0.$$

Under these circumstances F has the following representation [2] for $z\in D\cup\Gamma$.

$$\text{(1.2)}\qquad F(z)=-\frac{1}{\pi}\iint_D\Big[\frac{q(\zeta)}{\zeta-z}+\frac{z^3\overline{q(\zeta)}}{1-z\bar\zeta}-\beta(\zeta)z^2-B(\zeta)z+\overline{\beta(\zeta)}\Big]\,d\xi\,\delta\eta,$$

where

$$\beta(\zeta)=\Big(\frac{1}{i-\zeta}-\frac{i+\zeta}{1-\zeta^2}\Big)\frac{q(\zeta)}{2}+\Big(\frac{i+\bar\zeta}{1-\bar\zeta^2}-\frac{1}{i+\bar\zeta}\Big)\frac{\overline{q(\zeta)}}{2},$$

and

$$B(\zeta)=\frac{\overline{q(\zeta)}}{1-\bar\zeta^2}-\frac{q(\zeta)}{1-\zeta^2}.$$

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Our general purpose is to investigate firstly what functions F on Γ allow extensions to $D \cup \Gamma$ with bounded $\bar{\partial} F$, and, secondly, to minimize the sup norm $\|\bar{\partial} F\|_\infty$, when extensions are possible and F is subject to side-conditions on Γ. Some previous results for the first problem are found in [1] and [8].

It is well known ([10], Chapter I, Section 6, [3], Section 3) that $(Pq)(z)$ is (uniformly) "nearly-Lipshitz" in D; that is, there exists a number $C = C(q)$, such that

$$\left|(Pq)(z_2) - (Pq)(z_1)\right| \le C|z_2 - z_1| \log \frac{3}{|z_2 - z_1|}, \qquad \text{for any } z_1 \in D, z_2 \in D.$$

Therefore,

Theorem 1.1. *Every normalized quasiconformal deformation F is nearly-Lipshitz in $D \cup \Gamma$. In particular, a necessary condition on $F(e^{i\theta})$ to allow for an extension to a normalized quasiconformal deformation is that $F(e^{i\theta})$ is nearly-Lipshitz.*

A continuous function $\psi(x)$ is said to belong to the class Λ_* if there exists a constant A such that

$$\left|\psi(x+h) - 2\psi(x) + \psi(x-h)\right| \le Ah \tag{1.3}$$

for all x and all $h > 0$. This class was introduced by Zygmund [11] (See also [12], [4]), who showed, in particular, that every member of Λ_* is nearly-Lipshitz. On the other hand, not every nearly-Lipshitz function belongs to Λ_*. We will see (Theorem 2.2) that Λ_* is precisely the right class to characterize boundary values of quasiconformal deformations.

In Section 3 we will consider the problem of minimizing

$$\|\bar{\partial} F\|_\infty = \operatorname{ess\,sup}\left\{\left|\bar{\partial} F(z)\right| : z \in D\right\}$$

when F is normalized and F is specified at a finite number of points of Γ. This leads to a solution (Theorem 3.1) with a "Teichmüller"-type extremal function which can be found by a fairly constructive procedure. In Section 4, the corresponding problem when F is specified on *all* of Γ is considered. A necessary and sufficient condition for extremality of $\|\bar{\partial} F\|_\infty$ can in this case be expressed in terms of the norm of a linear functional over a Banach space of holomorphic functions. This condition is identical with a known condition from the theory of extremal quasiconformal mappings, thus providing a new characterization for the latter.

If in (1.3) we replace $O(h)$ by $o(h)$ we obtain Zygmund's class λ_* of so-called "smooth" functions. While Theorem 2.2 guarantees the *existence* of a normalized quasiconformal deformation if we are given boundary values $F(e^{i\theta})$ on Γ satisfying

(1.1) and belonging to Λ_*, we actually succeed in identifying the *optimal* extension (Theorem 4.2) if the hypothesis is strengthened by assuming that $F(e^{i\theta})$ belongs to λ_*.

In connection with the material in Section 3, the first-named author gratefully acknowledges helpful discussions with Professor L. Markus in Royal Leamington Spa and Professor V.D. Milman in Zurich that had the effect of convincing him to abandon the more cumbersome approach by way of the dual of $L^\infty(D)$.

