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A NOTE ON THE CONVERGENCE OF MOMENTS AND THE MARTINGALE CENTRAL LIMIT THEOREM

Esko Valkeila

University of Helsinki, Computing Centre, Teollisuuskatu 23
SF-00510 Helsinki, Finland

Abstract. We study the convergence of moments of square-integrable martingales, when the martingales converge to a (mixed) normal limit.

1. Introduction and basic facts

1.1. Let M^n, $n \geq 1$, be a sequence of square-integrable local martingales defined on a filtered space $(\Omega^n, F^n, \mathbf{F}^n, \mathbf{P}^n)$, with $M_0^n = 0$. Suppose that $G \subset \cap_n F_0^n$ is a σ-algebra and ξ is a random variable defined on $(\Omega, F, \mathbf{P})$ with

$$\mathbf{E}^G \exp\{it\xi\} = \exp\{-t^2\eta^2/2\},$$

where η is a finite G-measurable random variable. Consider the following two conditions: for a fixed time $T > 0$

$$(c_0) \qquad \langle M^n \rangle_T \xrightarrow{P} \eta^2$$

and for any $\varepsilon > 0$

$$(l_0) \qquad L_T^{n,\varepsilon} = |x|^2 1_{\{|x|>\varepsilon\}} \star \nu_T^n \xrightarrow{P} 0,$$

where ν^n is the $(\mathbf{F}^n, \mathbf{P}^n)$-compensator of the jump measure μ^n of M^n. The process $L^{n,\varepsilon}$ is the Lindeberg-process.

It is well known that conditions (c_0) and (l_0) imply the stable convergence of M_T^n to ξ (see for example Liptser and Shiryaev [4; p. 316] for a more general statement proving this fact).

Instead of condition (c_0) we assume that the following condition (c_q) is satisfied for an integer $q > 1$:

$$(c_q) \qquad \langle M^n \rangle_T \xrightarrow{L^q(P)} \eta^2.$$

We will also assume that

$$(a) \qquad (\Delta M^n)_T^* \xrightarrow{P} 0,$$

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where $\Delta X_t = X_t - X_{t-}$ for any cadlag process X and $X_t^* = \sup_{s\le t}|X_s|$. Condition (a) is equivalent to the following predictable condition:

$$(a^*) \qquad \nu^n\left(]0,T]\times\left\{|x|>\varepsilon\right\}\right) \xrightarrow{P} 0$$

(see Liptser and Shiryaev [4; p. 305]). Denote by $\xrightarrow{S}$ the stable convergence of random variables.

Our main result is the following

Theorem 1. *Suppose that M^n, $n\ge 1$, is a sequence of square integrable martingales satisfying (a) and (c_q) for some $q>1$. Then the following conditions are equivalent:*

a) $M_T^n \xrightarrow{S} \xi$ *and* $\mathbf{E}(M_T^n)^{2q} \to \mathbf{E}\xi^{2q}$,
b) $\mathbf{E}\left|[M^n]_T - \eta^2\right|^q \to 0$,
c) $\mathbf{E}|I|^{2q} \star \nu_T^n \to 0$,
d) $\left((\Delta M^n)_T^*\right)^{2q}$ *is uniformly integrable.*

Theorem 1 will be proved in the third section of this paper. It was proved for discrete time martingales by Hall. He assumed condition (c_q) and instead of (a) a somewhat stronger condition, and then showed the equivalence of a), b) and c) under his conditions (see Hall and Heyde [2; pp. 70–71]).

1.2. We next introduce some notation and definitions. We assume that $(\Omega, F, \mathbf{F}, \mathbf{P})$ is a given filtered space satisfying the usual assumptions. If X is a process on $(\Omega, F, \mathbf{F}, \mathbf{P})$, then X is adapted to the filtration $\mathbf{F}$. Moreover, we assume that the process X has right-continuous paths with left limits. If X^n is a sequence of stochastic processes, then its properties are defined with respect to a sequence $(\Omega^n, F^n, \mathbf{F}^n, \mathbf{P}^n)$. Denote by μ^X the jump measure of the process X and by ν^X the $(\mathbf{P},\mathbf{F})$-compensator of μ^X. If X is a process, then X^* is the increasing process $X_t^* = \sup_{s\le t}|X_s|$. For terminology not explained here we refer to Jacod and Shiryaev [3].

