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SOME REMARKS ON LEHTO'S DOMAIN CONSTANT

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Abstract. We show that there exist domains with arbitrary small domain constant in the sense of Lehto and which are not starlike with respect to the origin.

1. Introduction

Let A be any simply connected domain with at least one boundary point in $\mathbf{C}$. The *Poincaré density* (or *hyperbolic density*) of A is defined as

$$\eta_A(z) := \frac{|f'(z)|}{1-|f(z)|^2}, \qquad z \in A,$$

where $f(z)$ is a conformal mapping which maps A onto $\Delta = \{z : |z| < 1\}$. Clearly it is independent of the choice of $f(z)$.

Let

$$S(f,z) := \left(\frac{f''}{f'}\right)'(z) - \tfrac{1}{2}\left(\frac{f''}{f'}(z)\right)^2$$

be the *Schwarzian derivative* of f. We shall also write $S_f(z) \equiv S(f,z)$ if we do not want to emphasize z. It is well-known that $S(f,z) \equiv 0$ if and only if $f(z)$ is a Möbius transformation. Let $z = z(\zeta)$ be a conformal mapping of a domain $B \to A$. Then we have, after a simple computation

$$S(f \circ z, \zeta) = S(f,z)z'(\zeta)^2 + S(z,\zeta). \tag{1}$$

Let us define a norm on S_f as follows

$$\|S_f\|_A := \sup_{z\in A} |S(f,z)|\eta_A(z)^{-2}.$$

If $g(z)$ is also defined on A, then one can show that

$$\|S_f - S_g\|_A = \|S_{f\circ g^{-1}}\|_{g(A)}\,. \tag{2}$$

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Now we put $f(z) \equiv z$, then

(3) $$\|S_g\|_A = \|S_{g^{-1}}\|_{g(A)}.$$

Lehto introduced a notion of *domain constant* (see [9, p. 61])

(4) $$\delta(A) = \|S_f\|_A := \sup\left\{|S(f,z)|\eta_A(z)^{-2} : z \in A, f\colon A \to \Delta\right\}.$$

We note that $\delta(A)$ is independent of the choice of $f(z)$. We shall call two domains *Möbius equivalent* (or simply *equivalent*) if they differ by a Möbius transformation. Hence $\delta(A) \equiv 0$ if and only if A is Möbius equivalent to the unit disc. It is thus evident that $\delta(A)$ can be regarded as some kind of measure (or distance) of how much A deviates from Δ. Due to the invariance property (3) of the Schwarzian, we can rewrite the definition of the domain constant as $\delta(A) = \sup\{|S(f,z)|\eta_\Delta(z)^{-2} : z \in A, f\colon \Delta \to A \textit{ conformal}\}$. In fact

$$\delta(A) = \sup\{6|a_3 - a_2^2| : f\colon \Delta \to B \textit{ conformal}, B \textit{ equivalent to } A\},$$

where $f(z) = z + \sum_2^\infty a_n z^n$, provided f is invariant with respect to *Koebe transformation* i.e. the function

$$g(z) = \frac{(f\circ w)(z) - (f\circ w)(0)}{(f\circ w)'(0)}$$

where $w(z)$ is an automorphism of the unit disc, belongs to the same class as f. We can thus transform $\delta(A)$ to a coefficient problem (see also [9]). The following results, except the last one, are the consequences of this new characterization:

(i) *For all simply connected domains A, we have* $\delta(A) \le 6$ (Nehari [11], Lehto [9, p. 60]);

(ii) *If A is equivalent to a convex domain, then* $\delta(A) \le 2$ (Lehto [9], Nehari [12], Robertson [14]);

(iii) *If A is equivalent to a domain with bounded boundary rotation $\le k\pi$ ($k \le 4$) i.e.*

$$\lim_{r\to 1}\int_0^{2\pi} |\Re u(re^{i\theta})|\, d\theta \le k\pi$$

where $u(z) = 1 + zf''/f'$, then $\delta(A) \le (2k+4)/(6-k)$ (Lehto and Tammi [10], Lehto [9, p. 64]);

(iv) *If A is equivalent to a domain which is close-to-convex of order β i.e. there exists a convex conformal mapping g such that $|\arg f'/g'| \le \frac{1}{2}\beta\pi$, then*

$$\delta(A) \le \begin{cases} 2 + 4\beta, & \beta \le 1, \\ 2\beta^2 + 4\beta, & \beta \ge 1, \end{cases}$$

(Koepf [8]);

(v) *If A is equivalent to a strongly-starlike domain of order α (< 1) i.e. $|\arg zf'/f| \le \frac{1}{2}\alpha\pi$, then* $\delta(A) \le 6\sin(\frac{1}{2}\alpha\pi)$ (Chiang [5, p. 31]).