2. Existence of deformations with given boundary values

Suppose $F(z)$, $z \in \Gamma$, is a continuous complex-valued function. We ask whether an extension of F to a quasiconformal deformation in $D \cup \Gamma$ exists. We will see that a transformation T, defined below, which acts on $F(z)$, $z \in \Gamma$, to produce a complex-valued function with domain D, plays a key role in answering the question.

Let

$$(2.1) \qquad (TF)(z) = \frac{(1-|z|^2)^3}{2\pi i} \int_\Gamma \frac{F(\zeta)}{(1-\bar{z}\zeta)^3(\zeta - z)}\, d\zeta, \qquad (z \in D).$$

Lemma 2.1. [7] *(TF) defines a continuous extension from Γ to $D \cup \Gamma$.*

Proof. We have

$$(TF)(z) = \int_\Gamma F(\zeta) R(z,\zeta)\, |d\zeta|, \qquad R(z,\zeta) = \frac{(1-|z|^2)^3}{2\pi(1-\bar{z}\zeta)^2 |z-\zeta|^2}.$$

Thus, if $\Pi(z,\zeta)$ denotes the Poisson kernel,

$$R(z,\zeta) = \Big(\frac{1-|z|^2}{1-\bar{z}\zeta}\Big)^2 \Pi(z,\zeta) = \Big[1 + \frac{\zeta - z}{1-\bar{z}\zeta}\bar{z}\Big]^2 \Pi(z,\zeta).$$

Therefore, on the one hand,

$$\int_\Gamma R(z,\zeta)\, |d\zeta| = 1, \qquad (z \in D),$$

and on the other hand,

$$\big|R(z,\zeta)\big| \le 4\Pi(z,\zeta), \qquad (z \in D).$$

One can write down an explicit formula for (TF) in terms of the complex-harmonic extension of F. To see this, suppose first that $F(\zeta) = u(\zeta)$, $\zeta \in \Gamma$, where $u(\zeta)$ is real. Let $u(z)$, $z \in D$, denote the harmonic extension of u from Γ to

D, and let $f(z)$, $z \in D$, be holomorphic with $\operatorname{Re} f(z) = u(z)$. Let $\gamma(z) = z^2 f(z)$. Since

$$\frac{1}{2\pi i}\int_\Gamma \frac{u(\zeta)}{(1-\bar{z}\zeta)^3(\zeta - z)}\,d\zeta = \lim_{R\to 1-}\Big[\frac{1}{4\pi i}\int_\Gamma \frac{f(R\zeta)}{(1-\bar{z}\zeta)^3(\zeta - z)}\,d\zeta + \frac{1}{4\pi i}\int_\Gamma \frac{\overline{f(R\zeta)}}{(1-\bar{z}\zeta)^3(\zeta - z)}\,d\zeta\Big],$$

we get

$$\begin{aligned}(TU)(z) &= u(z) - \tfrac{1}{2}(1-|z|)^4\overline{f(z)} + \tfrac{1}{2}z(1-|z|^2)\overline{\gamma'(z)} + \tfrac{1}{4}(1-|z|^2)^2\overline{\gamma''(z)} \\ &= u(z) + \tfrac{1}{4}(1-|z|^2)\bar{z}\Big[2(2-|z|^2)\overline{f'(z)} + \bar{z}(1-|z|^2)\overline{f''(z)}\Big],\end{aligned}$$

or

$$(Tu)(z) = u(z) + \tfrac{1}{2}(1-|z|^2)\bar{z}\left[2(2-|z|^2)\bar{\partial}u(z) + \bar{z}(1-|z|^2)\bar{\partial}^2 u(z)\right]. \tag{2.2}$$

Since the operators T and $\bar{\partial}$ are both linear, (2.2) remains valid when $u(e^{i\theta})$ is a complex-valued continuous function, and $u(z)$, $z \in D$, denotes its complex-harmonic extension.

