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THE GROUP OF BIHOLOMORPHIC SELF-MAPPINGS OF SCHOTTKY SPACE

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1. Introduction

The Schottky space S_p of marked Schottky groups of genus $p \geq 2$ has very simple embeddings as a domain in $\mathbf{C}^n$, $n = 3p-3$, and is therefore a tempting place to study the Riemann space R_p of all closed Riemann surfaces of genus p. In fact every closed Riemann surface has many Schottky coverings and every Schottky group has many markings, so R_p is the quotient space of S_p obtained by considering points in S_p to be equivalent if they represent the same Riemann surface. The resulting quotient map from S_p to R_p is a branched covering, but the covering is not regular. In other words the group of cover transformations, which in this case is the full group $\mathrm{Aut}(S_p)$ of biholomorphic self-mappings of S_p, fails to act transitively on the fibers of the quotient map. Our purpose is to exhibit this non-transitivity very concretely by giving an explicit description of $\mathrm{Aut}(S_p)$. It consists entirely of the familiar mappings induced by changing the marking of the Schottky group.

We state our result formally in Section 3 as Theorem 1. Our proof of Theorem 1 depends on two topological observations, also stated in Section 3 as Theorems 2 and 3. The proofs are given in Sections 4, 5, and 6. They are quite straightforward. The interest of Theorem 1 lies not in the difficulty of its proof but in what it says about $\mathrm{Aut}(S_p)$: the biholomorphic self-mappings of S_p show us which points of S_p represent the same Schottky group with different markings, but they do nothing to show which points represent different Schottky coverings of the same Riemann surface.

We state a more concrete version of Theorem 1 in Section 7, where we describe $\mathrm{Aut}(S_p)$ in terms of the obvious action on S_p of the outer automorphisms of a free group. Finally, in Section 8 we give explicit formulas for a set of generators of $\mathrm{Aut}(S_p)$, using a standard set of global coordinates for S_p.

Sections 2 and 3 summarize the facts about Schottky space that we need in this paper. More information about Schottky groups can be found in Maskit's book

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[17] or Bers's paper [3]. Schottky space is a simple example of a quasiconformal deformation space of a Kleinian group. For information about these more general spaces the reader should consult [12], [13], [4], or [16], which is especially relevant to our considerations. The papers [11] and [18] also have some relevance to our work. Our exposition in Section 2 owes something to [11] as well as to [16].

I am indebted to Albert Marden, Gaven Martin, and Bernard Maskit for stimulating discussions about the topology of Schottky coverings. Finally, thanks are due to the people at the Institut Mittag-Leffler for their support and warm hospitality while I was writing this paper.

2. The Schottky space

Choose a closed Riemann surface X of genus $p \geq 2$ and a base point x_0 on X. Let $a_1, b_1, a_2, \ldots, b_p$ be a standard system of generators for $\pi_1(X, x_0)$. This means they are represented by simple loops on X that meet only at x_0, satisfy the standard relation

$$\prod_{j=1}^{p} a_j b_j a_j^{-1} b_j^{-1} = 1,$$

and are oriented so that $a_j \times b_j = 1$ for each j. We denote by N the normal subgroup of $\pi_1(X, x_0)$ generated by $b_1, b_2, \ldots, b_p$. The quotient group $\pi_1(X, x_0)/N$ is the free group of rank p generated by the images of $a_1, \ldots, a_p$ under the quotient map.

Let $\Omega \to X$ be the covering surface of X defined by the subgroup N (see [17]). It is classical that Ω can be mapped conformally into the complex plane and that any two such embeddings differ only by a Möbius transformation (see Theorem 2D and 19F in Chapter IV of [1]). The group $\pi_1(X, x_0)/N$ of cover transformations then becomes a Kleinian group (i.e. a discrete group of Möbius transformations) that acts freely and properly discontinuously on the plane region Ω. We fix such a conformal embedding once and for all, and we denote by G_p the Kleinian group of cover transformations. We also choose a system of free generators $g_1, \ldots, g_p$ for G_p; they will be used in Section 8.