All the domain constants mentioned above are known to be sharp except, possibly, for (v). In this paper we shall consider the converse problem.

2. A class of more general domains

A domain A is a K-*quasidisc* if it is an image of the unit disc under a K-*quasiconformal mapping* of the plane; its boundary is called K-*quasicircle*. Let A be a Jordan domain and its boundary $\mathscr{C}$ a Jordan curve and z_1 and $z_2 \in \mathscr{C}$ which divides it into two arcs $\mathscr{C}_1$ and $\mathscr{C}_2$. Then $\mathscr{C}$ is said to satisfy the *arc condition* if there exists a constant c (depending on A only) such that $\min_i \operatorname{diam}(\mathscr{C}_i) \leq c|z_1 - z_2|$ for all z_1 and $z_2 \in \mathscr{C}$. Ahlfors [1] gave a characterization of $\mathscr{C}$: *$\mathscr{C}$ satisfies the arc condition if and only if $\mathscr{C}$ is a K-quasicircle.*

It is also well-known that if $\delta(A) \leq 2k$, $k < 1$, then A is a $(1+k)/(1-k)$-quasidisc. On the other hand we have, if A is a K-quasidisc then

$$\delta(A) \leq 6(K^2 - 1)/(K^2 + 1)$$

(Lehto [9, p. 73]). It is therefore natural to ask if A has other geometrical properties provided $\delta(A)$ is close to zero. A result of this type was proved by Gabriel.

Theorem A (Gabriel [7]). *Let $f(z) = z + \sum_2^\infty a_n z^n$ and suppose that*

$$\left|S(f,z)\right| \leq 2c_0, \qquad \text{for each } z \in \Delta$$

where c_0 is the smallest positive root of the equation $2\sqrt{x} - \tan\sqrt{x} = 0$. Then $f(\Delta)$ is a starlike domain with respect to the origin.

On the other hand we prove below the following result of negative type.

Theorem 1. *Given $0 < \varepsilon < 1$ sufficiently small, then there exists a domain $A(\varepsilon)$ containing the origin such that*

$$\delta\big(A(\varepsilon)\big) < \varepsilon$$

and $A(\varepsilon)$ is not a starlike domain (let alone a convex domain).

3. Preliminaries to the proof of Theorem 1

The proof of Theorem 1 depends upon the theory of second order differential equation

$$y'' + Ay = 0 \tag{4}$$

and its linearly independent solutions. Similar to the Schwarzian derivatives we introduce the following notations and lemmas due to Bank and Laine. We set

$$\langle E, c\rangle = \tfrac{1}{4}\left\{\left(\frac{E'}{E}\right)^2 - 2\frac{E''}{E} - \left(\frac{c}{E}\right)^2\right\}$$

where E is a meromorphic function and c is a non-zero constant.

Lemma A (Bank and Laine [3]).

(a) *Let* A *be a function analytic in a region* D, *and assume that* f_1 *and* f_2 *be linearly independent solutions of equation* (4). *Then* $g := f_1/f_2$ *has the following properties:*

(i) *all zeros of* $g'(z)$ *in* D *are of even multiplicity;*

(ii) $A \equiv \frac{1}{2}S(g,z)$.

(b) *Conversely, let* $g(z)$ *be a non-constant analytic function defined in a simply connected domain* D *which possesses the property* (i) *and we define* A *by* (ii). *Then the equation* (4) *possesses two linearly independent solutions* f_1 *and* f_2 *such that* $g = f_1/f_2$.

Lemma B (Bank and Laine [3]).

