Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 17, 1992, 93–103

ON THE ANGLE FOR STATIONARY RANDOM FIELDS

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Abstract. The angle between past and future for stationary random fields on the lattice points of the plane is defined and it is shown that in contrast with other problems related to the past of random fields the positivity of the angle betweeen past and future is independent of different pasts which have been considered in the literature. It is shown that the positivity of the angle for random fields and processes is equivalent with that forming a Schauder basis. Besides this some analytic characterizations are also provided.

1. Introduction. A problem which has been proved to be useful in the prediction theory of stationary stochastic processes is the idea of the angle between past and future. Several authors have worked on this area and revealed its connection with the prediction theory of stationary stochastic processes, c.f. Helson and Szegő [4], Hunt, Muckenhoupt and Wheeden [5], Pousson [11], Sarason [13], Pourahmadi [10], and Miamee [8].

In this paper we introduce the definition of the angle for a stationary random field and prove that the crucial properties of the angle in the case of stationary processes have natural extensions to the case of stationary fields. In contrast to other problems concerning the past of random fields, we show that the positivity of the angle between past and future does not depend on the choice of different kind of pasts considered by different authors in the literature.

After setting up the necessary notation in Section 2 we prove our analytic and geometric characterization for positivity of the angle in Section 3.

The present paper is essentially based on the report [9]. In fact we reproduce some of the main results obtained already in [9]. Related results, based on [9], have been obtained also by Makagon and Salehi [7].

2. Preliminaries. In this section we introduce the notation and terminology needed in the rest of the paper. Let X_{mn}, $(m,n) \in Z^2$, be a double sequence of random variables on a probability space $(\Omega, \mathscr{B}, \mathbf{P})$ such that $\mathbf{E}X_{mn} = 0$ and $\mathbf{E}|X_{mn}|^2 < \infty$, for all $(m,n) \in Z^2$. The double sequence X_{mn}, $(m,n) \in Z^2$, is called a *stationary random field* if $\mathbf{E}X_{mn}\overline{X}_{rs}$ depends *only* on the differences

1991 Mathematics Subject Classification: Primary: 60G60.

Research was supported by AFOSR #F 49620 82 C 0009.

doi:10.5186/aasfm.1992.1712

$m-r$ and $n-s$; i.e. $\mathbf{E}X_{mn}\overline{X}_{rs} = \varrho(m-r, n-s)$. In this case the covariance function $\varrho(m,n) = \mathbf{E}X_{mn}\overline{X}_{rs}$ is a positive definite function on the group Z^2 of lattice points of plane. It is known (cf. for example Salehi and Scheidt [13]) that there exists a non-negative measure μ, defined on the Borel sets of the torus

$$T = \{\alpha : 0 \le \alpha \le 2\pi\} \times \{\beta : 0 \le \beta \le 2\pi\}$$

such that

$$\varrho(m,n) = \int e^{-i(m\alpha+n\beta)}d\mu, \qquad \text{for all } (m,n) \in Z^2. \tag{2.1}$$

The measure μ is called the *spectral measure* of the stationary random field X_{mn}. If μ is absolutely continuous with respect to the normalized Lebesgue measure

$$d\sigma = \frac{d\alpha\, d\beta}{4\pi^2}$$

its Radon–Nikodym derivative w is called the *spectral density* of the field.

By L^2_μ we denote the Hilbert space of all functions on the torus which are square summable with respect to the measure μ. From (2.1) it is clear that the operator

$$X_{mn} \to e^{-i(m\alpha+n\beta)}$$

extends to an isomorphism from $H_X =$ the closed linear subspace generated by all X_{mn}'s, onto L^2_μ. This is called the *Kolmogorov isomorphism* between the time domain and spectral domain.

For any subset M of Z^2 we define $H_X(M)$ to be the closed linear subspace of $L^2(\Omega, \mathscr{B}, \mathbf{P}) = H$, spanned by all X_{mn}'s with $(m,n) \in M$. The *vertical past-present* P^v_X and the *vertical future* F^v_X of the field X_{mn} is the subspace $H_X(S^v)$ and $H_X(\overline{S^v})$, respectively; where $S^v = \{(m,n) : m \le 0, n \in Z\}$. (Here and in what follows by $\overline{S}$ we denote the complement of a set $S \subset Z^2$; i.e. $\overline{S} = Z^2 \setminus S$.)