A useful formula is obtained by differentiating (2.1) with respect to $\bar{z}$. The result is

$$\bar{\partial}(TF) = \frac{3(1-|z|^2)^2}{2\pi i}\int_\Gamma \frac{F(\zeta)}{(1-\bar{z}\zeta)^4}\,d\zeta. \tag{2.3}$$

Lemma 2.2. *Let $F(z)$, $z \in \Gamma$, be a continuous complex-valued function. A necessary and sufficient condition that F has an extension to $D \cup \Gamma$ possessing a bounded $\bar{\partial}$-derivative in D is that*

$$\int_\Gamma \frac{F(\zeta)}{(1-\bar{z}\zeta)^4}\,d\zeta = O\Big[\frac{1}{(1-|z|)^2}\Big], \qquad z \in D. \tag{2.4}$$

(i) *Proof of sufficiency.* By Lemma 2.1 (TF) provides an extension of F, and if (2.4) holds, then, by (2.3), $\bar{\partial}(TF)$ is bounded.

(ii) *Proof of necessity.* Suppose $G(z)$, $z \in D$, is an extension of F, and $|\bar{\partial}G(z)| \le M$, $z \in D$. By Green's formula, we can rewrite (2.3) as

$$\bar{\partial}(TF)(z) = \frac{3(1-|z|^2)^2}{\pi}\iint_D \frac{\bar{\partial}G(\zeta)}{(1-\bar{z}\zeta)^4}\,d\zeta\,d\eta, \qquad (z \in D).$$

If we apply the Möbius transformation $w = (\zeta - z)/(1 - \bar{z}\zeta)$ in the integrand, we obtain

$$\overline{\partial}(TF)(z) = \frac{3}{\pi}\iint_D \frac{(1+\bar{z}w)^2}{(1+z\bar{w})^2}\overline{\partial}G(\zeta)\Big(\frac{w+z}{1+\bar{z}w}\Big)\,du\,dv.$$

Hence,

$$\big|\overline{\partial}(TF)(z)\big| \le 3M.$$

We now turn to considering functions on Γ belonging to the class Λ_*. We shall need to refer to two important facts about Λ_*, both due to Zygmund [11], that we list as Lemmas 2.3 and 2.4 below. The formulations here, taken from [4], are particularly convenient for our purpose.

Lemma 2.3. *Suppose $f(z)$ is holomorphic in D, and suppose $\operatorname{Re} f(z)$ is continuous in $D \cup \Gamma$. If $\operatorname{Re} f(e^{i\theta}) \in \Lambda_*$ then $\operatorname{Im} f(z)$ is continuous in $D \cup \Gamma$ and $\operatorname{Im} f(e^{i\theta}) \in \Lambda_*$.*

Lemma 2.4. *Suppose $f(z)$ is holomorphic in D. Then $f(z)$ is continuous in $D \cup \Gamma$ and $f(e^{i\theta}) \in \Lambda_*$ if and only if*

$$f''(re^{i\theta}) = O\Big[\frac{1}{1-r}\Big].$$

Theorem 2.1. *Let $F(z)$, $z \in \Gamma$, be a continuous complex-valued function. Suppose $F(e^{i\theta}) \in \Lambda_*$. Then $(TF)(z)$ provides an extension of F to $D \cup \Gamma$ with bounded $\overline{\partial}$-derivative in D.*

Proof. Without loss of generality, $F(z) = u(z)$, $z \in \Gamma$, $u(z) = \operatorname{Re} f(z)$, where $f(z)$ is holomorphic in D. We have

$$\frac{1}{2\pi i}\int_\Gamma \frac{u(\zeta)}{(1-\bar{z}\zeta)^4}\,d\zeta = \overline{\frac{1}{12}\frac{d^3}{dz^3}\big[z^2 f(z)\big]} = \frac{1}{12}\overline{\gamma'''(z)}, \quad (z \in D),$$

with $\gamma(z) = z^2 f(z)$. By Lemma 2.3, $f(z)$ has a continuous extension to $D \cup \Gamma$ and $f(e^{i\theta}) \in \Lambda_*$. It easily follows that $\gamma(e^{i\theta}) \in \Lambda_*$, also. Thus, by Lemma 2.4,

$$\gamma''(re^{i\theta}) = O\Big[\frac{1}{1-r}\Big].$$

By a classical result of Hardy and Littlewood [4, p. 80] it follows that

$$\gamma'''(re^{i\theta}) = O\Big[\frac{1}{(1-r)^2}\Big].$$

The conclusion of the theorem therefore follows by Lemma 2.2.

