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MARTINGALE-DIFFERENCE GIBBS RANDOM FIELDS AND CENTRAL LIMIT THEOREM

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Abstract. The notion of a martingale difference random field is introduced, and sufficient conditions for a Gibbs random field to possess the martingale difference property are studied. Various central limit theorems for such random fields are given.

Let $\mathbf{Z}^\nu$ be a lattice ($\nu \geq 1$) and W be the family of all its finite subsets.

Definition 1. We say that a random field ξ_t, $t \in \mathbf{Z}^\nu$, is a martingale-difference random field with respect to a given sequence of increasing finite subsets (s.i.f.s.) $V_i \in W$, $i = 1, 2 \ldots$, if $\mathbf{E}|\xi_t| < \infty$, $t \in \mathbf{Z}^\nu$, and

$$\mathbf{E}\big(\xi_t \mid \xi_s, s \in V_{i-1}\big) = 0, \qquad \text{a.s.} \tag{1}$$

for all $t \in V_i \setminus V_{i-1}$ and $i = 2, 3, \ldots$.

If (1) holds for all s.i.f.s. then the random field is called simply a martingale-difference random field.

Definition 2. We say that a random process S_V, $V \in W$, forms a martingale with respect to a s.i.f.s. V_i, $i = 1, 2, \ldots$, if $\mathbf{E}|S_V| < \infty$, $V \in W$, and the relation

$$\mathbf{E}\big(S_{V_i} \mid S_{V_1}, S_{V_2}, \ldots, S_{V_{i-1}}\big) = S_{V_{i-1}}, \qquad \text{a.s.} \tag{2}$$

is valid for all $i = 2, 3, \ldots$.

It is easy to see that if a random field ξ_t, $t \in \mathbf{Z}^\nu$, is a martingale-difference with respect to a s.i.f.s. V_i, $i = 1, 2, \ldots$, then the random process $S_V = \sum_{t \in V} \xi_t$, $V \in W$, is a martingale with respect to V_i, $i = 1, 2, \ldots$. Conversely, if a random process $S_V = \sum_{t \in V} \xi_t$, $V \in W$, is a martingale with respect to any s.i.f.s. then the random field ξ_t, $t \in \mathbf{Z}^\nu$, is a martingale-difference random field.

There is a wide spectrum of examples of random fields and processes satisfying the conditions in these definitions.

Example 1. Let V_i, $i = 1, 2, \ldots$, be a s.i.f.s. and $\Delta_j = V_j \backslash V_{j-1}$, $j = 2, 3, \ldots$. Suppose a random field ξ_t, $t \in \mathbf{Z}^\nu$ has the properties: $\mathbf{E}\xi_t = 0$, $t \in \mathbf{Z}^\nu$, and ξ_t, ξ_s are independent for $t \in \Delta_j$, $s \in \Delta_k$, $j \neq k$. Then it is a martingale-difference with respect to V_i, $i = 1, 2, \ldots$.

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Example 2. Let ξ_t, $t \in \mathbf{Z}^\nu$, be a random field taking values in a separable complete metric space X with a σ-finite measure μ, $\mu(X) > c$, defined on its Borel subsets. Suppose the finite dimensional distributions of this random field are absolutely continuous with respect to the product measures $\mu_v = \mu^{|v|}$, $v \in W$ ($|\cdot|$ stands for the number of points in a finite set) and suppose the densities $p_V(x_t, t \in V)$, $V \in W$, are strictly positive. Moreover, suppose $q_V(x_t, t \in V)$, $V \in W$, is another system of consistent densities. Then the process

$$S_V = \frac{q_V(\xi_t, t \in V)}{p_V(\xi_t, t \in V)}, \qquad V \in W,$$

is a martingale with respect to any s.i.f.s.

Example 3. Suppose ξ_t, $t \in \mathbf{Z}^\nu$, is a random field satisfying $\mathbf{E}|\xi_t| < \infty$, $t \in \mathbf{Z}^\nu$, and

$$\mathbf{E}\big(\xi_t \mid \xi_s, s \in \mathbf{Z}^\nu \setminus \{t\}\big) = 0 \qquad \text{a.s.}$$

Then ξ_t, $t \in \mathbf{Z}^\nu$, is a martingale-difference random field.