Recall that any local martingale M, defined on the space $(\Omega, F, \mathbf{F}, \mathbf{P})$, admits a representation

$$M = M^c + I \star (\mu^M - \nu^M),$$

where $I_t(x) = x$ and M^c is the continuous martingale part of M (which depends on $\mathbf{P}$). Here $\star$ stands for the Lebesgue–Stieltjes integration with respect to $dx\times dt$. The quadratic variation $[M]$ of M is defined by

$$[M]_t = \langle M^c\rangle_t + \sum_{s\le t}(\Delta X_s)^2.$$

If M is a square-integrable local martingale, then we can define the angle bracket $\langle M\rangle$ of M as the compensator of $[M]$. Recall that $\mathbf{E}\langle M\rangle_t = \mathbf{E}[M]_t = \mathbf{E}M_t^2$.

1.3. Let M be a local martingale. Recall that according to the Burkholder–Davis–Gundy inequalities: for a local martingale M given a stopping time T and $q > 2$ one has

$$c_q \mathbf{E}[M]_T^{q/2} \le \mathbf{E}(M_T^*)^q \le C_q \mathbf{E}[M]_T^{q/2}. \tag{1.1}$$

Here and in what follows c_q and C_q are constants with c_q decreasing and, respectively, C_q increasing as q increases. Following versions of (1.1) were proved in [1]:

$$c_q \mathbf{E}\Big(\langle M\rangle_T^{q/2} + ((\Delta M)_T^*)^q\Big) \le \mathbf{E}(M^*)_T^q \le C_q \mathbf{E}\Big(\langle M\rangle_T^{q/2} + ((\Delta M)_T^*)^q\Big), \tag{1.2}$$

$$c_q \mathbf{E}\Big(\langle M\rangle_T^{q/2} + |I|^q \star \nu_T^M\Big) \le \mathbf{E}(M_T^*)^q \le C_q \mathbf{E}\Big(\langle M\rangle_T^{q/2} + |I|^q \star \nu_T^M\Big). \tag{1.3}$$

1.4. In the next section we present some auxiliary results. A proof of Theorem 1 will be given in Section 3. In the last section we prove characterizations for the functional convergence under the additional requirement that also the moments convergence.

2. Auxiliary results

2.1. Suppose $M^n \in \mathscr{M}^2_{\text{loc}}$ with $M_0^n = 0$. Define $\nu^n = \nu^{M^n}$ and $V^n = I^2 \star \nu^n$. Recall that the sequence V_T^n is tight if

$$\sup_n \mathbf{P}(V_T^n \ge \lambda) \to 0,$$

as $\lambda \to \infty$. Define $M_T^{n,\varepsilon} = M_T^n + x1_{\{|x|\le\varepsilon\}} \star \mu_T^n$. Note that, in general, $M^{n,\varepsilon}$ is not a martingale.

Lemma 1. *Suppose that the the sequence V_T^n is tight for a fixed time point T, and that the Lindeberg condition (l_0) holds. Then*

$$\sup_{s\le T} \big|[M^n]_s - \langle M^n\rangle_s\big| \xrightarrow{P} 0, \tag{2.1}$$

as $n \to \infty$.