The construction above can be applied to any closed Riemann surface of genus p, with any standard set of generators for π_1. The resulting covering surfaces are called Schottky coverings, and the associated Kleinian groups are the Schottky groups of genus p. By definition, a *marked Schottky group of genus* p is an isomorphism $\theta\colon G_p \to G$ of our distinguished Schottky group G_p onto some Schottky group G. Two marked Schottky groups θ and θ' are *equivalent* if and only if there is a Möbius transformation A such that

$$\theta'(g) = A\theta(g)A^{-1} \qquad \text{for every } g \text{ in } G_p.$$

The set of equivalence classes $[\theta]$ is the Schottky space S_p. It is clear that the space S_p does not depend in an essential way on our choice of G_p, for the choice of an isomorphism $\theta\colon G_p \to G$ amounts to the same thing as the choice of a system of free generators $\theta(g_1), \ldots, \theta(g_p)$ for the Schottky group G.

A remarkable observation of Chuckrow [6] says that for any marked Schottky group $\theta\colon G_p \to G$ there is a quasiconformal homeomorphism f of the Riemann sphere such that $\theta(g) = fgf^{-1}$ for every g in G_p. Therefore S_p coincides with the quasiconformal deformation space (see [12] or [16]) of G_p, and the theory of these deformation spaces applies to S_p. In particular, S_p is a complex manifold of dimension $n = 3p - 3$, and any injective holomorphic map of S_p into $\mathbf{C}^n$ is a biholomorphic map onto a region in $\mathbf{C}^n$. Such injections are easily defined in terms of fixed points and multipliers of the transformations $\theta(g)$. See for example [3], [11], [13], or Section 8.

3. The theorems

A fundamental theorem of Maskit (see Corollary 8 in [16]) states that the universal covering space of S_p is the Teichmüller space T_p and identifies the group of cover transformations. (Bers [2] had also proved that T_p covers S_p, but his description of the cover transformations was not explicit.) We shall describe that group presently. First we review the definition and some properties of T_p.

Let $\mathcal{M}(X)$ be the space of all smooth (class C^∞) conformal structures on X, with the usual topology of C^∞ convergence (see [7], [8], or [18]). The group $\mathrm{Diff}^+(X)$ of all sense-preserving smooth diffeomorphisms of X acts (from the right) on $\mathcal{M}(X)$ by pullback. The space T_p is the quotient space

$$T_p = \mathcal{M}(X)/\mathrm{Diff}_0(X)$$

of $\mathcal{M}(X)$ by the normal subgroup $\mathrm{Diff}_0(X)$, which consists of the diffeomorphisms that are homotopic to the identity. T_p inherits a complex analytic structure from $\mathcal{M}(X)$ and is a contractible complex manifold of dimension $3p-3$, homeomorphic to $\mathbf{C}^{3p-3}$. The quotient group $\mathrm{Mod}(X) = \mathrm{Diff}^+(X)/\mathrm{Diff}_0(X)$ acts properly discontinuously on T_p as a group of biholomorphic maps, and a deep theorem of Royden says that every biholomorphic self-mapping of T_p is induced by some member of $\mathrm{Mod}(X)$. All this is classical (see [7], [9], [12], or [19]).

Now recall from Section 2 the normal subgroup N of $\pi_1(X, x_0)$, which determines the Schottky covering surface $\Omega \to X$. Following Maskit, we introduce some subgroups of $\mathrm{Diff}^+(X)$ and $\mathrm{Mod}(X)$. Let $\mathrm{Diff}^+(X, N)$ be the group of all f in $\mathrm{Diff}^+(X)$ that can be lifted to a diffeomorphism $\tilde{f}\colon \Omega \to \Omega$, and let $\mathrm{Diff}_0(X, N)$ be the subgroup consisting of all f that can be lifted to a diffeomorphism $\tilde{f}\colon \Omega \to \Omega$ that commutes with the group G_p.

Since $\mathrm{Diff}_0(X)$ is a subgroup of $\mathrm{Diff}_0(X, N)$, we can form the quotient groups

$$\mathrm{Mod}^*(X, N) = \mathrm{Diff}^+(X, N)/\mathrm{Diff}_0(X),$$
$$\mathrm{Mod}_*(X, N) = \mathrm{Diff}_0(X, N)/\mathrm{Diff}_0(X).$$

These are subgroups of $\mathrm{Mod}(X)$, so they act properly discontinuously on T_p. Maskit proved

Theorem A (Maskit [16]). *The group* $\operatorname{Mod}_*(X, N)$ *acts freely on* T_p, *and* S_p *equals the quotient space* $T_p/\operatorname{Mod}_*(X, N)$.