(a) *Let* $A(z)$ *be analytic in a region* D, *and assume that equation* (4) *possesses two linearly independent analytic solutions* f_1 *and* f_2 *in* D. *Set* $E := f_1 f_2$ *and* $c := W(f_1, f_2)$ *(the Wronskian of* f_1 *and* f_2*). Then*

(i) *all zeros of* $E(z)$ *are simple;*

(ii) *at any zero* z_1 *of* $E(z)$ *in* D, *the number* $c/E'(z_1)$ *is an odd integer;*

(iii) $A \equiv \langle E, c\rangle$.

(b) *Conversely, let* $E(z) \not\equiv 0$ *be an analytic function defined in a simply connected region* D, *and let* c *be a non-zero constant such that* (i), (ii) *above hold. Then if we define* $A(z)$ *by* (iii) *the equation* (4) *possesses two linearly independent solutions* f_1 *and* f_2 *such that* $E = f_1 f_2$ *and* $c = W(f_1, f_2)$.

We note that Lemma A is mostly well-known. We shall only make use of part (a) of Lemma A and (b) of Lemma B, the rest is stated here for completeness. We also note that in [3], Bank and Laine consider the above $A(z)$ to be meromorphic and the results seem to have an independent interest. They also have many applications in the theory of differential equations (see [3]).

4. Proof of Theorem 1

We shall prove the Theorem by constructing an explicit counter-example. In view of the new characterization of $\delta(A)$, it is sufficient to show that given $\varepsilon > 0$ between 0 and 1 there exists a conformal mapping f (dependent on ε) in Δ such that $(1-|z|^2)^2|S(f,z)| < \varepsilon$ and $f(\Delta) = A$ is the required domain which is not starlike with respect to the origin.

Given $\varepsilon > 0$, let $E(z) = z/(1-z^2)^\lambda$ in Δ, where $\lambda = i\mu$, $0 < \mu < 2\varepsilon/7$. We note that $E(0) = 0$ and this is the only zero of E in Δ and

$$E'(z) = \frac{1 + (2\lambda - 1)z^2}{(1-z^2)^{1+\lambda}}, \qquad E'(0) = 1.$$

Therefore E clearly satisfies (i), (ii) of Lemma B with $c = -1$. Let $A \equiv \langle E, c\rangle$, then the corresponding equation (4) has two linearly independent solutions f_1, f_2

analytic in Δ such that $E = f_1 f_2$ and

$$W(f_1, f_2) \equiv c = -1 = f_1(0)f_2'(0) - f_2(0)f_1'(0).$$

Since $E(0) = 0$ we may assume $f_1(0) = 0$. Now

$$A(z) = \langle E, -1\rangle = \langle E, 1\rangle = \tfrac{1}{4}\Big\{\Big(\frac{E'}{E}\Big)^2 - 2\frac{E''}{E} - \Big(\frac{c}{E}\Big)^2\Big\} = \tfrac{1}{4}\Big\{2\Big(\frac{E'}{E}\Big)' - \Big(\frac{E'}{E}\Big) - \frac{1}{E^2}\Big\}.$$

We calculate $A(z)$. Differentiate $\log E$ and we obtain

$$\begin{aligned}
4A(z) &= -\Big\{2\Big(-\frac{1}{z^2} + \frac{2\lambda + 2\lambda z^2}{(1-z^2)^2}\Big) + \Big(\frac{1}{z} + \frac{2\lambda z}{1-z^2}\Big)^2 + \frac{(1-z^2)^{2\lambda}}{z^2}\Big\} \\
&= \frac{2}{z^2} - \frac{2(2\lambda + 2\lambda^2 z^2)}{(1-z^2)^2} - \frac{1}{z^2} - \frac{4\lambda}{1-z^2} - \frac{4\lambda^2 z^2}{(1-z^2)^2} - \frac{(1-z^2)^{2\lambda}}{z^2} \\
&= \frac{1}{z^2} - \frac{4\lambda + 4\lambda z^2 + 4\lambda(1-z^2) + 4\lambda^2 z^2}{(1-z^2)^2} - \frac{(1-z^2)^{2\lambda}}{z^2} \\
&= \frac{1}{z^2} - \frac{8\lambda + 4\lambda^2 z^2}{(1-z^2)^2} - \frac{1}{z^2}\Big(1 - 2\lambda z^2 + \frac{2\lambda(2\lambda-1)}{2!}z^4 - \cdots\Big) \\
&= -\frac{8\lambda + 4\lambda^2 z^2}{(1-z^2)^2} + \Big(2\lambda - \frac{2\lambda(2\lambda-1)}{2!}z^2 + \cdots\Big).
\end{aligned}$$