Proof. The Lindeberg condition together with the Lenglart–Rebolledo inequality imply that

$$[M^{n,\varepsilon}]_T - [M^n]_T \xrightarrow{P} 0.$$

For the predictable bracket process $\langle M^{n,\varepsilon}\rangle$ one has

$$\langle M^{n,\varepsilon}\rangle_T = I^2 1_{\{|x|\le\varepsilon\}} \star \nu^n_T - \sum_{s\le T}\Big(\int_{\{|x|\le\varepsilon\}} x\nu^n(\{s\},dx)\Big)^2.$$

Now,

$$\int_{\{|x|\le\varepsilon\}} x\nu^n(\{s\},dx) = -\int_{\{|x|>\varepsilon\}} x\nu^n(\{s\},dx)$$

yielding

$$\sum_{s\le T}\Big(\int_{\{|x|\le\varepsilon\}} x\nu^n(\{s\},dx)\Big)^2 = \sum_{s\le T}\Big(-\int_{\{|x|>\varepsilon\}} x\nu^n(\{s\},dx)\Big)^2 \le |I|^2 1_{\{|x|>\varepsilon\}} \star \nu^n_T. \tag{2.2}$$

Using (2.2) combined with the condition (l_0) it is easy to see that

$$\langle M^n\rangle_T - \langle M^{n,\varepsilon}\rangle_T \xrightarrow{P} 0,$$

as $n\to\infty$.

To prove (2.1) we have to show that

$$\sup_{s\le T}\big|[M^{n,\varepsilon}]_s - \langle M^{n,\varepsilon}\rangle_s\big| \xrightarrow{P} 0.$$

Define $Y^{n,\varepsilon} = [M^{n,\varepsilon}] - \langle M^{n,\varepsilon}\rangle$ and $\tau_n = \inf\{t \mid V^n_t > \lambda\}$. Clearly, the process $2|I|^2 1_{\{|x|\le\varepsilon\}} \star \nu^n$ dominates the process $Y^{n,\varepsilon}$. We have

$$P\big(\sup_{s\le T}|Y^{n,\varepsilon}_s| > \delta\big) \le \delta^{-2}\mathbf{E}(Y^{n,\varepsilon}_{\tau_n\wedge t})^2 + P(\tau_n < T) \tag{2.3}$$

with

$$E(Y^{n,\varepsilon}_{\tau_n\wedge t})^2 \le 4\varepsilon^2(\lambda+\varepsilon^2).$$

The lemma now follows, since the right-hand side of (2.3) can be made arbitrary small by using the tighness assumption of V^n_T and by an appropriate choice of ε.

The next lemma is obvious.

Lemma 2. *Suppose that X^n and Y^n are sequences of random variables with*

$$X^n \xrightarrow{P} 0 \quad \text{and} \quad \mathbf{P}(|Y^n|\ge\gamma)\to 0,$$

where $\gamma>0$ is a finite random variable. Then

$$X^nY^n \xrightarrow{P} 0.$$

Lemma 3. *Suppose*

$$\langle M^n\rangle_T \xrightarrow{P} \zeta^2, \tag{2.4}$$

where ζ *is a finite random variable and suppose the sequence*

$$((\Delta M^n)^*_T)^p \tag{2.5}$$

is uniformly integrable for some $p > 2$. *Then the following conditions are equivalent:*

a) *condition* (l_0) *holds,*

b) *for any* $p \geq q > 2$ *one has*

$$|I|^q \star \nu^n_T \xrightarrow{P} 0,$$

c) *for any* $q > 2$ *one has*

$$|I|^q \star \mu^n_T \xrightarrow{P} 0.$$

Proof. Suppose the Lindeberg condition holds. We show that condition c) holds. By (2.4) the sequence V^n_T is tight. Hence by Lemma 1 we have

$$[M^n]_T \xrightarrow{P} \zeta$$

as $n \to \infty$. Thus, for any ε

$$\lim_n \mathbf{P}\big(|I|^2 \star \mu^n_T > \zeta^2 + \varepsilon\big) \to 0.$$

The Lindeberg condition (l_0) implies the condition (a),

$$(\Delta M^n)^*_T \xrightarrow{P} 0,$$

and so for any $q > 2$

$$((\Delta M^n)^*_T)^q \xrightarrow{P} 0.$$

The inequality

$$|I|^q \star \mu^n_T \leq ((\Delta M^n)^*_T)^{q-2} |I|^2 \star \mu^n_T$$

together with Lemma 2 imply c).