Note that the functions $u(z)$, harmonic in D, for which $u(e^{i\theta}) \in \Lambda_*$ need not have $\overline{\partial} u$ bounded. An example is $u(z) = \operatorname{Re} f(z)$,

$$f(z) = z + (1 - z)\log(1 - z).$$

Since $f''(z) = (1 - z)^{-1}$, it follows by Lemma 2.4 that $u(e^{i\theta}) \in \Lambda_*$, but $\overline{\partial} u(z) = -(1/2)\log(1 - \bar{z})$ is unbounded in D. It is also clear that if $F(z)$ is continuous in $D \cup \Gamma$ and $\overline{\partial} F(z)$ is bounded in D, it does not follow that $F(e^{i\theta}) \in \Lambda_*$. For example, if $F(z) = (1 - z)^{1/2}$ then $\overline{\partial} F = 0$, but $F(e^{i\theta})$ does not belong to Λ_*. However, as the following result shows, if we normalize $F(e^{i\theta})$ by condition (1.1), then the requirement that $F(e^{i\theta}) \in \Lambda_*$ becomes both necessary and sufficient.

Theorem 2.2. *Suppose $F(z)$, $z \in \Gamma$, is a continuous complex-valued function, with*

$$\operatorname{Re}\left[\bar{z}F(z)\right] = 0, \qquad (z \in \Gamma), \qquad F(1) = F(i) = F(-1) = 0.$$

Then $F(z)$ has a continuous extension to $D \cup \Gamma$ with bounded generalized $\overline{\partial}$-derivative if and only if $F(e^{i\theta}) \in \Lambda_*$.

Proof. Theorem 2.1 tells us that the condition $F(e^{i\theta}) \in \Lambda_*$ is sufficient. It remains to verify that this condition is necessary. (This also follows from Theorem 2 of [1].)

For $z \in \Gamma$, (1.2) becomes

$$F(z) = \frac{2z}{\pi i} g(z) + \frac{1}{\pi} \iint_D \left[\beta(\zeta) z^2 + B(\zeta) z - \overline{\beta(\zeta)}\right] d\xi \, d\eta, \tag{2.5}$$

where

$$g(z) = \operatorname{Im} \iint_D \frac{g(\zeta)}{z(\zeta - z)} \, d\xi \, d\eta.$$

It suffices to show that $g(e^{i\theta}) \in \Lambda_*$. Letting

$$\delta(h) = e^{ih} + e^{-ih} - 2,$$

we have

$$\begin{aligned} g(e^{i(\theta+h)}) &- 2g(e^{i\theta}) + g(e^{i(\theta-h)}) \\ &= \operatorname{Im} \iint_D e^{-i\theta} q(\zeta) \frac{\zeta(\zeta + e^{i\theta})\delta(h) - e^{i\theta}(\zeta - e^{i\theta})\delta(2h)}{(\zeta - e^{i\theta})(\zeta - e^{i(\theta+h)})(\zeta - e^{i(\theta-h)})} \, d\xi \, d\eta \\ &= I_1 + I_2. \end{aligned}$$

$$|I_1| \le C\big|\delta(h)\big| \iint_D \frac{d\xi\, d\eta}{\big|(\zeta - e^{i\theta})(\zeta - e^{i(\theta+h)})(\zeta - e^{i(\theta-h)})\big|}$$
$$\le C|e^{ih} - 1| \iint_{\mathbf{C}} \frac{du\, dw}{\big|w(w-1)(w+e^{-ih})\big|}$$

where the substitution, $\zeta - e^{i\theta} = (e^{ih} - 1)e^{i\theta} w$, was used. Clearly, $I_1 = O(h)$, as $h \to 0$.

$$|I_2| \le \|q\|_\infty \big|\delta(2h)\big| \iint_D \frac{d\xi\, d\eta}{\big|(\zeta - e^{2ih})(\zeta - 1)\big|}.$$

Setting $\zeta - 1 = (e^{2ih} - 1)w$, so that $\zeta - e^{2ih} = (e^{2ih} - 1)(w - 1)$, one verifies easily that $I_2 = O(-h^2 \log h)$ as $h \to 0$. Therefore,

$$g\big(e^{i(\theta+h)}\big) - 2g(e^{i\theta}) + g\big(e^{i(\theta-h)}\big) = O(h),$$

uniformly with respect to θ.

