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OPTIMAL TRANSMISSION OF GAUSSIAN SIGNALS INVOLVING COST OF FEEDBACK

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Methods to analyse the problem of signal transmission through a noiseless feedback channel using the results of nonlinear filtering theory are given in [2]. In [1] we constructed an optimal transmission model for Gaussian signals taking into account the cost of transmission. The aim of this paper is to construct an optimal transmission scheme for a Gaussian signal involving the cost of feedback.

Suppose the transmitted message is a Gaussian random variable θ with $\mathbf{E}\theta = m$ and $\mathbf{E}(\theta - m)^2 = \gamma > 0$, and the transmission of θ is carried out according to the following scheme

$$(1)\qquad \begin{aligned} d\xi_t &= \Big\{\beta_t\,[A_0(t,\xi) + A_1(t,\xi)\theta] + (1-\beta_t)\,[B_0(t) + B_1(t)\theta]\Big\}\,dt + dW_t,\\ \xi_0 &= 0, \quad t \in [0,T], \end{aligned}$$

where $\beta = (\beta_t)_{t\le T}$ is a $(\mathscr{F}_t^\xi)_{t\le T}$-adapted stochastic process taking two values 0 and 1. Under $\beta_t = 1$ the transmission is with feedback and under $\beta_t = 0$ it is without feedback. Let

$$\delta(t) = \inf_{A_0,A_1,B_0,B_1,\beta,\hat\theta} \mathbf{E}\Big[(\theta - \hat\theta_t)^2 + c\int_0^t \beta_s\,ds\Big], \qquad c > 0,$$

where c is the cost of the feedback.

Let the following moment conditions

$$(2)\qquad \mathbf{E}\Big\{\big[A_0(t,\xi) + A_1(t,\xi)\theta\big]^2 \big| \mathscr{F}_t^\xi\Big\} \le P, \qquad \mathbf{E}\big[B_0(t) + B_1(t)\theta\big]^2 \le P$$

be satisfied, $P > 0$.

The functionals A_0, A_1, B_0, B_1, β are supposed to be such that equation (1) has a unique strong solution.

Denote

$$m_t = \mathbf{E}(\theta \mid \mathscr{F}_t^\xi), \qquad \gamma_t = \mathbf{E}\big[(\theta - m_t)^2 \mid \mathscr{F}_t^\xi\big].$$

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The equations for m_t and γ_t are of the following form

$$
\begin{aligned}
dm_t &= \gamma_t\big[\beta_t A_1(t,\xi) + (1-\beta_t)B_1(t)\big]\cdot \\
&\quad\cdot\Big\{d\xi_t - \big[\beta_t A_0(t,\xi) + (1-\beta_t)B_0(t) + \big(\beta_t A_1(t,\xi) \\
&\quad + (1-\beta_t)B_1(t)\big)m_t\big]\,dt\Big\}, \qquad m_0 = m,
\end{aligned} \tag{3}
$$

$$
d\gamma_t = -\gamma_t^2\big[\beta_t A_1(t,\xi) + (1-\beta_t)B_1(t)\big]^2\,dt, \qquad \gamma_0 = \gamma. \tag{4}
$$

Theorem. *Suppose the transmission of a Gaussian variable θ is carried out according to scheme* (1).

If $c \ge \frac{1}{4}\gamma P$, then the optimal strategy is $\beta_s^* = 0$, $s \in [0,t]$, i.e., transmission without feedback. The optimal coding is $B_1^*(s) = \sqrt{P/\gamma}$, $B_0^*(s) = -m\sqrt{P/\gamma}$, and optimal decoding m_t^*, for given β^*, B_0^*, B_1^*, is determined through (2) *and* (3). *In that case*

$$
\delta(t) = \frac{\gamma}{1+Pt}.
$$

Exactly the same statement is true if $c < \frac{1}{4}\gamma P$ and $t \le a$ or $t \ge b$ with

$$
a = \frac{\gamma P - 2c - \sqrt{\gamma^2P^2 - 4c\gamma P}}{2cP}, \qquad b = \frac{\gamma P - 2c + \sqrt{\gamma^2P^2 - 4c\gamma P}}{2cP}.
$$

If $c < \frac{1}{4}\gamma P$ and $t \in (a,b)$, then the optimal strategy has the form $\beta_s^* = I(t - x_0 \le s \le t)$, where x_0 is the unique solution of the equation

$$
ce^{Px}\big[1 + P(t-x)\big]^2 = \gamma P^2(t-x)
$$

in the interval $(0,t)$.