Assume next that c) holds with $p \geq q > 2$. The inequality $((\Delta M^n)^*_T)^q \leq |I|^q \star \mu^n$ can be used to show that we then have (a), and hence also

$$((\Delta M^n)^*_T)^q \xrightarrow{P} 0,$$

for any $p \geq q > 2$. Since the process $|I|^q \star \mu^n$ dominates the process $|I|^q \star \nu^n$ we get, using the Lenglart–Rebolledo inequality,

$$\mathbf{P}\big(|I|^q \star \nu_T^n \geq a\big) \leq \frac{1}{a}\mathbf{E}\left(|I|^q \star \mu_T^n \wedge \left((\Delta M^n)_T^*\right)^q + b\right) + \mathbf{P}\big(|I|^q \star \mu_T^n \geq b\big).$$

The last inequality combined with (2.5) and (a) gives b).

Assume finally that b) holds. Using the inequality

$$\varepsilon^{q-2}\int_0^T \int_{\{|x|>\varepsilon\}} |x|^2 \nu^n(ds,dx) \leq |I|^q \star \nu_T^n$$

we see that b) implies a).

Remark 1. Note that in the previous lemma we have proved the following three implications:

a) 2.4) & (l_0) $\Rightarrow$ c),
b) c) & (2.5) $\Rightarrow$ b),
c) b) $\Rightarrow$ (l_0).

Lemma 4. *Let M^n be a sequence of square-integrable local martingales and let T be a stopping time. The following conditions are equivalent:*

a) *the sequence $|M_T^n|^p$ is uniformly integrable for $p > 2$,*
b) *the sequence $\left((M^n)_T^*\right)^p$ is uniformly integrable for $p > 2$.*

Proof. Assume b). Since $|M_T^n| \leq (M^n)_T^*$ it is clear that a) holds.

Assume a). We first note that the sequence $\mathbf{E}\left((M^n)_T^*\right)^p$ is bounded, since

$$\mathbf{E}\left((M^n)_T^*\right)^p \leq C_p|M_T^n|^p.$$

For $c > 0$ put $A_c^n = \left\{(M^*)_T^n \geq c\right\}$. Recall that from Doob's inequality one gets

$$\mathbf{P}(A_c^n) \leq \frac{1}{c}\mathbf{E}1_{A_c^n}|M_T^n|,$$

yielding

$$\mathbf{E}\Big(\left((M^*)_T^n\right)^p 1_{A^n_{c^{1/p}}}\Big) = \int_{c^{1/p}}^\infty \mathbf{P}(A^n_{a^{1/p}})\,da \leq \mathbf{E}|M_T^n|\int_c^{((M^*)_T^n)^p} a^{-1/p}da$$
$$= \frac{p}{p-1}\mathbf{E}|M_T^n|\left((M^*)_T^n\right)^p 1_{A^n_{c^{1/p}}}.$$

Hölder's inequality then gives

$$\mathbf{E}\Big(\left((M^*)_T^n\right)^p 1_{A^n_{c^{1/p}}}\Big) \leq C_p\mathbf{E}\left(|M^n|_T^p 1_{A_c^n}\right). \tag{2.6}$$

Since $\sup_n \lim_{c\to\infty} P(A_c^n) \to 0$ and since the sequence $|M_T^n|$ is uniformly integrable, we have

$$\sup_n \lim_{c\to\infty} \mathbf{E}\left(|M^n|_T^p 1_{A_c^n}\right) = 0.$$

It clearly follows from (2.6) that the same is true for $\left((M_T^n)^*\right)^p$, yielding b).

In what follows we often make use of the following principle.