3. Extremal deformations for the N-point problem

Suppose z_1, z_2, ..., z_N, ($N \ge 3$), are distinct given points of Γ, and $F(z_n)$, $n = 1, 2, \ldots, N$, are given complex numbers consistent with condition (1.1); that is,

$$F(z_n) = i\alpha_n z_n, \qquad n = 1, 2, \ldots, N, \alpha_n \in \mathbf{R}. \tag{3.1}$$

Let $\mathcal{F} = \mathcal{F}(z_1, z_2, \ldots . z_N; \alpha_1, \alpha_2, \ldots, \alpha_N)$ denote the class of functions $q \in L^\infty(D)$ for which $F(z)$ as defined by (1.2) satisfies (1.1) and (3.1); that is, $\mathcal{F}$ is the class of $\overline{\partial}$-derivatives of normalized quasiconformal deformations for which (3.1) holds on Γ. It is obvious that one can determine $F(e^{i\theta})$ in Λ_* so that (1.1) and (3.1) hold. Thus, by Theorem 2.2, $\mathcal{F}$ is certainly non-empty.

Our problem is to "determine"

$$M_0 = \inf\big\{\, \|q\|_\infty : q \in \mathcal{F} \big\}, \tag{3.2}$$

and to describe extremal members of $\mathcal{F}$, if any, for which the infimum is attained. The solution is as follows.

Theorem 3.1. *$\mathcal{F}(z_1, z_2, \ldots, z_N; \alpha_1, \alpha_2, \ldots, \alpha_N)$ contains a unique element q_0 for which $\|q_0\|_\infty = M_0$. Unless $q_0(z) \equiv 0$, $q_0(z)$ is of the form*

$$q_0 = M_0 \frac{\overline{\varphi_0(z)}}{\big|\varphi_0(z)\big|},$$

where $\varphi_0(z)$ is a rational function, holomorphic in D, possessing at worst simple poles at the points 1, i, -1, and z_1, z_2, ..., z_N.

Proof. In (2.5) we considered the expression for $F(z)$ which results from (1.2) for $z \in \Gamma$. More explicitly,

$$F(z) = iz \operatorname{Re} \iint_D A(z,\zeta) q(\zeta)\, d\xi\, d\eta, \qquad (z \in \Gamma),$$

where

$$(3.3)\quad A(z,\zeta) = (2i/\pi)\Big[\frac{1}{z(\zeta - z)} - \frac{z}{(\zeta^2-1)(\zeta - i)} - \frac{\zeta^2 - i\zeta - 1}{(\zeta^2-1)(\zeta-i)z} - \frac{1}{\zeta^2-1}\Big].$$

So, if we put $A_n(z) = A(z_n, z)$, we can write

$$\begin{aligned}\mathcal{F} &= \mathcal{F}(z_1, z_2, \ldots, z_N; \alpha_1 \alpha_2, \ldots, \alpha_N)\\ &= \Big\{ q \in L^\infty(D) : \operatorname{Re} \iint_D A_n(z) q(z)\, dx\, dy = \alpha_n, \qquad n = 1,2,\ldots,N \Big\}.\end{aligned}$$

In this guise, problem (3.2) is identical with the classical one solved by F. Riesz in 1910 [9]. The text-book procedure would be to consider the vector space over the reals spanned by $\{A_n(z)\}$ as a subspace of $L^1(D)$, and apply the Hahn–Banach theorem. We prefer to outline the pre-Hahn–Banach procedure used by Riesz because of its more constructive nature. (In order to avoid complications, it is best to avoid the natural temptation of interpreting (3.2) in the geometry of $L^\infty(D)$.)

For $\mu_n \in \mathbf{R}$, $n = 1,2,\ldots,N$, set

$$V(\mu_1,\mu_2,\ldots,\mu_N) = \Big|\sum_{n=1}^N \mu_n \alpha_n\Big|, \quad \Phi(\mu_1,\mu_2,\ldots,\mu_N) = \iint_D \Big|\sum_{n=1}^N \mu_n A_n(z)\Big|\, dx\, dy.$$

Let

$$\begin{aligned}M_1 &= \sup\Big\{ \frac{V(\mu_1,\mu_2,\ldots,\mu_N)}{\Phi(\mu_1,\mu_2,\ldots,\mu_N)} : \mu_n \in \mathbf{R}, n = 1,2,\ldots,N \Big\}\\ &= \sup\big\{ V(\mu_1,\mu_2,\ldots,\mu_N) : \Phi(\mu_1,\mu_2,\ldots,\mu_N) = 1 \big\}.\end{aligned}$$