As a measure of the *angle* between the vertical past-present and future subspaces of the field X_{mn} we take its *vertical-cosine* defined by

$$\varrho^v_X = \sup\left\{\left|(Y,Z)\right| : Y \in P^v_X, Z \in F^v_X, \|Y\| = \|Z\| = 1\right\};$$

and the subspaces P^v_X and F^v_X are said to be at *positive angle* if $\varrho^v_X < 1$.

The *horizontal past-present* subspace P^h_X, the *horizontal future* subspace F^h_X and the *horizontal-cosine* of the angle between these subspaces ϱ^h_X are defined similarly. Finally we define

$$\varrho_X = \max(\varrho^v_X, \varrho^h_X).$$

For any nonnegative measure on the torus $\varrho^v_\mu, \varrho^h_\mu$ and ϱ_μ can be defined in a similar way. However, if μ is the spectral measure of our stationary random field X_{mn}, then by the Kolmogorov isomorphism it is evident that $\varrho^v_X = \varrho^v_\mu$, $\varrho^h_X = \varrho^h_\mu$, and $\varrho_X = \varrho_\mu$.

3. Characterizations for the positivity of the angle. In this section we present some geometric and analytic properties of stationary random fields X_{mn} with $\varrho_X < 1$. This will include the generalizations of some well known geometric and analytic characterizations for the stationary random processes. Our Theorem 3.4 which shows that $\varrho_X < 1$ if and only if X_{mn} forms a Schauder basis for its time domain seems to be new even in the case of random processes.

The proof of the following lemma is similar to the corresponding result in the case of stationary processes (cf. Helson and Szegő [4]) and is therefore omitted.

3.1. Lemma. *Let X_{mn} be a stationary random field, then*

(a) $\varrho_X^v < 1$ *if and only if there exists a constant N such that*

$$\Big\| \sum_{(m,n)\in S^v} a_{mn}X_{mn} \Big\|_H \le N \Big\| \sum_{(m,n)\in Z^2} a_{mn}X_{mn} \Big\|_H,$$

where $\{a_{mn}\}$ is any double sequence of scalars with finitely many non-zero elements.

(b) $\varrho_X^h < 1$ *if and only if there exists a constant M such that*

$$\Big\| \sum_{(m,n)\in S^h} a_{mn}X_{mn} \Big\|_H \le M \Big\| \sum_{(m,n)\in Z^2} a_{mn}X_{mn} \Big\|_H,$$

where $\{a_{mn}\}$ is as in (a).

Now we can prove the following:

3.2. Lemma. *Let X_{mn} be a stationary random field. Then $\varrho_X < 1$ if and only if there exists a constant K such that for any double sequence as in Lemma* 3.1(a) *and any integers m_0, m_1, n_0, and n_1 we have*

$$\Big\| \sum_{m=m_0}^{m_1} \sum_{n=n_0}^{n_1} a_{mn}X_{mn} \Big\|_H \le K \Big\| \sum_{(m,n)\in Z^2} a_{mn}X_{mn} \Big\|_H. \tag{3.3}$$

Proof. Using Lemma 3.1(a) and considering the fact that our field is stationary we have

$$\begin{aligned}\Big\| \sum_{m=m_0}^{m_1} \sum_{n=-\infty}^{\infty} a_{mn}X_{mn} \Big\|_H &\le N \Big\| \sum_{m=-\infty}^{m_1} \sum_{n=-\infty}^{\infty} a_{mn}X_{mn} \Big\|_H \\ &\le N^2 \Big\| \sum_{(m,n)\in Z^2} a_{mn}X_{mn} \Big\|_H.\end{aligned}$$

Applying Lemma 3.1(b) in a similar fashion, we get

$$\Big\| \sum_{m=m_0}^{m_1} \sum_{n=n_0}^{n_1} a_{mn}X_{mn} \Big\|_H \le N^2M^2 \Big\| \sum_{(m,n)\in Z^2} a_{mn}X_{mn} \Big\|_H.$$

To prove the converse take any double sequence $\{a_{mn}\}$ with finitely many nonzero elements. We know (3.3) is valid for this sequence $\{a_{mn}\}$. Now taking $m_1 = 0$ and m_0, n_0, n_1 large enough one can write (3.3) as

$$\Big\| \sum_{(m,n)\in S^v} a_{mn}X_{mn} \Big\|_H \le K \Big\| \sum_{(m,n)\in Z^2} a_{mn}X_{mn} \Big\|_H,$$

which by Lemma 3.1 implies $\varrho_X^v < 1$. Similarly one can get $\varrho_X^h < 1$ and hence $\varrho_X < 1$.