$\operatorname{Diff}_0(X, N)$ is a normal subgroup of $\operatorname{Diff}^+(X, N)$, so the quotient group

$$\Gamma_p = \operatorname{Diff}^+(X, N)/\operatorname{Diff}_0(X, N) = \operatorname{Mod}^*(X, N)/\operatorname{Mod}_*(X, N)$$

acts on S_p as a group of biholomorphic self-mappings. Our main result is

Theorem 1. *Every biholomorphic self-mapping of* S_p *is induced by some member of* Γ_p.

As J.A. Gentilesco pointed out under more general circumstances (see Theorem VIII of [10]), Theorem 1 is an easy consequence of Royden's analogous theorem about T_p and the purely topological

Theorem 2. *The normalizer of* $\operatorname{Mod}_*(X, N)$ *in* $\operatorname{Mod}(X)$ *is* $\operatorname{Mod}^*(X, N)$.

We deduce Theorem 2 from the following topological theorem, which provides a tight relationship between the groups N and $\operatorname{Mod}_*(X, N)$.

Theorem 3. *Let* c *in* $\pi_1(X, x_0)$ *be represented by a simple loop. Then* $c \in N$ *if and only if the Dehn twist* $\tau(c)$ *on* c *belongs to* $\operatorname{Mod}_*(X, N)$.

Remarks. 1) For a discussion of Dehn twists see [5].

2) Theorems 1 and 2 measure the failure of $\operatorname{Aut}(S_p)$ to act transitively on the fibers of the quotient map from S_p to R_p. The Riemann space is the quotient

$$R_p = T_p/\operatorname{Mod}(X),$$

and $S_p/\operatorname{Aut}(S_p) = T_p/\operatorname{Mod}^*(X, N)$, so the map from $S_p/\operatorname{Aut}(S_p)$ to R_p has fibers generically isomorphic to the set of cosets of $\operatorname{Mod}^*(X, N)$ in $\operatorname{Mod}(X)$.

It is known that $\operatorname{Mod}^*(X, N)$ is a rather thin subgroup of $\operatorname{Mod}(X)$. In fact Masur [18] showed that $\operatorname{Mod}^*(X, N)$ acts properly discontinuously on a nontrivial open subset of Thurston's sphere $P\mathcal{F}$ of projective measured foliations. In contrast, $\operatorname{Mod}(X)$ acts minimally, even ergodically, on $P\mathcal{F}$ (see[18]).

The Dehn twist $\tau(a_1)$ provides an obvious example of an element of $\operatorname{Mod}(X)$ that does not belong to $\operatorname{Mod}^*(X, N)$.

3) The action of the Schottky group G_p extends to hyperbolic 3-space, and the quotient of hyperbolic 3-space by G_p is a solid handlebody H bounded by the surface X. $\operatorname{Diff}^+(X, N)$ is just the subgroup of $\operatorname{Diff}^+(X)$ that can be extended to H, and $\operatorname{Diff}_0(X, N)$ consists of the diffeomorphisms whose extensions to H are homotopic to the identity in H.

4) According to an interesting theorem of Luft [14], the Dehn twists $\tau(c)$ in Theorem 3 generate the group $\operatorname{Mod}_*(X, N)$.

5) Hejhal [11] characterizes the covering group $\operatorname{Mod}_*(X, N)$ by a lifting property that differs slightly from (3.2). Maskit's characterization in terms of (3.2) is more useful for us here.

4. Proof of Theorem 1

For the reader's convenience we shall derive Theorem 1 from Theorem 2, following the method of Gentilesco [10]. Let $\varphi\colon S_p \to S_p$ be a biholomorphic self-mapping of S_p. The quotient map from T_p to S_p in Maskit's Theorem A is a holomorphic universal covering, so φ lifts to a biholomorphic self-mapping ψ of T_p. By Royden's theorem, ψ is induced by an element θ of $\mathrm{Mod}(X)$. Since θ and θ^{-1} both induce the maps on S_p, we must have $\theta\sigma\theta^{-1} \in \mathrm{Mod}_*(X,N)$ and $\theta^{-1}\sigma\theta \in \mathrm{Mod}_*(X,N)$ for all σ in $\mathrm{Mod}_*(X,N)$. Therefore θ belongs to the normalizer of $\mathrm{Mod}_*(X,N)$, which, by Theorem 2, equals $\mathrm{Mod}^*(X,N)$. QED

5. Proof of Theorem 2

Since $\mathrm{Diff}_0(X) \subset \mathrm{Diff}_0(X,N)$, Theorem 2 is equivalent to the statement that the normalizer of $\mathrm{Diff}_0(X,N)$ in $\mathrm{Diff}^+(X)$ is $\mathrm{Diff}^+(X,N)$. That is what we shall prove.