Example 4. Suppose $\mathbf{Z}^\nu = \cup_j T_j$, $T_j \cap T_k = \emptyset$, $j \neq k$; and suppose ξ_t, $t \in \mathbf{Z}^\nu$, is a random field having the property: $S_V = \sum_{t \in V} \xi_t$, $V \subset T_j$, is a martingale with respect to any s.i.f.s. V_i, $i = 1, 2, \ldots$, $V_i \subset T_j$ for any fixed $j = 1, 2, \ldots$. If S_V and $S_{\tilde{V}}$ are, in addition, independent for $V \subset T_j$ and $\tilde{V} \subset T_k$, $j \neq k$, then the random field ξ_t, $t \in \mathbf{Z}^\nu$, is a martingale-difference.

The following example is a special case of Example 4.

Example 5. Suppose a random field ξ_t, $t \in \mathbf{Z}^2$, $\mathbf{E}|\xi_t| < \infty$, $t \in \mathbf{Z}^2$ has the property: for any p, $k \in \mathbf{Z}^1$

$$\mathbf{E}\big(\xi_{(p,k)} \mid \ldots \xi_{(p-1,k)}, \xi_{(p+1,k)} \ldots\big) = 0 \qquad \text{a.s.}$$

and $\xi_{(p,k)}$, $\xi_{(q,j)}$ are independent for $k \neq j$. Then the random field ξ_t, $t \in \mathbf{Z}^2$, is a martingale-difference.

Note that each of the examples considered here has an analogy in the theory of martingales.

We introduce next a new construction of martingale-difference random fields which is useful also in the theory of Gibbs random fields.

Suppose Y, $Y \subset R^1$, is a symmetric set with respect to the origin (i.e., if $y \in Y$ then $-y \in Y$) and $\mathscr{B}(Y)$ is the σ-algebra of its Borel subsets. Consider a symmetric measure μ on $\mathscr{B}(Y)$ (i.e., $\mu(B) = \mu(-B)$, $B \in \mathscr{B}(Y)$) satisfying

$$\int_Y |y|\, \mu(dy) < \infty.$$

Lemma 1. *Let ξ_t, $t \in \mathbf{Z}^\nu$, be a random field taking values in Y. Suppose its finite-dimensional distributions are absolutely continuous with respect to the* *product-measures* $\mu^{|v|}$, $V \in W$, *with densities* $p_V(y_{t_1}, y_{t_2}, \ldots, y_{t_{|V|}})$, $V \in W$, *satisfying*

$$p_V(\theta_1 y_{t_1}, \theta_2 y_{t_2}, \ldots, \theta_{t_{|V|}} y_{t_{|V|}}) = p_V(y_{t_1}, y_{t_2}, \ldots, y_{t_{|V|}}), \qquad V \in W,$$

for any $\theta_i \in \{-1, 1\}$, $i = 1, 2, \ldots, |V|$ *(the superparity property). Then the random field* ξ_t, $t \in \mathbf{Z}^\nu$, *is a martingale-difference.*

Proof. It is sufficient to show that $S_V = \sum_{t \in V} \xi_t$, $V \in W$, is a martingale with respect to an arbitrary s.i.f.s. That is, for any sequence $V_i \in W$, $V_i \subset V_{i+1}$, $i = 1, 2, \ldots$, one has

$$\sum_{t \in V_i \setminus V_{i-1}} \mathbf{E}(\xi_t \mid S_{V_1}, S_{V_2}, \ldots, S_{V_{i-1}}) = 0 \qquad \text{a.s.}$$

or

$$\sum_{t \in V_i \setminus V_{i-1}} \int_A \xi_t \mathbf{P}(d\omega) = 0 \qquad \text{for any } A \in \sigma(\xi_s, s \in V_{i-1}).$$

This relation can be rewritten in the following form: for any

$$B \in \mathscr{B}(Y_{V_{i-1}}), \qquad V_{i-1} \in W,$$

$$\sum_{t \in V_i \setminus V_{i-1}} \int_{B \times Y_t} x p_{V_{i-1} \cup \{t\}}(y, x) \mu_t(dx) \mu_{V_{i-1}}(dy) = 0.$$

However, taking into account the superparity of the densities p_V and the symmetry of the measure μ_t we have

$$\int_{Y_t} x p_{V_{i-1} \cup \{t\}}(y, x) \mu_t(dx) = 0.$$

The lemma is proven.

