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GENERALIZATIONS OF ROSENTHAL'S INEQUALITIES

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1. We shall prove some generalizations of two well-known inequalities for moments of sums of independent random variables obtained by Rosenthal [5], [6]. Instead of the classical power moments we consider moments belonging to a more general class. Another generalization is connected with one-sided moments. We prove some inequalities for generalized moments of this type for the maximum of partial sums of independent random variables.

2. Let $X_1, X_2, \dots, X_n$ be independent random variables, $S_k = \sum_{i=1}^k X_i$. Let G_0 be the set of non-negative even functions $g(x)$, $x \in \mathbf{R}$, non-decreasing on the positive half-axis and satisfying $g(0)=0$.

Theorem 1. *Suppose*

$$\mathbf{E}X_k = 0, \qquad k = 1, \dots, n, \tag{1}$$

and

$$0 < B_n < \infty, \tag{2}$$

where

$$B_n = \sum_{k=1}^n \mathbf{E}X_k^2. \tag{3}$$

If

$$\mathbf{E}g(X_k) < \infty, \qquad k = 1, \dots, n, \tag{4}$$

for some $g \in G_0$, then

$$\mathbf{E}g\Bigl(\max_{1\le k\le n} S_k\Bigr) \le \sum_{k=1}^n \mathbf{E}g(rX_k) + 2e^r \int_0^\infty \Bigl(1 + \frac{x^2}{rB_n}\Bigr)^{-r} dg(x) \tag{5}$$

for every $r > 0$.

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Remark. For $g(x) = |x|^p$, $x \in \mathbf{R}$, $p \geq 2$, we obtain

$$\mathbf{E}\Big|\max_{1\leq k\leq n} S_k\Big|^p \leq r^p M_{p,n} + 2pe^r r^{p/2} B\Big(\frac{p}{2}, r - \frac{p}{2}\Big) B_n^{p/2} \tag{6}$$

for every $p \geq 2$ and $r > \frac{1}{2}p$, where

$$M_{p,n} = \sum_{k=1}^{n} \mathbf{E}|X_k|^p \tag{7}$$

and $B(x,y)$ is the Beta-function.

Let X be a random variable with the distribution function $F(x)$, $x \in \mathbf{R}$. In what follows we use the notation

$$\mathbf{E}^+ g(X) = \int_0^\infty g(x)\, dF(x).$$

Theorem 2. *Suppose*

$$\mathbf{E}^+ g(X_k) < \infty, \qquad k = 1, \ldots, n, \tag{8}$$

for some $g \in G_0$. *If the conditions* (1) *and* (2) *are satisfied, then*

$$\mathbf{E}^+ g\Big(\max_{1\leq k\leq n} S_k\Big) \leq \sum_{k=1}^{n} \mathbf{E}^+ g(rX_k) + e^r \int_0^\infty \Big(1 + \frac{x^2}{rB_n}\Big)^{-r} dg(x) \tag{9}$$

for every $r > 0$.

Theorem 3. *Suppose*

$$0 < D_n < \infty \tag{10}$$

where

$$D_n = \sum_{k=1}^{n} \mathbf{E}|X_k|. \tag{11}$$

If the condition (4) *is satisfied for some* $g \in G_0$, *then*

$$\mathbf{E} g\Big(\max_{1\leq k\leq n} S_k\Big) \leq \sum_{k=1}^{n} \mathbf{E} g(rX_k) + 2e^r \int_0^\infty \Big(1 + \frac{x}{D_n}\Big)^{-r} dg(x) \tag{12}$$

for every $r > 0$.

Remark. For $g(x) = |x|^p$, $x \in \mathbf{R}$, $p > 1$, we get

$$\mathbf{E}\big|\max_{1\le k\le n} S_k\big|^p \le r^p M_{p,n} + 2pe^r B(p, r-p) D_n^p \tag{13}$$

for every $r > p > 1$, where $B(x,y)$ is the Beta-function and $M_{p,n}$ is defined by (7).

Theorem 4. *If the conditions* (10) *and* (8) *are satisfied for some* $g \in G_0$, *then*

$$\mathbf{E}^+ g\Big(\max_{1\le k\le n} S_k\Big) \le \sum_{k=1}^n \mathbf{E}^+ g(rX_k) + e^r \int_0^\infty \Big(1 + \frac{x}{D_n}\Big)^{-r} dg(x) \tag{14}$$

for every $r > 0$.