Let f belong to $\mathrm{Diff}^+(X)$ and let c belong to $\pi_1(X,x_0)$. Since N is a normal subgroup, the statement that $f(c)$ belongs to N makes sense even though f need not preserve the base point x_0. Covering space theory then tells us that $f \in \mathrm{Diff}^+(X,N)$ if and only if both $f(c) \in N$ and $f^{-1}(c) \in N$ whenever $c \in N$. It obviously suffices to have $f(b_j) \in N$ and $f^{-1}(b_j) \in N$ for $1 \le j \le p$.

Now let f belong to the normalizer of $\mathrm{Diff}_0(X,N)$. Let $\tau(b_j)$ be (a representative in $\mathrm{Diff}^+(X)$ of) the Dehn twist on b_j. By Theorem 3, $\tau(b_j) \in \mathrm{Diff}_0(X,N)$, so $f\tau(b_j)f^{-1} \in \mathrm{Diff}_0(X,N)$. But $f\tau(b_j)f^{-1}$ is (represents) the Dehn twist on $f(b_j)$ (see [5]). Therefore, by Theorem 3, $f(b_j) \in N$. The same reasoning applied to f^{-1} shows that $f^{-1}(b_j)$ also belongs to N. Therefore $f \in \mathrm{Diff}^+(X,N)$. QED

6. Proof of Theorem 3

We will do the trivial implication first. Suppose the simple geodesic loop γ represents c in N. Choose a small collar C about γ. By definition of the covering surface $\pi\colon \Omega \to X$, each connected component of $\pi^{-1}(C)$ in Ω is mapped homeomorphically onto C by π.

Now choose a diffeomorphism f that equals the identity in $X \setminus C$ and represents the Dehn twist $\tau(c)$. Lift f to a diffeomorphism $\tilde f\colon \Omega \to \Omega$ by putting $\tilde f = \mathrm{id}$ in $\Omega \setminus \pi^{-1}(C)$ and $\tilde f = (\pi \mid \tilde C)^{-1} \circ f \circ (\pi \mid \tilde C)$ in each connected component $\tilde C$ of $\pi^{-1}(C)$. Since $\tilde f$ commutes with G_p, $f \in \mathrm{Diff}_0(X,N)$ and $\tau(c) \in \mathrm{Mod}_*(X,N)$ as required.

Conversely, suppose $\tau(c) \in \mathrm{Mod}_*(X,N)$. As before, we choose a small collar C about a simple geodesic loop γ that represents c, and we represent $\tau(c)$ by a diffeomorphism f that equals the identity in $X \setminus C$. By hypothesis, f has a lift $\tilde f\colon \Omega \to \Omega$ that commutes with the group G_p.

We shall assume that $c \notin N$ and look for a contradiction. Let β_1 and β_2 be the two boundary loops of C. Since $c \notin N$, each connected component of

$\pi^{-1}(\beta_1 \cup \beta_2)$ is a simple arc β in Ω. Let φ_β be a generator of the infinite cyclic group of all φ in G_p such that $\varphi(\beta) = \beta$. Then β connects the two fixed points of φ_β (which are boundary points of Ω).

First we shall prove that $\tilde{f} = \mathrm{id}$ in $\Omega \setminus \pi^{-1}(C)$. Let Y be a connected component of $\Omega \setminus \pi^{-1}(C)$, and let H be the subgroup of G_p that maps Y onto itself. Since $f = \mathrm{id}$ on $X \setminus C$, there is some ψ in G_p such that $\tilde{f} = \psi$ on Y. Since $\tilde{f}$ commutes with G_p, ψ commutes with the subgroup H. Suppose $\psi \neq id$. Then H is cyclic and all nontrivial elements of H have the same two fixed points. Now the boundary of Y in Ω consists of arcs β in $\pi^{-1}(\beta_1 \cup \beta_2)$. Each φ_β belongs to H, and β connects its fixed points, so Y must be a Jordan region bounded by the union of two arcs β, β' and their common endpoints. Therefore Y/H is an annulus. Since Y/H is a connected component of $X \setminus C$ and X has genus $p \geq 2$, this is nonsense. Therefore $\psi = \mathrm{id}$, so $\tilde{f}$ is the identity in Y and hence in $\Omega \setminus \pi^{-1}(C)$.