Lemma 5. *Suppose that X^n is a sequence of random variables with*

$$X^n \xrightarrow{d} \gamma,$$

where γ is a random variable having all the moments and $\xrightarrow{d}$ stands for convergence in distribution; and suppose

$$\mathbf{E}(X^n)^2 \to \mathbf{E}\gamma^2.$$

If for $q > 2$ the sequence $\mathbf{E}(X^n)^q$ is bounded, then we have

$$\mathbf{E}(X^n)^q \to \mathbf{E}\gamma^q.$$

Proof. Introduce the following two topologies τ and σ:

a sequence Y^n converges with respect to τ, if Y^n converges in distribution to a random variable κ and $\mathbf{E}(Y^n)^2 \to \mathbf{E}\kappa^2$,

a sequence Y^n converges with respect to σ, if Y^n converges in distribution to a random variable κ and $\mathbf{E}(Y^n)^q \to \mathbf{E}\kappa^q$

It is clear that convergence with respect to σ is stronger than convergence with respect to τ.

After these definitions we continue with the proof. If the sequence $\mathbf{E}(X^n)^q$ is bounded, there is a subsequence with $\mathbf{E}(X^{n_k})^q \to c$. Then, obviously $c = \mathbf{E}\gamma^q$. This also shows that there are no other possible cluster points for the sequence $\mathbf{E}(X^n)^q$, finishing the proof.

Remark 2. The above lemma is a modification of Zolotarev [5; pp. 100–101].

3. Main results and proofs

3.1. We start the proof of Theorem 1 with

Theorem 2. *Suppose M^n is a sequence of square-integrable local martingales for which conditions (c_1) and (a) hold. Then the following conditions are equivalent:*

a) $\mathbf{E}\big|[M^n]_T - \eta^2\big| \to 0$,
b) *Lindeberg's condition (l_0) holds in $L^1(P)$,*
c) $\big((\Delta M^n)^*_T\big)^2$ *is uniformly integrable,*
d) $M^n_T \xrightarrow{S} \xi$ *and* $\mathbf{E}M^2_T \to \mathbf{E}\xi^2$.

Proof. Assume a). Then

$$[M^n]_T - [M^{n,\varepsilon}] = |I|^2 1_{\{|x|\geq\varepsilon\}} \star \mu^n_T.$$

Moreover,

$$|I|^2 1_{\{|x|\geq\varepsilon\}} \star \mu^n_T \leq (\Delta M^n)^*_T \frac{1}{\varepsilon} |I|^2 1_{\{|x|\geq\varepsilon\}} \star \mu^n_T.$$

Thus, by a) and condition (a) we get $|I|^2 1_{\{|x|\gamma e\varepsilon\}} \star \mu^n_T \xrightarrow{P} 0$. By a) the sequence $[M^n]_T$ is uniformly integrable. Thus, the same is true also for the sequence $|I|^2 1_{\{|x|\geq\varepsilon\}} \star \mu^n_T$. Hence $\mathbf{E}|I|^2 1_{\{|x|\geq\varepsilon\}} \star \nu^n_T \to 0$, proving b).

Assume b). Since $|I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T \xrightarrow{P} 0$, the sequence $|I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T$ is uniformly integrable. Moreover, since $(\Delta M^n)^2 \leq \varepsilon^2 + |I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T$ also the sequence $\left((\Delta M^n_T)^*\right)^2$ is uniformly integrable, giving c).

Assume c). We have

$$|I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T \leq \left((M^n)^*_T\right) \frac{1}{\varepsilon} |I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T.$$

Note that the random variable $|I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T$ is integrable by (c_1) and by the equality $\mathbf{E}[M^n]_T = \mathbf{E}\langle M^n\rangle_T$. This together with (a) then implies that

$$|I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T \xrightarrow{P} 0. \tag{2.7}$$

By the Lenglart–Rebolledo inequality we get

$$\mathbf{P}\left(|I|^2 1_{\{|x|>\varepsilon\}} \star \nu^n_T > a\right) \leq \frac{1}{a}\mathbf{E}\Big(b + \left((\Delta M^n)^*_T\right)^2\Big) \wedge |I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T$$
$$+ \mathbf{P}\left(|I|^2 1_{\{|x|>\varepsilon\}} \star \mu^n_T \geq b\right).$$

This, together with (2.7), (a) and c) shows that the Lindeberg condition (l_0) is satisfied. So by Lemma 1

$$[M^n]_T \xrightarrow{P} \eta^2.$$

But by (2.7) $\lim \mathbf{E}[M^n]_T = \mathbf{E}\xi^2$ and so the sequnece $[M^n]_T$ is uniformly integrable, giving a).