Next we show the following useful theorem which states that for a stationary random field X_{mn} the property $\varrho_X < 1$ is equivalent to the fact that X_{mn} is a Schauder basis for H_X. Recall that a double sequence c_{mn} is called a *Schauder basis* for a Hilbert space H if for any element Z in H there exists a uniquely determined set of coefficients $c_{mn}(Z)$ such that

$$Z = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn}(Z)c_{mn}.$$

We should mention here that by the convergence of a double series

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn}$$

to a limit we mean that the double sequence of its partial sums converges to that limit and by the partial sums we mean the rectangular partial sums, defined by

$$S_{m_0n_0}^{m_1n_1} = \sum_{m=m_0}^{m_1} \sum_{n=n_0}^{n_1} a_{mn}.$$

Having Lemma 3.2 proved the proof of the following theorem is a standard Schauder basis argument and, in particular, can be given similar to the proof given on pages 102 and 103 of [6], and hence it is omitted.

3.4. Theorem. *A stationary random field X_{mn} is a Schauder basis for H_X if and only if $\varrho_X < 1$.*

The following lemma shows that the notion of the positivity of angle is stronger than some other measures of independence. The lemma is also useful in our proof of the analytic characterization given in Theorem 3.6.

3.5. Lemma. *Let X_{mn} be a stationary random field. If $\varrho_X < 1$, then*

(a) *X_{mn} is horizontally nondeterministic, i.e. $\cap_{p\in Z} H_X(\{(m,n) : m \le p, n \in Z\}) = \{0\}$,*

(b) *X_{mn} is vertically nondeterministic, i.e. $\cap_{q\in Z} H_X(\{(m,n) : m \in Z, n \le q\}) = \{0\}$,*

(c) *X_{mn} is strongly nondeterministic, i.e. $\cap_{p,q\in Z} H_X(\{(m,n) : m \le p, n \le q\}) = \{0\}$,*

(d) *for any two subsets A and B of Z^2 we have $H_X(A\cap B) = H_X(A) \cap H_X(B)$.*

Proof. Since proof of (b) and (c) is similar to (a) we just prove (a) and (d). For the proof of (a) suppose that

$$Y \in \bigcap_{p\in Z} H_X(\{(m,n) : m \le p, n \in Z\}).$$

Then $Y \in H_X(\{(m,n) : m \le p, n \in Z\})$ for all $p \in Z$. It follows from Theorem 3.4 that X_{mn} is a Schauder basis for H_X. Hence $\{X_{mn}, m \le p, n \in Z\}$ is Schauder basis for $H_X(\{(m,n) : m \le p, n \in Z\})$ for each p (cf. the characterization in [6; p. 103]). Thus for each $p \in Z$ we have a representation of the form

$$Y = \sum_{m=-\infty}^{p} \sum_{n=-\infty}^{\infty} b^p_{mn} X_{mn}.$$

But since these representations for Y must be unique, we conclude that $b^p_{mn} = b^q_{mn}$ for all p, q, m and n, yielding $b^p_{mn} = 0$. Thus, $Y = 0$, which completes the proof of (a).

To see (d) take any Y in $H_X(A) \cap H_X(B)$. Then $Y \in H_X(A)$ and $Y \in H_X(B)$ so we have

$$Y = \sum_{(m,n)\in A} a^A_{mn} X_{mn} \qquad \text{and} \qquad Y = \sum_{(m,n)\in B} a^B_{mn} X_{mn},$$

since the elements X_{mn}, $(m,n) \in A$, and X_{mn}, $(m,n) \in B$, form a Schauder basis in the spaces $H_X(A)$ and $H_X(B)$, respectively (cf. [6; p. 102]). Since the sequence X_{mn}, $(m,n) \in Z^2$, is a Schauder basis in H_X (cf. Theorem 3.4), it follows from the uniqueness property of the Schauder basis that

$$a^A_{mn} = a^B_{mn} \qquad \text{for all } (m,n) \in Z^2.$$

Especially

$$a^A_{mn} = a^B_{mn} = 0 \qquad \text{for all } (m,n) \notin A \cap B,$$

yielding

$$Y = \sum_{(m,n)\in A\cap B} a^A_{mn} X_{mn}.$$

Thus, $Y \in H_X(A \cap B)$. This shows that $H_X(A) \cap H_X(B) \subseteq H_X(A \cap B)$, and the other inclusion is obviously always correct.