Let

$$P(z) = 2\lambda - \frac{2\lambda(2\lambda-1)}{2!}z^2 + \cdots$$

be the term in the braces above. The nth coefficient of $P(z)$ is equal to

$$\frac{2\lambda(2\lambda-1)(2\lambda-2)\cdots(2\lambda-n+1)}{n!} = 2\lambda\Big(\frac{2\lambda-1}{2}\Big)\Big(\frac{2\lambda-2}{3}\Big)\cdots\Big(\frac{2\lambda-(n-1)}{n}\Big).$$

It is not difficult to see that the modulus of each term of the right hand side of the above equality is strictly less than 1 as long as $|\lambda|$ is chosen to be sufficiently small. Hence we may assume all the coefficients of $P(z)$ are bounded by $2|\lambda| = 2\mu$. We deduce

$$\begin{aligned}
(1-|z|^2)^2|4A(z)| &\le (1-|z|^2)^2\Big|-\frac{8\lambda + 4\lambda^2 z^2}{(1-z^2)^2}\Big| \\
&\quad + (1-|z|^2)^2\Big|2\lambda - \frac{2\lambda(2\lambda-1)}{2!}z^2 + \cdots\Big| \\
&\le (1-|z|^2)^2\frac{8|\lambda| + 4|\lambda^2|}{|1-z^2|^2} \\
&\quad + (1-|z|^2)^2(2|\lambda| + 2|\lambda||z|^2 + 2|\lambda||z|^4 + \cdots) \\
&\le 12|\lambda| + (1-|z|^2)^2\frac{2|\lambda|}{1-|z|^2} \\
&= 12|\lambda| + 2|\lambda|(1-|z|^2) \le 14|\lambda|.
\end{aligned}$$

Hence

$$(1-|z|^2)^2|A(z)| = (1-|z|^2)^2|\langle E,1\rangle| \le \tfrac{14}{4}|\lambda| = \tfrac{7}{2}\mu < \varepsilon. \tag{5}$$

Since $A(z)$ is analytic in Δ, and f_1, f_2 considered above are linearly independent solutions of (4), by Lemma A (a), the function $f(z)$ defined by

$$f(z) = \frac{f_1(z)}{f_2(z)} \tag{6}$$

satisfies the identity $A(z) \equiv \frac{1}{2}S(f,z)$. According to (4), f_1 and f_2 are analytic and so is their product $E = f_1 f_2$ and since the only zero of $E(z)$ is when $z=0$ which is therefore the only zero of f_1. We conclude that f_1 has only one zero and f_2 has no zero in Δ. Hence we deduce that f must be analytic in Δ (in fact $f'(z) = -W(f_1,f_2)/f_2^2 \neq 0$ in Δ, so f is locally univalent). Now the inequality

$$(1-|z|^2)^2|S(f,z)| = (1-|z|^2)^2|2A| \le 2\varepsilon < 2$$

implies, by the classical result of Nehari [11], that f is univalent in Δ.

The necessary and sufficient condition for an analytic function to be starlike (see [13]) is that $\Re(zf'/f) > 0$, $z \in \Delta$. However

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z}{f_1(z)f_2(z)} = \frac{z}{E(z)} = (1-z^2)^\lambda \\ &= \exp\{\lambda\log(1-z^2)\} = \exp\{i\mu\log|1-z^2| - \mu\arg(1-z^2)\} \\ &= \exp\big(-\mu\arg(1-z^2)\big)\exp\big(i\mu\log|1-z^2|\big).\end{aligned}$$