Assume b). We have then (l_0) and (c_0) implying $M^n_T \xrightarrow{S} \xi$. By (c_1) we have $\mathbf{E}(M^n)^2_T \to \mathbf{E}\xi^2$, giving d).

Assume d). Then the sequence $(M^n_T)^2$ is uniformly integrable. By Lemma 4 the same holds for the sequence $\left((M^n)^*_T\right)^2$. Thus, also the sequence $\left((\Delta M^n)^*_T\right)^2$ is uniformly integrable, giving c).

3.2. We prove now Theorem 1. Thus, we assume that for some $q > 1$ the conditions (c_q) and (a) hold. Note that (c_q) implies (c_1) and hence Theorem 2 is applicable.

Assume a). By Theorem 2 we have $\mathbf{E}\big|[M^n]_T - \eta^2\big| \to 0$. The sequence $\mathbf{E}\big((M^n)^*_T\big)^{2q}$ is by a) bounded, and b) follows then by using similar arguments as in the the proof of Lemma 5.

Assume b). Then the sequence $[M^n]^q_T$ is uniformly integrable. The inequality $\big((\Delta M^n)^*_T\big)^{2q} \le [M^n]^q_T$ then gives d).

Assume d). By Theorem 2 (l_0) holds, and hence by Lemma 3

$$|I|^{2q} \star \nu^n_T \xrightarrow{P} 0.$$

From d), (1.2), (1.3) and (c_q) we get that the sequence $|I|^{2q} \star \nu^n_T$ is bounded in $L^1(P)$. Using similar arguments as in the proof of Lemma 5 we then see that c) holds.

Assume c). We then have $\mathbf{E}L^{n,\varepsilon}_T \to 0$ (cf. Remark 2.1. c)), and by Theorem 2 we get $M^n_T \xrightarrow{S} \xi$ and $\mathbf{E}(M^n_T)^2 \to \mathbf{E}\xi^2$. Now use (1.3) together with c) and (c_q) to check that $\mathbf{E}(M^n_T)^{2q}$ is bounded. Hence a) follows from Lemma 2.5, finishing the proof of Theorem 1.

For the discrete time we have the following

Corollary 1. *Suppose M^n_i, $i = 1, \dots, n_k$, is a sequence of square integrable martingales with*

$$[M^n]_{n_k} = \sum_{i=1}^{n_k} (\Delta M^n_i)^2$$

and

$$\langle M^n \rangle_{n_k} = \sum_{i=1}^{n_k} E^{F^n_{i-1}} (\Delta M^n_i)^2,$$

where $\Delta M^n_i = M^n_i - M^n_{i-1}$. Assume that (c_1) and (a) hold. Then Theorem 2 is true. If (c_q) and (a) hold, Theorem 1 is true. Condition (a) can be replaced by the corresponding condition (a^*):

$$\sum_{i=1}^{n_k} \mathbf{P}\big(|\Delta M^n_i| > \varepsilon \big| F^n_{i-1}\big) \xrightarrow{P} 0.$$

Remark 3. Hall (see Hall and Heyde [2; pp. 70–71]) assumes instead of (a) the following condition:

$$\max_i \mathbf{E}^{F^n_{i-1}} (\Delta M^n_i)^2 \xrightarrow{P} 0. \tag{3.1}$$

It is easy to see that (3.1) implies (a). Hall also assumes that $\mathbf{E}(M^n_{n_k})^2 = 1$.