It is now easy to reach the desired contradiction. Let $\tilde{C}$ be a connected component of $\pi^{-1}(C)$. Then $\tilde{C}$ is a Jordan domain whose boundary is the union of two arcs β and β' as above, and their common endpoints. The stabilizer of $\tilde{C}$ in G_p is the cyclic group generated by φ_β. Since $\tilde{f} = \mathrm{id}$ in $\Omega \setminus \pi^{-1}(C)$, $\tilde{f}$ maps $\tilde{C}$ onto itself and equals the identity on the boundary of $\tilde{C}$. In addition, $\tilde{f}$ commutes with φ_β.

It is easy to construct a φ_β-equivariant homotopy of $\tilde{f}$ to the identity in $\tilde{C}$, holding the boundary of $\tilde{C}$ pointwise fixed. (For instance there is a conformal map that takes $\tilde{C}$ to a closed horizontal strip $\{z = x + iy; |y| \leq r\}$ and φ_β to $z \mapsto z+1$. We can then set $\tilde{f}_t(z) = tz + (1-t)\tilde{f}(z)$ in the closed strip.) Projecting that homotopy to the collar C we find that f is homotopic to the identity in X, contradicting the fact that f represents $\tau(c)$. This contradiction implies that $c \in N$. QED

7. The action of the outer automorphism group

Let $\mathrm{Aut}(G_p)$ be the group of all automorphisms of the group G_p, and let $\mathrm{Inn}(G_p)$ be the normal subgroup of inner automorphisms. $\mathrm{Aut}(G_p)$ acts in an obvious way on the set of marked Schottky groups: if $\theta\colon G_p \to G$ is a marked Schottky group and $\alpha \in \mathrm{Aut}(G_p)$, then $\theta \cdot \alpha$ is the marked Schottky group $\theta \circ \alpha\colon G_p \to G$. This action obviously preserves equivalence classes and induces the action

$$[\theta] \cdot \alpha = [\theta \circ \alpha] \qquad \text{if } [\theta] \in S_p \quad \text{and } \alpha \in \mathrm{Aut}(G_p) \tag{7.1}$$

of $\mathrm{Aut}(G_p)$ on S_p. The subgroup $\mathrm{Inn}(G_p)$ acts trivially on S_p, so (7.1) defines an action of the outer automorphism group

$$\mathrm{Outer\,Aut}(G_p) = \mathrm{Aut}(G_p)/\,\mathrm{Inn}(G_p)$$

on S_p. In terms of these actions Theorem 1 takes the concrete and explicit form

Theorem 1′. *For each α in $\mathrm{Aut}(G_p)$ the self-mapping of S_p defined by (7.1) is biholomorphic. Every biholomorphic self-mapping of S_p has the form (7.1) for some α.*

The proof is simply a matter of being explicit about the action of Γ_p in Theorem 1. First we must describe the standard map of $\mathcal{M}(X)$ onto S_p. Each μ in $\mathcal{M}(X)$ defines a new Riemann surface structure on X, which determines a new G_p-invariant Riemann surface structure on Ω via the covering map $\pi\colon \Omega \to X$. We denote the resulting Riemann surfaces by Ω^μ and X^μ. Since $\pi\colon \Omega^\mu \to X^\mu$ is a Schottky covering there is a conformal mapping w^μ of Ω^μ into the complex plane. The group $G^\mu = w^\mu G_p (w^\mu)^{-1}$ is a Schottky group, and the isomorphism

$$g \mapsto \theta^\mu(g) = (w^\mu) \circ g \circ (w^\mu)^{-1} \qquad \text{if} \quad g \in G \tag{7.2}$$

defines a marked Schottky group. Its equivalence class $[\theta^\mu]$ depends only on μ because the conformal map w^μ is unique up to composition with a Möbius transformation. The map $\mu \mapsto [\theta^\mu]$ from $\mathcal{M}(X)$ to S_p factors through T_p ($= \mathcal{M}(X)/\mathrm{Diff}_0(X)$) to produce Maskit's universal covering map (see [12] or [16]).

Recall that $\mathrm{Diff}^+(X)$ acts on $\mathcal{M}(X)$ by pullback: the map $f\colon X^{\mu\cdot f} \to X^\mu$ is conformal for every f in $\mathrm{Diff}^+(X)$ and μ in $\mathcal{M}(X)$. The subgroup $\mathrm{Diff}^+(X,N)$ acts on S_p by

$$[\theta^\mu]\cdot f = [\theta^{\mu\cdot f}] \qquad \text{if} \quad [\theta^\mu] \in S_p \quad \text{and} \quad f \in \mathrm{Diff}^+(X,N). \tag{7.3}$$

The normal subgroup $\mathrm{Diff}_0(X,N)$ acts trivially, and (7.3) induces the action of the quotient group Γ_p in Theorem 1.