The optimal coding rules are determined as follows:

$$
B_1^*(s) = \sqrt{\frac{P}{\gamma}}, \qquad B_0^*(s) = -m\sqrt{\frac{P}{\gamma}},
$$

$$
A_1^*(s,\xi) = \sqrt{\frac{P}{\gamma}}\exp\left\{\frac{P}{2}\int_0^s \beta_u^*\,du\right\}\left[1+\gamma\int_0^s (1-\beta_u^*)\big(B_1^*(u)\big)^2\exp\left\{-P\int_0^u \beta_r^*\,dr\right\}du\right],
$$

$$
A_0^*(t,\xi) = -A_1^*(t,\xi)m_t^*,
$$

where m_t^* and γ_t^* are determined through equations (3) *and* (4). *In this case*

$$
\delta(t) = \frac{\gamma e^{-Px_0}}{1+P(t-x_0)} + cx_0.
$$

To prove this theorem we need the following lemma.

Lemma. *Let $\beta = (\beta_t)_{t \geq 0}$ be a real function with values in $[0,1]$ and let P be a positive constant. Then the following inequality is valid*

$$\int_0^t e^{-P\int_0^s \beta_u\,du}\,ds \leq t - \int_0^t \beta_s\,ds + \frac{1}{P}\Big(1 - e^{-P\int_0^t \beta_s ds}\Big).$$

If β has the form $\beta_s = I(u \leq s \leq t)$, $u \in [0,t]$, then the inequality reduces into an equality.

Proof. Let $\int_0^t \beta_s\,ds = v$ and $\beta^*_s = I(t - v \leq s \leq t)$. Then $\int_0^t \beta^*_s\,ds = v$. Let us first show that

$$\int_0^s \beta^*_u\,du \leq \int_0^s \beta_u\,du$$

for any $s \in [0,1]$. This inequality is evident for $s < t - v$. Assume that for some $t_1 \geq t - v$ an inverse inequality occurs. Then

$$v = \int_0^t \beta^*_u\,du = \int_0^{t_1} \beta^*_u\,du + \int_{t_1}^t \beta^*_u\,du > \int_0^{t_1} \beta_u\,du + \int_{t_1}^t \beta_u\,du = \int_0^t \beta_u\,du,$$

which is not true. Thus,

$$\int_0^t e^{-P\int_0^s \beta_u\,du} ds \leq \int_0^t e^{-P\int_0^s \beta^*_u du}\,ds.$$

One can easily see that

$$\int_0^t e^{-P\int_0^s \beta^*_u du} ds = t - v + \frac{1}{P}\big(1 - e^{-Pv}\big),$$

and since $v = \int_0^t \beta_s\,ds$, we obtain the required inequality.

The validity of the second assertion of the lemma follows by a direct verification.

Proof of the theorem. For fixed coding functionals A_0, A_1, B_0, B_1 and strategy β the optimal decoding is m_t. Hence,

$$\delta(t) = \inf_{A_0,A_1,B_0,B_1,\beta} \mathbf{E}\Big[\gamma_t + c\int_0^t \beta_s\,ds\Big].$$

We construct first the optimal coding functionals for a fixed strategy β. Rewrite the moment condition in the following form

$$\text{(5)}\quad \big[A_0(t,\xi)+A_1(t,\xi)m_t\big]^2+\gamma_t A_1^2(t,\xi) \leq P, \qquad \big[B_0(t)+B_1(t)m\big]^2+\gamma B_1^2(t) \leq P.$$

The equation for γ_t can be written in the form

$$\gamma_t = \gamma \exp\Big\{ - \int_0^t \beta_s \gamma_s A_1^2(s,\xi)\, ds - \int_0^t (1-\beta_s)\gamma_s B_1^2(s)\, ds \Big\}.$$

From (5) it follows that $\gamma_t A_1^2(t,\xi) \leq P$. Therefore,

$$\gamma_t \geq \gamma \exp\Big\{ - P \int_0^t \beta_s\, ds - \int_0^t (1-\beta_s) B_1^2(s)\, ds \Big\}. \tag{6}$$

If we choose $A_1^*(t,\xi) = \sqrt{P/\gamma_t^*}$, where γ_t^* is a solution of (4) for a given functional A_1^* and a fixed B_1, then the inequality reduces into an equality. Hence A_1^* is the optimal coding. The moment condition is satisfied for $A_0^*(t,\xi) = -A_1^*(t,\xi) m_t^*$, where m_t^* is a solution of (3) for given A_0^*, A_1^*, γ_t^*. If we solve the equation (4) for $A_1^*(t,\xi) = \sqrt{P/\gamma_t^*}$, we obtain

$$\gamma_t^* = \frac{\gamma e^{-P\int_0^t \beta_s ds}}{1 + \gamma \int_0^t (1-\beta_s) B_1^2(s) e^{-P\int_0^s \beta_u du} ds},$$

and therefore the optimal coding has the form

$$A_1^*(t,\xi) = \sqrt{\frac{P}{\gamma} e^{P/2\int_0^t \beta_s ds}\Big[1 + \gamma \int_0^t (1-\beta_s) B_1^2(s) e^{-P\int_0^s \beta_u du} ds\Big]}.$$