Since the functions $\{A_n(z)\}$ are obviously linearly independent, we have $M_1 < \infty$. It is also clear that $(\mu_1,\mu_2,\ldots,\mu_N)$ can be restricted to a compact subset of R^N without affecting the supremum, and that the supremum is therefore attained at some point $(\mu_1,\mu_2,\ldots,\mu_N) = (m_1,m_2,\ldots,m_N)$. Therefore, for some real Lagrange multiplier c,

$$\frac{\partial V}{\partial \mu_n}(m_1,m_2,\ldots,m_N) = c \frac{\partial \Phi}{\partial \mu_n}(m_1,m_2,\ldots,m_N), \qquad n = 1,2,\ldots,N.$$

Thus,

$$\frac{\sum_n m_n\alpha_n}{\left|\sum_n m_n\alpha_n\right|}\alpha_n = c\,\mathrm{Re}\iint_D \frac{\sum_n m_n\overline{A_n(z)}}{\left|\sum_n m_n A_n(z)\right|}A_k(z)\,dx\,dy, \quad k=1,2,\ldots,N, \tag{3.4}$$

and

$$\Phi(m_1,m_2,\ldots,m_N)=1. \tag{3.5}$$

Multiplying both sides of (3.4) by m_k, and summing over k and using (3.5), we obtain

$$\alpha_k = \Big(\sum_n m_n\alpha_n\Big)\,\mathrm{Re}\iint_D \frac{\sum_n m_n\overline{A_n(z)}}{\left|\sum_n m_n A_n(z)\right|}A_k(z)\,dx\,dy,\ k=1,2,\ldots,N. \tag{3.6}$$

So, if we define $q_0(z)$ by

$$q_0 = \Big(\sum_{n=1}^N m_n\alpha_n\Big)\frac{\sum_{n=1}^N m_n\overline{A_n(z)}}{\left|\sum_{n=1}^N m_n A_n(z)\right|}, \qquad z\in D,$$

we see that $|q_0(z)| = M_1$ a.e. in D. (At the same time, we have an independent proof that $\mathcal{F}\neq\emptyset$, since (3.6) shows that $\mathcal{F}$ contains q_0.)

Now, if q is an arbitrary element of $\mathcal{F}$, and $\mu_n\in\mathbf{R}$, $n=1,2,\ldots,N$, then

$$\sum_{n=1}^N \mu_n\alpha_n = \mathrm{Re}\iint_D \Big[\sum_{n=1}^N \mu_n A_n(z)\Big]q(z)\,dx\,dy.$$

Thus,

$$V(\mu_1,\mu_2,\ldots,\mu_N)\le \|q\|_\infty\,\Phi(\mu_1,\mu_2,\ldots,\mu_N). \tag{3.7}$$

Therefore, $\|q\|_\infty \ge M_1$ for all $q\in\mathcal{F}$. In particular,

$$M_1 \le \|q_0\|_\infty = V(\mu_1,\mu_2,\ldots,\mu_N) = M_1.$$

It follows that $M_1 = M_0$, and this establishes $q_0(z)$ as an extremal function for (3.2). The assertion regarding uniqueness is proved by tracing back the implications of equality in (3.6).

4. Boundary values on all of Γ

In contrast with Section 3, we now suppose that the role of $\{z_1, z_2, \ldots, z_N\}$ is taken over by the complete circle Γ. Suppose the function $F(e^{i\theta})$, $0 \le \theta \le 2\pi$, belongs to Λ_* and satisfies the normalization conditions (1.1). Let $\mathcal{F} = \mathcal{F}[F]$ denote the class of functions $q \in L^\infty(D)$ that are $\overline{\partial}$-derivatives of quasiconformal deformations of $D \cup \Gamma$ with boundary values F on Γ. By Theorem 2.2, $\mathcal{F}$ is non-empty. We again consider the extremal problem (3.2). In view of the representation (1.2) the dominated convergence theorem guarantees the existence of extremal functions $q_0 \in \mathcal{F}[F]$ for which $\|q_0\|_\infty = \inf\{\, \|q\|_\infty : q \in \mathcal{F}[F]\}$. We will see that an extremal q can be characterized in a manner familiar [5] from the theory of extremal quasiconformal mappings with specified boundary values.