4. On the functional central limit theorem

4.1. Denote by $\xrightarrow{\mathscr{L}}$ the weak convergence in the Skorohod space. In this section we study the moment convergence under the functional central limit theorem. Assume that D is a dense subset of R_+. It is well known that if we have for each $T \in D$ $\big(c_0(D)\big)$:

$$\langle M^n \rangle_T \xrightarrow{P} C_T$$

and for each $T \in D$ $(l_0(D))$:

$$L_T^{n,\varepsilon} \xrightarrow{P} 0,$$

then

$$M^n \xrightarrow{\mathscr{L}} M, \tag{4.1}$$

where M is a continuous Gaussian martingale with $\langle M \rangle_T = C_T$ and C is an increasing continuous function. Instead of (4.1) consider the following condition, valid for each $T \in D$ and some $q \geq 1$:

$$(\mathscr{L}_q) \qquad M^n \xrightarrow{\mathscr{L}} M \quad \text{and} \quad \mathbf{E}(M_T^n)^{2q} \to \mathbf{E}M_T^{2q}.$$

We have the following:

Theorem 3. *The following conditions are equivalent:*

a) $(\mathscr{L}_q)$ *holds for some* $q \geq 1$;
b) *for* $T \in D$ *one has* $\mathbf{E}\big|[M^n]_T - C_T\big|^q \to 0$;
c) *for* $T \in D$ *one has* $\mathbf{E}\big|\langle M^n \rangle_T - C_T\big|^q \to 0$ *and* $\mathbf{E}\big((\Delta M^n)_T^*\big)^{2q} \to 0$.

If $q > 1$ *then* c) *is equivalent to*

d) *for* $T \in D$ *one has* $\mathbf{E}\big|\langle M^n \rangle_T - C_T\big|^q \to 0$ *and* $\mathbf{E}|I|^{2q} \star \nu_T^n \to 0$.

Proof. Assume a) and $q = 1$. Then

$$\big|[M^n]_T - C_T\big| \xrightarrow{P} 0 \tag{4.2}$$

(see Liptser and Shiryaev [4; p. 417]). Condition b) then follows from $E(M_T^n)^2 \to C_T$. If $q > 1$ we still have (4.2). The fact that (4.2) is valid also in $L^q(P)$ follows from Theorem 1, giving b).

Assume b). Then, the continuity of C implies that

$$(\Delta M^n)_T^* \xrightarrow{P} 0. \tag{4.3}$$

Because $\left((\Delta M^n)^*_T\right)^2 \le [M^n]_T$ and $[M^n]^q_T$ is uniformly integrable by b), we get that (4.3) holds in $L^{2q}(P)$. Since for any $\delta > 0$

$$x^2 1_{\{|x|>\delta\}} \star \mu^n_T \le \frac{(\Delta M^n)^*_T}{\delta}[M^n]_T,$$

we get by b), (4.3) and Lemma 2 that $x^2 1_{\{|x|>\delta\}} \star \mu^n_T \xrightarrow{P} 0$. The Lenglart–Rebolledo inequality and the fact that (4.3) holds in $L^2(P)$ then imply that the Lindeberg condition $(l_0)(D)$ holds. So, by Lemma 1, we have

$$|\langle M^n\rangle_T - C_T| \xrightarrow{P} 0. \tag{4.4}$$

Using b) it is easy to check that (4.4) holds in $L(P)$ and as in the proof of Theorem 1 we get that (4.4) holds in $L^q(P)$, giving c).

Assume c). By Theorem 2 we have $(l_0)(D)$, and hence $M^n \xrightarrow{\mathscr{L}} M$. The rest of a) follows then from Theorem 2 for $q = 1$ and from Theorem 1 for $q = 2$.

Assume d) and $q > 1$. It is obvious that also c) holds.

Assume c) and $q > 1$. Then d) follows as in the proof of Theorem 1.

Remark 4. Condition $\mathscr{L}_q$ is equivalent to

$$M^n \xrightarrow{\mathscr{L}} M \qquad \text{and} \qquad (M^n_T)^{2q} \text{ is uniformly integrable.}$$

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