We now prove the following generalization of a well-known analytic characterization for the positivity of the angle between past and future for stationary random processes due to Helson and Szegő [4] and Hunt, Muckenhoupt and Wheeden [5]. (For the matricial form of this result one can see Pousson [11], Pourahmadi [10] and Miamee [8].)

3.6. Theorem. *Let X_{mn} be a stationary random field with a spectral measure μ on the torus. Then, $\mu_X < 1$ if and only if*

(a) *μ is absolutely continuous with respect to the normalized Lebesgue measure $d\sigma = d\alpha\, d\beta / 4\pi^2$, with spectral density w,*
(b) *$L^2_w \subset L^1$, where $L^2_w = L^2_{wd\sigma} = L^2_\mu$,*
(c) *the Fourier series of any $f \in L^2_w$ converges to f in the norm of L^2_w.*

Proof. By Lemma 3.5(c) X_{mn} is strongly nondeterministic in the sense of Soltani [14] and hence by Theorem 3.4 in [14] μ is absolutely continuous with respect to the Lebesgue measure and its spectral density w has the log property, i.e. $\log w \in L^1$. This shows (a).

To see (b) let I be an operator defined on the polynomials

$$P = \sum_{(m,n)\in Z^2} a_{mn} e^{i(m\alpha + n\beta)}$$

by

$$I(P) = \iint a_{00} w(\alpha, \beta)\, d\sigma.$$

The operator is bounded because

$$\begin{aligned} |I(P)| &= \left| \iint a_{00} w(\alpha,\beta)\, d\sigma \right| = \left| \iint a_{00} \sqrt{w}\sqrt{w}\, d\sigma \right| \\ &\le \left(\iint |a_{00}|^2 w\, d\sigma \right)^{1/2} \left(\iint w\, d\sigma \right)^{1/2}. \end{aligned}$$

Hence by Lemma 3.2 and the Kolmogorov isomorphism we get

$$|I(P)| \le K \, \|P\|_{L^2_w} \sqrt{\iint w\, d\sigma}.$$

Thus I can be extended to a bounded functional on L^2_w. Hence there exists a function g in L^2_w such that

$$I(f) = (f, g)_{L^2_w} \qquad \text{for all } f \in L^2_w.$$

In particular we have $I\big(e^{i(m\alpha+n\beta)}\big) = \big(e^{i(m\alpha+n\beta)}, g\big)_{L^2_w}$. On the other hand

$$I\big(e^{i(m\alpha+n\beta)}\big) = \begin{cases} 0, & \text{if } (m,n) \neq (0,0) \\ 1, & \text{if } (m,n) = (0,0). \end{cases}$$

Thus we have

$$\iint e^{i(m\alpha+n\beta)} w(\alpha,\beta)\overline{g(\alpha,\beta)}\, d\sigma = \begin{cases} 0, & \text{if } (m,n) \neq (0,0) \\ 1, & \text{if } (m,n) = (0,0), \end{cases}$$

which means $w\overline{g} \equiv 1$. Hence $w^{-1} = \overline{g} \in L^2_w$; implying $L^2_w \subset L^1$ because for any $h \in L^2_w$,

$$\iint |h|\, d\sigma = \iint |h|\sqrt{w}\sqrt{w^{-1}}\, d\sigma \leq \Big(\iint |h|^2 w\, d\sigma\Big)^{1/2}\Big(\iint w^{-1}\, d\sigma\Big)^{1/2}.$$