Now we introduce the notion of a Gibbs random field. Note that the definition given below is not general.

Let $(Y_t, \mathscr{B}_t, \mu_t)$, $t \in \mathbf{Z}^\nu$, be a copy of $(Y, \mathscr{B}, \mu)$. A system of measurable functions $\Phi = \{\Phi_V, V \in W\}$, defined on $(Y_V, \mathscr{B}_V, \mu_V)$, is called a potential. Here

$$Y_V = \otimes_{t \in V} Y_t, \qquad \mathscr{B}_V = \otimes_{t \in V} \mathscr{B}_t, \qquad \mu_V = \otimes_{t \in V} \mu_t,$$

($\otimes$ is here also the symbol of the Cartesian product).

For any $v \in W$ and $\bar{y} \in Y_{\mathbf{Z}^\nu \setminus V}$ we define a function

$$U_V^{\bar{y}}(y) = \sum_{J \subset V} \sum_{\tilde{J} \subset \mathbf{Z}^\nu \setminus V} \Phi_{J \cup \tilde{J}}(y_J, \bar{y}_{\tilde{J}}), \; y \in Y_V, \qquad y_J = (y_s, s \in J), \; \bar{y}_{\tilde{J}} = (\bar{y}_s, s \in \tilde{J}),$$

which is called the potential energy. This quantity is finite if for example the condition

$$\sup_{a\in \mathbf{Z}^\nu} \sum_{J: a\in J\in W} \sup_{y\in Y_J} \left|\Phi_J(y)\right| < \infty$$

is satisfied.

Define

$$\left(q_V^{\bar{y}}\right)_I(y) = \frac{\int_{Y_{V\setminus I}} \exp\left\{ -U_V^{\bar{y}}(y,z)\right\}\mu_{V\setminus I}(dz)}{\int_{Y_V} \exp\left\{ -U_V^{\bar{y}}(z)\right\}\mu_V(dz)}, \qquad y\in Y_I,\ \bar{y}\in Y_{\mathbf{Z}^\nu\setminus V}.$$

Suppose that for any $I \in W$ there exist an increasing sequence $V_k \in W$, $k = 1,2,\ldots$, satisfying $\cup_k V_k = \mathbf{Z}^\nu$, and boundary conditions $\bar{y}_k \in Y_{\mathbf{Z}^\nu\setminus V_k}$ such that the limes (uniform with respect to $y \in Y_I$)

$$\lim_{k\to\infty} \left(q_{V_k}^{\bar{y}_k}\right)_I(y) = p_I(y)$$

exists. Then the system of finite-dimensional densities $\{p_I, I \in W\}$ is consistent, and hence there exists a random field called a Gibbs random field.

Definition 3. We say that a potential Φ is a superparity potential if for any $y = (y_{t_1},\ldots,y_{t_{|V|}})$, $y_{t_i} \in Y$, $V \in W$, the relation

$$\Phi_V(\theta_1 y_{t_1},\ldots,\theta_{|V|} y_{t_{|V|}}) = \Phi(y_{t_1},\ldots,y_{t_{|V|}}), \qquad \theta_i \in \{-1,1\},$$

holds.

Lemma 2. *Let Φ be a potential satisfying*

$$\sup_{a\in \mathbf{Z}^\nu} \sum_{J: a\in J\in W} \sup_{y\in Y_J} \left|\Phi_J(y)\right| < \infty.$$

If the potential Φ has the superparity property, then the corresponding Gibbs random field is a martingale-difference.

This lemma follows from the superparity of the potential Φ combined with Lemma 1.

One can construct many examples of superparity potentials $\Phi = \{\Phi_V,\ V \in W\}$. For example, for $V \in W$ define

$$\Phi_V(y_t, t\in V) = \begin{cases} \prod_{t\in V} |y_t|(\operatorname{Diam} V)^{-\gamma}, & \gamma > 0,\ |V| \le 2, \\ 0, & |V| > 2; \end{cases}$$

or

$$\Phi_V(y_t, t\in V) = \exp\left\{ -\sup_{t\in V} |y_t||V|^\gamma \right\}, \qquad \gamma > 0.$$

There are several well-known articles dealing with the central limit theorem for martingale-difference random processes.

The next theorem is a direct consequence of the results of Brown [3] and Dvoretsky [5].