In order to go ahead it is best to alter the approach somewhat. Instead of starting with $F(z)$, $z \in \Gamma$, as "given", assume that we are given a function $q(z)$ belonging to $L^\infty(D)$. Using the function $A(z,\zeta)$ from (3.3), let

$$\alpha(e^{i\theta}) = \operatorname{Re} \iint_D A(e^{i\theta},\zeta) q(\zeta)\, d\xi\, d\eta, \qquad F_q(e^{i\theta}) = ie^{i\theta}\alpha(e^{i\theta}), \qquad 0 \le \theta \le 2\pi.$$

The subscript q is used to indicate that the normalized boundary function $F_q(e^{i\theta})$, $0 \le \theta \le 2\pi$, is induced by q. The question we ask is whether or not

$$\|q\|_\infty = \inf\{\, \|Q\|_\infty : Q \in \mathcal{F}[F]\}.$$

If the answer is *yes*, then q is an extremal $\overline{\partial}$-derivative of the induced normalized quasiconformal deformation, or we say, for short, that *q is extremal.*

To proceed with the characterization of extremality, let $\mathcal{B}$ denote the set of functions holomorphic in D that belong to $L^1(D)$.

Lemma 4.1. *$q_1 \in \mathcal{F}[F_q]$ if and only if*

$$\iint_D q_1(z)\varphi(z)\, dx\, dy = \iint_D q(z)\varphi(z)\, dx\, dy \qquad \textit{for all } \varphi \in \mathcal{B}.$$

Proof. Let $F_1(z)$, $F(z)$, $(z \in D)$, be determined for $q_1(z)$, $q(z)$, respectively, by means of (1.2). Since $\varphi(z)$ can be approximated in the norm of $L^1(D)$ by $\varphi(Rz)$, $R \to 1-$, there is no loss of generality in assuming that $\varphi(z)$ is holomorphic in $D \cup \Gamma$. Evidently, $\overline{\partial}(F_1 - F) = q_1 - q$ in the sense of distributions. So, the assertion follows from Green's formula, applied to $[F_1(z) - F(z)]\varphi(z)$.

Theorem 4.1. *Suppose $q \in L^\infty(D)$. Then q is extremal if and only if*

$$\sup_{\varphi \in \mathcal{B}} \frac{\left| \iint_D q(z)\varphi(z)\, dx\, dy \right|}{\iint_D |\varphi(z)|\, dx\, dy} = \|q\|_\infty. \tag{4.1}$$

(i) *Proof that* (4.1) *is necessary for extremality.* The left side of (4.1) is the norm of the linear functional

$$\mathcal{L}_q[\varphi] = \iint_D q(z)\varphi(z)\,dx\,dy,$$

over the Banach space $\mathcal{B}$. If, contrary to the assertion of the theorem, the norm of $\mathcal{L}_q$ satisfied $\|\mathcal{L}_q\| = \varrho < \|q\|_\infty$, then, forming the Hahn–Banach extension of $\mathcal{L}_q$ from $\mathcal{B}$ to $L^1(D)$, we arrive at a function $q_1 \in L^\infty(D)$ with $\|q_1\|_\infty = \varrho$ and such that

$$\iint_D q_1(z)\varphi(z)\,dx\,dy = \iint_D q(z)\varphi(z)\,dx\,dy \qquad \text{for all } \varphi \in \mathcal{B}.$$

By Lemma 4.1, $q_1 \in \mathcal{F}[F_q]$. Since q is extremal, we must therefore have

$$\varrho = \|q_1\|_\infty \geq \|q\|_\infty,$$

a contradiction.