3. Proof of Theorems 1 and 2.

Lemma 1. *Let* $y_1, \dots, y_n$ *be positive numbers,* $y = \max\{y_1, \dots, y_n\}$. *If the condition* (2) *holds, then*

$$\mathbf{P}\Big(\max_{1\le k\le n} S_k \ge x\Big) \le \sum_{k=1}^n \mathbf{P}(X_k \ge y_k) + \exp\Big\{\frac{x}{y} - \frac{x}{y}\log\Big(1 + \frac{xy}{B_n}\Big)\Big\}$$

and

$$\mathbf{P}\Big(\big|\max_{1\le k\le n} S_k\big| \ge x\Big) \le \sum_{k=1}^n \mathbf{P}(|X_k| \ge y_k) + 2\exp\Big\{\frac{x}{y} - \frac{x}{y}\log\Big(1 + \frac{xy}{B_n}\Big)\Big\}$$

for every $x > 0$.

Lemma 1 follows from inequalities of Fuk and Nagaev [2] and a result of Borovkov [1] (see also Lemma 13, inequality (5.5) and Supplement 16 (Section 6) in Chapter 3 of [4]).

Lemma 2. *If* X *is a random variable and* $\mathbf{E}^+ g(X) < \infty$ *for some* $g \in G_0$, *then*

$$\mathbf{E}^+ g(X) = \int_0^\infty \mathbf{P}(X \ge x)\, dg(x).$$

If $\mathbf{E}g(X) < \infty$ *for some* $g \in G_0$, *then*

$$\mathbf{E}g(X) = \int_0^\infty \mathbf{P}(|X| \ge x)\, dg(x).$$

It is easy to prove this lemma by integrating by parts the expressions appearing on the right-hand sides of the last two equalities. We take into account also the relations $g(0)=0$ and $\lim_{x\to+\infty} g(x)P(X\geq x)=0$, which follows from the inequality

$$g(x)\mathbf{P}(X\geq x)\leq\int_x^\infty g(y)\,dF(y),\qquad x>0,$$

where $F(y)$, $y\in\mathbf{R}$, stands for the distribution function of X.

Let $x>0$ and $r>0$. To prove Theorem 2 we put in Lemma 1 $y_k=y=x/r$, $k=1,\ldots,n$. We then have

$$\mathbf{P}\Big(\max_{1\leq k\leq n} S_k\geq x\Big)\leq\sum_{k=1}^n\mathbf{P}(rX_k\geq x)+\exp\Big\{r-r\log\Big(1+\frac{x^2}{rB_n}\Big)\Big\}$$

and

$$\int_0^\infty\mathbf{P}\Big(\max_{1\leq k\leq n} S_k\geq x\Big)\,dg(x)\leq I_1+I_2$$

where

$$I_1=\sum_{k=1}^n\int_0^\infty\mathbf{P}(rX_k\geq x)\,dg(x),\qquad I_2=e^r\int_0^\infty\Big(1+\frac{x^2}{rB_n}\Big)^{-r}dg(x).$$

Applying Lemma 2 we get

$$\mathbf{E}^+g\Big(\max_{1\leq k\leq n} S_k\Big)\leq\sum_{k=1}^n\mathbf{E}^+g(rX_k)+I_2,$$

finishing the proof of Theorem 2.

Theorem 1 can be proved using the other inequalities in Lemma 1 and Lemma 2.

4. The proofs of Theorems 3 and 4 are similar to the proofs of Theorems 1 and 2. Instead of Lemma 1 it is possible to apply the consequences of more general probabilistic inequalities stated in [2], [3] and [1].

5. Lemma 1 remains true if we replace $\max_{1\leq k\leq n} S_k$ by S_n. Therefore under the conditions of Theorem 1 the same upper bound for $\mathbf{E}g(S_n)$ holds as the one given in (5). In particular, we have

$$\mathbf{E}|S_n|^p\leq C(p)\big(M_{p,n}+B_n^{p/2}\big),\qquad p\geq 2,\tag{15}$$

and, taking into account (6),

$$\mathbf{E}\Big|\max_{1\leq k\leq n} S_k\Big|^p\leq C(p)\big(M_{p,n}+B_n^{p/2}\big),\qquad p\geq 2.\tag{16}$$

Here $C(p)$ is a positive constant depending only on p.

Inequality (15) is due to Rosenthal [5], [6]. Of course Theorems 2, 3 and 4 remain true also if we replace $\max_{1\le k\le n} S_k$ by S_n. In particular, under the conditions of Theorem 3 we have

$$\mathbf{E}|S_n|^p \le C(p)\big(M_{p,n} + D_n^p\big), \qquad p > 1, \tag{17}$$

and, as a consequence of (13),

$$\mathbf{E}\Big|\max_{1\le k\le n} S_k\Big|^p \le C(p)\big(M_{p,n} + D_n^p\big), \qquad p > 1. \tag{18}$$

Inequality (17) was proved by Rosenthal [5], [6].

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