Thus

$$\delta(t) = \inf_{B_0, B_1, \beta} \mathbf{E}\left[\frac{\gamma e^{-P\int_0^t \beta_s ds}}{1 + \gamma \int_0^t (1-\beta_s) B_1^2(s) e^{-P\int_0^s \beta_u du} ds} + c \int_0^t \beta_s\, ds \right].$$

If follows from (5) that $B_1^2(t) \leq P/\gamma$. If we choose $B_1^*(t) = \sqrt{P/\gamma}$ and $B_0^*(t) = -m\sqrt{P/\gamma}$, then the condition (5) is fulfilled and

$$\delta(t) = \inf_{\beta} \mathbf{E}\left[\frac{\gamma e^{-P\int_0^t \beta_s ds}}{1 + P \int_0^t (1-\beta_s) e^{-P\int_0^s \beta_u du} ds} + c \int_0^t \beta_s\, ds \right].$$

Thus, for a fixed strategy β, our construction gives the optimal coding functionals and the optimal decoding.

Now we construct the optimal strategy. It can be easily shown that

$$\delta(t) = \inf_{\beta} \mathbf{E}\left[\frac{\gamma}{1 + P e^{P\int_0^t \beta_s ds} \int_0^t e^{-P\int_0^s \beta_u du} ds} + c \int_0^t \beta_s\, ds \right].$$

We denote the expression in the brackets by $\delta_\beta(t)$. Using the inequality from the previous lemma we obtain

$$\delta_\beta(t) \geq \frac{\gamma e^{-P\int_0^t \beta_s ds}}{1 + P(t - \int_0^t \beta_s ds)} + c\int_0^t \beta_s\, ds.$$

Consider the function

$$f(x) = \frac{\gamma e^{-Px}}{1 + P(t-x)} + cx, \qquad x \in [0,t].$$

The properties of the derivatives of this function easily show that this function increases for all $x \in [0,1]$, if $c \geq \frac{1}{4}\gamma P$ and therefore $f(x) \geq f(0) = \gamma/(1+Pt)$. If $c < \gamma P/4$ and $t \notin (a,b)$, where the numbers a and b have been defined in the enunciation of the theorem, then again $f(x) \geq \gamma/(1+Pt)$, $x \in [0,t]$. If $c < \frac{1}{4}\gamma P$ and $t \in (a,b)$, then $f'(x) < 0$ for $x \in [0, x_0)$ and $f'(x) > 0$ for $x \in (x_0, t]$, i.e., $f(x) \geq f(x_0)$, $x \in [0,t]$, where x_0 is the unique solution of the equation

$$ce^{Px}\big[1 + P(t-x)\big]^2 = \gamma P^2(t-x).$$

Using the properties of the function f we can obtain the following statements: If $c \geq \frac{1}{4}\gamma P$, then for any strategy β we have

$$\delta_\beta(t) \geq \frac{\gamma}{1+Pt}, \qquad \text{(note that } 0 \leq \int_0^t \beta_s\, ds \leq t\text{)},$$

and for $\beta_s^* = 0$, $s \in [0,t]$,

$$\delta_{\beta^*}(t) = \frac{\gamma}{1+Pt},$$

i.e., β^* is the optimal strategy.

If $c < \frac{1}{4}\gamma P$ and $t \leq a$ or $t \geq b$, then the same results hold.

If $c < \frac{1}{4}\gamma P$ and $t \in (a,b)$, then

$$\delta_\beta(t) \geq \frac{\gamma e^{Px_0}}{1 + P(t-x_0)} + cx_0.$$

Let $\beta_s^* = I(t - x_0 \leq s \leq t)$. Then, according to the second statement in the previous lemma, we obtain

$$\begin{aligned}\delta_{\beta^*}(t) &= \frac{\gamma e^{-P\int_0^t \beta_s^* ds}}{1 + P(t - \int_0^t \beta_s^* ds)} + c\int_0^t \beta_s^*\, ds \\ &= \frac{\gamma e^{-Px_0}}{1 + P(t-x_0)} + cx_0,\end{aligned}$$

which completes the proof of the theorem.

References

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