(ii) *Proof that* (4.1) *is sufficient for extremality.* Let

$$M_0 = \inf\left\{ \|\tilde{q}\|_\infty : \tilde{q} \in \mathcal{F}[F_q]\right\},$$

and let q_0, be an extremal function in the class $\mathcal{F}[F_q]$. By Lemma 4.1,

$$\iint_D q_0(z)\varphi(z)\,dx\,dy = \iint_D q(z)\varphi(z)\,dx\,dy \qquad \text{for all } \varphi \in \mathcal{B}.$$

Therefore,

$$\left|\iint_D q(z)\varphi(z)\,dx\,dy\right| \leq \|q_0\|_\infty \iint_D |\varphi(z)|\,dx\,dy = M_0 \iint_D |\varphi(z)|\,dx\,dy$$

for all $\varphi \in \mathcal{B}$. By (4.1), this implies that $\|q\|_\infty \leq M_0$. But $q \in \mathcal{F}[F_q]$. Hence, by the definition of M_0, $\|q\|_\infty \geq M_0$. Thus, $\|q\|_\infty = M_0$.

Using Theorem 4.1 one sees that a simple example of an extremal q is obtained if we take

$$q(z) = k\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}, \tag{4.2}$$

where $k \in \mathbf{R}$ and $\varphi_0 \in \mathcal{B}$. Since condition (4.1) is identical with the one obtained for extremality of quasiconformal mappings with given boundary values, known results (e.g. [5], [6], as well as classical work of Strebel referred to in [5]) imply that $q(z)$ may be extremal even though $|q(z)|$ is not constant, and a given class $\mathcal{F}[F_q]$ may contain more than one extremal. Not every normalized boundary value function $F(e^{i\theta})$ that is of class Λ_* can be induced by a $q(z)$ of type (4.2). The following result shows, however, that if $F(e^{i\theta})$ belongs to λ_* (In particular, this means that "corners" are ruled out.) then a function q of type (4.2) that induces $F(e^{i\theta})$ *does* exist. Such a q is then automatically extremal.

Theorem 4.2. *Suppose $F(e^{i\theta})$ satisfies (1.1) and belongs to λ_*. There exist $M_0 \geq 0$ and $\varphi_0 \in \mathcal{B}$ such that $q_0(z) = M_0\overline{\varphi_0(z)}/|\varphi_0(z)|$ determines an extremal normalized quasiconformal deformation with boundary values $F(e^{i\theta})$.*

Proof. Let q_0 be extremal in the class $\mathcal{F}$ determined by $F(e^{i\theta})$, and let $M_0 = \|q_0\|_\infty$. We can assume that $M_0 > 0$; otherwise the result is trivial.

If we replace Λ_* by λ_* in the hypothesis of Lemma 2.3 then [11, 12] we can replace Λ_* by λ_* in the conclusion. Also [11, Theorem 13], in the conclusion of Lemma 2.4 O can then be replaced by o. Proceeding as in the proof of Theorem 2.1 we obtain $(1-r)^2\gamma'''(re^{i\theta}) = o(1)$, as $r \to 1-$. This means that

$$q_1(z) = \overline{\partial}(TF)(z) = o(1) \qquad \text{as } z \to \Gamma, \ (z \in D). \tag{4.3}$$

Since $q_1 \in \mathcal{F}$,

$$\iint_D q_0(z)\varphi(z)\,dx\,dy = \iint_D q_1(z)\varphi(z)\,dx\,dy, \qquad \varphi \in \mathcal{B}. \tag{4.4}$$

By Theorem 4.1, there exist $\varphi_n \in \mathcal{B}$, with

$$\iint_D |\varphi_n(z)|\,dx\,dy = 1, \qquad n = 1, 2, \ldots,$$

such that

$$\lim_{n\to\infty} \iint_D q_0(z)\varphi_n(z)\,dx\,dy = M_0.$$

Therefore, by (4.4),

$$\lim_{n\to\infty} \iint_D q_1(z)\varphi_n(z)\,dx\,dy = M_0. \tag{4.5}$$

The possibility that $\lim \varphi_n(z) = 0$ locally uniformly in D can be ruled out since, by (4.3), it would lead to the conclusion that the left side of (4.5) is zero. Under these circumstances the corollary of Lemma 0.3 of [6] implies that q_0 must have the form (4.2).

Addendum. After submission of this paper, Aimo Hinkkanen kindly called the attention of the authors to the fact that a number of their results are contained in results of the manuscript, "Symmetric structures on a closed curve", by Frederick P. Gardiner and Dennis P. Sullivan. The paper by Gardiner and Sullivan is to appear in the American Journal of Mathematics. While there is an overlap in the motivation and applications of the two papers, their scope, methods and emphasis are rather different.

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