To prove (c), by Lemma 3.2 the operators S_{mn} defined on the polynomials $\sum_{(m,n)\in Z^2} a_{mn} e^{i(m\alpha+n\beta)}$ by

$$S_{mn}\Big(\sum_{(p,q)\in Z^2} a_{pq} e^{i(p\alpha+q\beta)}\Big) = \sum_{p=-m}^{m}\sum_{q=-n}^{n} a_{pq} e^{i(p\alpha+q\beta)}$$

are bounded, with common bound K. Hence each of the operators S_{mn} can be extended to a bounded operator on L^2_w, so that they have the same norm K. It can be seen that these operators are just the symmetric Fourier partial sum operators. Letting $f \in L^2_w$, to complete the proof of this part we just have to show that

$$S_{mn}(f) \to f, \text{ in the } L^2_w \text{ sense.}$$

Given $\varepsilon > 0$, we take a polynomial P such that $\|f - P\|_{L^2_w} < \varepsilon/(K+1)$, we then have

$$\begin{aligned} \|S_{mn}(f) - f\|_{L^2_w} &\leq \|(S_{mn} - I)(f - P)\|_{L^2_w} + \|S_{mn}(P) - P\|_{L^2_w} \\ &\leq \|S_{mn} - I\|\, \|f - P\|_{L^2_w} + \|S_{mn}(P) - P\|_{L^2_w}. \end{aligned}$$

Thus

$$\|S_{mn}(f) - f\|_{L^2_w} \leq (K+1)\frac{\varepsilon}{K+1} + \|S_{mn}(P) - P\|_{L^2_w}.$$

Now if we take m and n large enough we get $S_{mn}(P) = P$, and hence

$$\|S_{mn}(f) - f\|_{L^2_w} < \varepsilon.$$

Now, to prove the other half assume that the conditions (a), (b), and (c) in the statement of the theorem hold. Then by (b) any function f in L^2_w belongs to L^1 and, as such, has a Fourier series

$$f \sim \sum_{(p,q)\in Z^2} a_{pq} e^{i(p\alpha+q\beta)}.$$

We consider the partial sum operator $S^{m_1 n_1}_{m_0 n_0}\colon L^2_w \to L^2_w$ defined by

$$S^{m_1 n_1}_{m_0 n_0}(f) = \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{n_1} a_{pq} e^{i(p\alpha+q\beta)}.$$

For any $f \in L^2_w$ we can write

$$\left\| S^{m_1 n_1}_{m_0 n_0}(f) \right\|_{L^2_w} = \left\| \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{n_1} a_{pq} e^{i(p\alpha+q\beta)} \right\|_{L^2_w} \le \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{n_1} \left\| a_{pq} \right\|_{L^2_w}.$$

Hence

$$\begin{aligned} \left\| S^{m_1 n_1}_{m_0 n_0}(f) \right\|_{L^2_w} &= \sqrt{\iint w\, d\sigma} \sum_{p=-m_0}^{m_1} \sum_{q=-n_0}^{n_1} |a_{pq}| \\ &\le (2m_0 m_1 + 2n_0 n_1 + 1) \sqrt{\iint w\, d\sigma}\, \|f\|_{L^1}. \end{aligned}$$

Now since $L^2_w \subset L^1$, Lemma 3.1 of Miamee [8] implies that there exists a constant K such that $\|f\|_{L^1} \le K \|f\|_{L^2_w}$. This means that all operators $S^{m_1 n_1}_{m_0 n_0}$ are bounded. On the other hand, by (c) for each $f \in L^2_w$ we have

$$S^{m_1 n_1}_{m_0 n_0}(f) \to f, \text{ in } L^2_w.$$

Hence by the uniform boundedness principle there exists a constant M such that

$$\left\| S^{m_1 n_1}_{m_0 n_0} \right\| \le M, \qquad \text{for all } m_0, m_1, n_0, n_1 \ge 0.$$

This means that for any $f \in L^2_w$ we have

$$\left\| S^{m_1 n_1}_{m_0 n_0}(f) \right\|_{L^2_w} \le M \|f\|_{L^2_w}, \qquad \text{for all } m_0, m_1, n_0, n_1 \ge 0.$$

In particular for any polynomial P we have

$$\left\| S^{m_1 n_1}_{m_0 n_0}(P) \right\|_{L^2_w} \le M \|P\|_{L^2_w}, \qquad \text{for all } m_0, m_1, n_0, n_1 \ge 0.$$

But this, up to the Kolmogorov isomorphism, is just (3.3). Hence by Lemma 3.2 we deduce $\varrho_X < 1$.