According to Theorem 1 every biholomorphic self-mapping of S_p has the form (7.3). To compute $[\theta^{\mu\cdot f}]$ for f in $\mathrm{Diff}^+(X,N)$ we choose a lift $\tilde f\colon \Omega \to \Omega$ and observe that $\tilde f\colon \Omega^{\mu\cdot f} \to \Omega^\mu$ is conformal. Therefore $w^\mu \circ \tilde f$ maps $\Omega^{\mu\cdot f}$ conformally into the complex plane, so

$$\theta^{\mu\cdot f}(g) = w^\mu \circ (\tilde f \circ g \circ \tilde f^{-1}) \circ (w^\mu)^{-1} = \theta^\mu(\tilde f \circ g \circ \tilde f^{-1})$$

for all g in G. Thus (7.3) takes the form

$$[\theta^\mu]\cdot f = [\theta^\mu \circ \alpha_{\tilde f}] \qquad \text{if} \quad [\theta^\mu] \in S_p \quad \text{and} \quad f \in \mathrm{Diff}_+(X,N), \tag{7.3$'$}$$

where $\alpha_{\tilde f}$ in $\mathrm{Aut}(G_p)$ is the automorphism $g \mapsto \tilde f \circ g \circ \tilde f^{-1}$. We see that every biholomorphic self-mapping of S_p is indeed of the form (7.1).

To verify that all the maps (7.1) are biholomorphic we must show that every α in $\mathrm{Aut}(G_p)$ is of the form $\alpha_{\tilde f}$. That is the content of Chuckrow's observation (Theorem 2 of [6]), which for any given α guarantees the existence of a sense-preserving diffeomorphism $\tilde f\colon \Omega \to \Omega$ such that $\alpha(g) = \tilde f \circ g \circ \tilde f^{-1}$ for all g in G_p. The map $\tilde f$ covers a diffeomorphism $f\colon X \to X$. By definition, $f \in \mathrm{Diff}^+(X,N)$ and $\alpha = \alpha_{\tilde f}$. The proof is complete.

Remarks. 1) The outer automorphism $[\alpha_{\tilde{f}}]$ depends only on f, and the map $f \mapsto [\alpha_{\tilde{f}}]$ induces an isomorphism between the groups Γ_p and Outer Aut(G_p).

2) It is a striking fact that every outer automorphism of G_p is induced by a sense-preserving diffeomorphism of X and that sense-reversing diffeomorphisms are not required. The geometric reason for this is that there is a sense-reversing diffeomorphism $\tilde{f}\colon \Omega \to \Omega$ that commutes with the group G_p. This is easy to see if we take G_p to be a Fuchsian group of the second kind and $\tilde{f}$ to be inversion in the fixed circle.

8. Explicit formulas

Finally, we shall borrow a set of global coordinates for S_p from Hejhal [11] and give formulas in these coordinates for a set of generators of Aut(S_p). Following [11], for any marked Schottky group $\theta\colon G_p \to G$ we put $L_j = \theta(g_j)$ for $1 \le j \le p$ and we set a_j, b_j, and λ_j equal to the attracting fixed point, repelling fixed point, and multiplier of L_j, defining the multiplier so that $0 < |\lambda_j| < 1$. Replacing θ by an equivalent isomorphism, we can normalize the p-tuple $(L_1, \dots, L_p)$ so that $a_1 = 0$, $a_2 = 1$, and $b_1 = \infty$. The injective holomorphic map

$$[\theta] \mapsto (a_3, \dots, a_p, b_2, \dots, b_p, \lambda_1, \dots, \lambda_p) \in \mathbf{C}^{3p-3}$$

then defines a global coordinate system for S_p (see [11] and [13]), allowing us to interpret S_p as a region in $\mathbf{C}^{3p-3}$ and Aut(S_p) as the group of biholomorphic self-mappings of that region.