Theorem 1. *Let S_V, $V \in W$, be a martingale asociated with a s.i.f.s. V_i, $i = 1, 2, \dots$. Suppose the following conditions hold:*

1) $\mathbf{D}\eta_{\Delta_j} \geq \sigma^2 |\Delta_j|$, $\sigma^2 > 0$, $\mathbf{E}|\eta_{\Delta_j}|^{2+\delta} \leq C|\Delta_j|^{1+\delta/2}$, $0 < C < \infty$, $\delta > 0$; *here* $\mathbf{D}$ *stands for the variance, and* $\Delta_j = V_j \setminus V_{j-1}$, $\eta_{\Delta_j} = S_{V_j} - S_{V_{j-1}}$, $j = 2, 3, \dots$;
2) $\lim_{n\to\infty}(\mathbf{D}S_{V_n})^{-1} \sum_{j=2}^{n} \operatorname{Cov}(\eta^2_{\Delta_j}, \operatorname{sgn} I_j) = 0$, *where* $I_j = \mathbf{E}(\eta^2_{\Delta_j} \mid \eta_{\Delta_1}, \dots, \eta_{\Delta_{j-1}}) - \mathbf{E}\eta^2_{\Delta_j}$, $j = 2, 3, \dots$.

Then for any $x \in R^1$

$$\lim_{n\to\infty} \mathbf{P}\big((\mathbf{D}S_{V_n})^{-1/2}(S_{V_n} - \mathbf{E}S_{V_n}) < x\big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

i.e., the central limit theorem holds.

The conditions of this type can be checked for the martingales S_V, $V \in W$, which are sums of weakly dependent random variables (see [1], [6–8]). In this paper we restrict ourselves to the consideration of martingales which are sums of components of a weakly dependent random field. We use the following mixing coefficients

$$\alpha_{m,n}(p) = \sup_{m,n\in\mathbf{N}\cup\{\infty\}} \big\{\alpha(M_I, M_V); I, V \in W, \varrho(I,V) \geq p, |I| \leq m, |V| \leq n\big\},$$

$$\alpha(M_I, M_V) = \sup_{A\in M_I, B\in M_V} \big|P(AB) - P(A)P(B)\big|,$$

$$M_I = \sigma(\xi_t, t \in I), \quad M_V = \sigma(\xi_t, t \in V),$$

$$\varrho(I,V) = \inf_{s\in V, t\in I} \varrho(s,t), \quad \varrho(s,t) = \max_{1\leq i\leq \nu} |s^{(i)} - t^{(i)}|, \quad s,t \in \mathbf{Z}^\nu.$$

Theorem 2. *Let ξ_t, $t \in \mathbf{Z}^\nu$, be a martingale-difference field satisfying* $\mathbf{E}\xi_t = 0$, $\inf_{t\in\mathbf{Z}^\nu} \mathbf{E}\xi_t^2 = \sigma_0^2 > 0$, *and*

$$|\xi_t| < C, \qquad t \in \mathbf{Z}^\nu, \quad \text{a.s.}$$

Suppose

$$\alpha_{m,n}(p) \leq f(m)\alpha(p)$$

where $f(m)$ is some function and $\alpha(p) \to 0$ as $p \to \infty$. Then for any increasing sequence of cubes $V_n \subset \mathbf{Z}^\nu$, $n = 1, 2, \dots$

$$\lim_{n\to\infty} \mathbf{P}\left(\Big(\mathbf{D}\sum_{t\in V_n} \xi_t\Big)^{-1/2} \sum_{t\in V_n} \xi_t < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

Note that in contrast to the well-known central limit theorems for weakly dependent random fields (see for example [2], [4], [9–11]) the mixing coefficients in Theorem 2 do not depend on the dimension of the lattice $\mathbf{Z}^\nu$.

Theorem 3. *Let Φ be a translation invariant potential having the superparity property. Suppose Φ has a sufficiently small norm*

$$\|\Phi\| = \sum_{J:0\in J\in W} |J| \sup_{y\in Y_J} \left|\Phi_J(y)\right|.$$

Then the central limit theorem holds for the corresponding Gibbs random field for any increasing sequence of cubes $V_n \subset \mathbf{Z}^\nu$, $n = 1, 2, \ldots$.

This theorem is a consequence of Theorem 2 and Theorem 9.1.1 of [10].

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