3.7. Remark. The proof of Theorem 3.6 shows that $w^{-1} \in L^1$ and this in turn means that (cf. for example Salehi and Scheidt [12]) the random field X_{mn} with spectral measure $d\mu = w\,d\sigma$ is minimal, i.e.

$$X_{mn} \notin H_X(\{(p,q) : (p,q) \neq (m,n)\}).$$

This in particular implies that X_{mn} is purely nondeterministic in the Helson–Lowdenslager sense [2]. Hence we have $\iint \log w\,d\sigma > -\infty$. This also shows that the positivity of angle is stronger than some other measures of independence for a random field.

The following lemma shows that in sharp contrast to other prediction problems for random fields the positivity of the angle is independent of the kind of past one may take, whether it is taken to be the usual half plane [1], the Helson–Lowdenslager's half plane [2], [3], or the quarter plane.

3.8. Lemma. *If X_{mn} is a stationary random field, then $\varrho_X < 1$ if and only if the angle between $H_X(U^v)$ and $H_X(\overline{U^v})$ as well as the angle between $H_X(U^h)$ and $H_X(\overline{U^h})$ are positive, where*

$$U^v = \{(m,n) : m \leq -1, n \in Z\} \cup \{(0,n) : n \leq -1\}$$

and

$$U^h = \{(m,n) : m \in Z, n \leq -1\} \cup \{(m,0) : m \leq -1\}.$$

If this is the case then the angle between $H_X(Q)$ and $H_X(\overline{Q})$ is also positive, where Q is the third quadrant, namely

$$Q = \{(m,n) : m \leq 0, n \leq 0\}.$$

Proof. We break the proof of our lemma into the following steps:

Step 1. The angle between $H_X(U^v)$ and $H_X(\overline{U^v})$ is positive if and only if there exists a constant N such that

$$\Big\| \sum_{(m,n)\in U^v} a_{mn} X_{mn} \Big\|_H \leq N \Big\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \Big\|_H,$$

where $\{a_{mn}\}$ is any double sequence of scalars with finitely many non-vanishing elements. This statement can be proved similar to Lemma 3.1.

Step 2. The angle between $H_X(U^v)$ and $H_X(\overline{U^v})$ as well as the angle between $H_X(U^h)$ and $H_X(\overline{U^h})$ is positive if and only if there exists a constant L such that

$$\Big\| \sum_{(m,n)\in R} a_{mn} X_{mn} \Big\|_H \leq L \Big\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \Big\|_H, \tag{3.9}$$

where $\{a_{mn}\}$ is any double sequence of scalars with finitely many non-vanishing elements, and the first summation ranges over any *generalized rectangle* of the form

$$R = U^v_{m_0 n_0} \cap \overline{U}^v_{m_1 n_1} \cap U^h_{p_0 q_0} \cap \overline{U}^h_{p_1 q_1}, \tag{3.10}$$

where

$$U^v_{m,n} = \{(r,s) : r \leq m-1, s \in Z\} \cup \{(m,s) : s \leq n-1\},$$

and

$$U^h_{m,n} = \{(r,s) : r \in Z, s \leq n-1\} \cup \{(r,n) : r \leq m-1\}.$$

The proof of this step is similar to that of Lemma 3.2 and hence it is again omitted.

Step 3. We note that the generalized rectangles contain all the usual rectangles. Thus to complete the proof of the lemma it suffices to show that $\varrho_X < 1$ implies (3.9). To see this we observe that any region R in (3.9), that is any region R of the form (3.10), can be represented as a disjoint union of at most 5 usual rectangles, say R_i, $i = 1,2,3,4,5$. Thus we can take the L in Step 2 to be simply $5M$. In fact, for any R of the form in (3.9) or (3.10) we can write

$$\begin{aligned}\Big\| \sum_{(m,n)\in R} a_{mn} X_{mn} \Big\|_H &\leq \sum_{i=1}^{5} \Big\| \sum_{(m,n)\in R_i} a_{mn} X_{mn} \Big\|_H \\ &\leq \sum_{i=1}^{5} M \Big\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \Big\|_H \\ &\leq 5M \Big\| \sum_{(m,n)\in Z^2} a_{mn} X_{mn} \Big\|_H.\end{aligned}$$

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