Now every member of Aut(S_p) is induced by an automorphism of the free group G_p, and a theorem of Nielsen (see Corollary N1 in Section 3.5 of [15]) implies that Aut(G_p) is generated by these four automorphisms:

$$\alpha_1(g_1) = g_p, \quad \text{and} \quad \alpha_1(g_j) = g_{j-1} \quad \text{if} \quad j > 1, \tag{8.1}$$

$$\alpha_2(g_1) = g_2, \quad \alpha_2(g_2) = g_1, \quad \text{and} \quad \alpha_2(g_j) = g_j \quad \text{if} \quad j > 2, \tag{8.2}$$

$$\alpha_3(g_1) = g_1^{-1}, \quad \text{and} \quad \alpha_3(g_j) = g_j \quad \text{if} \quad j > 1, \tag{8.3}$$

$$\alpha_4(g_1) = g_1, \quad \alpha_4(g_2) = g_2^{-1} g_1, \quad \text{and} \quad \alpha_4(g_j) = g_j \quad \text{if} \quad j > 2. \tag{8.4}$$

We must calculate their effect on the region S_p.

The automorphism α_1 transforms $(L_1, \dots, L_p)$ to $(L_p, L_1, \dots, L_{p-1})$. Conjugation by

$$T(z) = \frac{b_p(z - a_p)}{a_p(z - b_p)} \tag{8.5}$$

produces the normalized p-tuple

$$(TL_pT^{-1}, TL_1T^{-1}, \ldots, TL_{p-1}T^{-1})$$

and replaces each fixed point by its image under T, so α_1 induces the map

$$(8.1') \qquad (a, b, \lambda) \mapsto \big(T(a_2), \ldots, T(a_{p-1}), T(b_1), \ldots, T(b_{p-1}), \lambda_p, \lambda_1, \ldots, \lambda_{p-1}\big).$$

(We use (a, b, λ) as an abbreviation for $(a_3, \ldots, a_p, b_2, \ldots, b_p, \lambda_1, \ldots, \lambda_p)$, and we remind the reader that $a_2 = 1$ and $b_1 = \infty$.)

The automorphism α_2 produces the p-tuple $(L_2, L_1, L_3, \ldots, L_p)$, which is conjugated to normalized form by the transformation

$$(8.6) \qquad S(z) = \frac{b_2(z-1)}{z - b_2}.$$

Therefore α_2 induces the map

$$(8.2') \qquad (a, b, \lambda) \mapsto \big(S(a_3), \ldots, S(a_p), S(b_1), S(b_3), \ldots, S(b_p), \lambda_2, \lambda_1, \lambda_3, \ldots, \lambda_p\big).$$

The automorphism α_3 leads to $(L_1^{-1}, L_2, L_3, \ldots, L_p)$. The transformation L_1^{-1} has multiplier λ_1, attracting fixed point ∞, and repelling fixed point 0. Conjugation by $z \mapsto 1/z$ produces a normalized p-tuple, so α_3 induces the map

$$(8.3') \qquad (a, b, \lambda) \mapsto \Big(\frac{1}{a_3}, \ldots, \frac{1}{a_p}, \frac{1}{b_2}, \ldots, \frac{1}{b_p}, \lambda_1, \ldots, \lambda_p\Big).$$

All this was very easy, but α_4 requires more effort. We must normalize the p-tuple $(L_1, L_2^{-1}L_1, L_3, \ldots, L_p)$, so we need to find a^*, b^*, and λ^*, the attracting fixed point, repelling fixed point, and multiplier of $L_2^{-1}L_1$. We know that $L_1(z) = \lambda_1 z$ and that L_2 is represented by the matrix

$$\begin{pmatrix} 1 - b_2\lambda_2 & b_2(\lambda_2 - 1) \\ 1 - \lambda_2 & \lambda_2 - b_2 \end{pmatrix}$$

in $GL(2, \mathbf{C})$. The matrix

$$(8.7) \qquad A = \begin{pmatrix} \lambda_1(\lambda_2 - b_2) & b_2(1 - \lambda_2) \\ \lambda_1(\lambda_2 - 1) & 1 - b_2\lambda_2 \end{pmatrix}$$

then represents $L_2^{-1}L_1$. (Our matrices do not necessarily have determinant one.)

The multiplier λ^* of $L_2^{-1}L_1$ satisfies

$$\lambda^* + (\lambda^*)^{-1} + 2 = \operatorname{trace}(A)^2 / \det(A),$$

so it is a solution of the quadratic equation

$$(8.8)\quad \lambda_1\lambda_2(b_2-1)^2\left[(\lambda^*)^2+1\right] = \left[\lambda_1^2(\lambda_2-b_2)^2+(1-b_2\lambda_2)^2-2b_2\lambda_1(\lambda_2-1)^2\right]\lambda^*.$$

In fact λ^* is the unique solution of (8.8) satisfying $|\lambda^*| < 1$. Thus λ^* is an algebraic, but not a rational, function of λ_1, λ_2, and b_2.

Since $L_2^{-1}L_1$ is represented by the matrix A in (8.7), the product a^*b^* of its fixed points equals b_2/λ_1. Put $\zeta = \lambda_1 b^*$. Then $\zeta a^* - b_2 = \zeta - \lambda_1 b^* = 0$, so the matrix

$$\begin{aligned} B &= \begin{pmatrix} \zeta & b_2 \\ \lambda_1 & \zeta \end{pmatrix} \begin{pmatrix} \lambda^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta & -b_2 \\ -\lambda_1 & \zeta \end{pmatrix} \\ &= \begin{pmatrix} \lambda^*\zeta^2 - b_2\lambda_1 & b_2(1-\lambda^*)\zeta \\ \lambda_1(\lambda^*-1)\zeta & \zeta^2 - b_2\lambda_1\lambda^* \end{pmatrix} \end{aligned}$$

also represents $L_2^{-1}L_1$.

Now the equation $(\lambda_2-1)B = (\lambda^*-1)\zeta A$ yields a pair of quadratic equations for ζ. Eliminating ζ^2 between them we find that

$$\zeta = \frac{b_2\lambda_1(1-\lambda_2)(\lambda^*+1)}{(1-b_2\lambda_2)\lambda^* - \lambda_1(\lambda_2-b_2)},$$

$$(8.9)\qquad a^* = \frac{(1-b_2\lambda_2)\lambda^* - \lambda_1(\lambda_2-b_2)}{\lambda_1(1-\lambda_2)(\lambda^*+1)},$$

and

$$(8.10)\qquad \frac{b^*}{a^*} = \frac{\lambda_1(b_2-\lambda_2)\lambda^* + (1-b_2\lambda_2)}{(1-b_2\lambda_2)\lambda^* + \lambda_1(b_2-\lambda_2)}.$$

(To obtain (8.10) we use (8.8) to simplify the right hand side of the equation $b^*/a^* = \zeta^2/b_2\lambda_1$.) The map induced by α_4 is therefore

$$(8.4')\qquad (a,b,\lambda) \mapsto \left(\frac{a_3}{a^*},\dots,\frac{a_p}{a^*},\frac{b^*}{a^*},\frac{b_3}{a^*},\dots,\frac{b_p}{a^*},\lambda_1,\lambda^*,\lambda_3,\dots,\lambda_p\right),$$

where λ^* is the unique solution of (8.8) with $|\lambda^*| < 1$, a^* is given by (8.9), and b^*/a^* by (8.10).

We sum up our results in a final

Proposition. *The group of all biholomorphic self-mappings of the region* S_p *in* $\mathbf{C}^{3p-3}$ *is generated by the four transformations* (8.1′) *through* (8.4′).

Remark. The special case $p = 2$ has particular interest, both because Kleinian groups with two generators have been much studied and because the results become simpler and even more explicit. Notice first that the automorphisms α_1 and α_2 coincide when $p = 2$, so α_2, α_3, and α_4 generate $\mathrm{Aut}(G_2)$. Secondly, another theorem of Nielsen (see Corollary N4 in Section 3.5 of [15]) tells us that $\mathrm{Outer\,Aut}(G_2)$ is canonically isomorphic to the automorphism group of the abelianization of G_2, so $\mathrm{Outer\,Aut}(G_2)$ is isomorphic to the group $GL(2,Z)$ of two-by-two unimodular integer matrices. Finally, since every closed Riemann surface of genus two is hyperelliptic, the automorphism of G_2 defined by $g_j \mapsto g_j^{-1}$, $j = 1$ or 2, acts trivially on the Schottky space S_2. Therefore the group of biholomorphic self-mappings of S_2 is isomorphic to

$$PGL(2,Z) = GL(2,Z)/\{\pm I\}.$$

The generating matrices

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \pm\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad \pm\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

correspond to the self-mappings

$$(b_2,\lambda_1,\lambda_2) \mapsto (b_2,\lambda_2,\lambda_1), \qquad (b_2,\lambda_1,\lambda_2) \mapsto \left(\frac{1}{b_2},\lambda_1,\lambda_2\right),$$

and

$$(b_2,\lambda_1,\lambda_2) \mapsto \left(\frac{b^*}{a^*},\lambda_1,\lambda^*\right)$$

respectively, with λ^* and b^*/a^* determined as in (8.4′).

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