Now the argument of zf'/f is $\mu\log|1-z^2|$ which tends to $-\infty$ as $z \to 1$. Therefore, there exists infinitely many $z \in \Delta$ such that $\Re(zf'/f) < 0$. This shows that f cannot be starlike and also completes the proof. □

From the definition of the arc condition, a quasidisc does not allow any cusps on the boundary of $f(\Delta)$. Fait, Krzyż and Zygmunt [6] constructed an explicit quasiconformal extension for the class of strongly-starlike functions of order α (< 1). This may characterize all the starlike domains without cusps. Thus it would be interesting to find a quasidisc which is starlike but not strongly-starlike. Theorem 1 shows that no matter how small that $\delta(A)$ may be, it is not necessarily a starlike domain.

5. An analogue for logarithmic derivatives

The logarithmic derivative does not share the same properties as the Schwarzian derivatives, for it is only invariant with respect to linear mappings, i.e. $f''/f' \equiv 0$ if and only if $f(z) = az + b$. We can also define $T(f, z) = zf''/f'$ to be the logarithmic derivative and a norm on it by

$$\|T_f\|_A = \sup_{z \in A} \big|T(f, z)\big| \eta_A(z)^{-1}.$$

By a similar argument as in Section 1, we see that $\|T_f\|_A = \big\|T_{f^{-1}}\big\|_{f(A)}$. Hence if we define another constant $\Omega(A) := \|T_f\|_A = \big\|T_{f^{-1}}\big\|_{f(A)}$ where $f\colon A \to \Delta$ is conformal. Again, this is a well defined constant for it is independent of the choices of f. The following results are known:

(i) *for all simply connected domains A, we have* $\Omega(A) \le 6$ [13];
(ii) *if A is a convex domain, then* $\Omega(A) \le 4$ (W.K. Hayman, see Ahlfors [2, p. 5]);
(iii) *if A is close-to-convex of order β domain, then* $\Omega(A) \le 4 + 2\beta$ (Chiang [5, p. 36], Koepf [8]);
(iv) *if A is strongly-starlike of order α (< 1) domain, then* $\Omega(A) \le 6\alpha$ (Chiang [5, p. 33]).

All the above estimates are sharp. Also by a well-known result of J. Becker [4], if $\Omega(A) \le k < 1$, then A is a K-quasidisc where $K = (1 + k)/(1 - k)$. We also ask the similar question here as in Section 2.

Theorem 2. *Given $0 < \varepsilon < 1$, there exists a domain $A(z)$ containing the origin, such that*

$$\Omega\big(A(\varepsilon)\big) < \varepsilon$$

and $A(\varepsilon)$ is not a starlike domain (let alone a convex domain).

6. Proof of Theorem 2

We simply consider the function $f(z)$ constructed in the proof of Theorem 1, then $E = f_1 f_2 = z/(1 - z^2)^\lambda$ and $|\lambda| < \varepsilon/3$. It has been shown that we can make $\big(1 - |z|^2\big)^2 \big|S(f, z)\big|$ arbitrarily small and it is not difficult to show that $\big(1 - |z|^2\big)\big|T(f, z)\big|$ can also be made arbitrarily small. Since

$$G(z) := \frac{zf'}{f} = (1 - z^2)^\lambda$$

then

$$\frac{zf''}{f'} = G(z) - 1 + \frac{zG'(z)}{G(z)} = (1 - z^2)^\lambda - 1 - \frac{2\lambda z^2}{1 - z^2}.$$

Now

$$\begin{aligned}(1-|z|^2)\left|\frac{zf''}{f'}\right| &\le (1-|z|^2)\left|(1-z^2)^\lambda-1\right|+(1-|z|^2)\frac{2|\lambda z^2|}{|1-z^2|}\\ &=(1-|z|^2)\left|\left(1-\lambda z^2+\frac{\lambda(\lambda-1)}{2!}z^4-\cdots\right)-1\right|+2|\lambda z^2|\frac{1-|z|^2}{|1-z^2|}\\ &\le (1-|z|^2)|\lambda|(|z|^2+|z|^4+\cdots)+2|\lambda|\\ &=(1-|z|^2)\frac{|\lambda z^2|}{1-|z|^2}+2|\lambda|=|\lambda z^2|+2|\lambda|\le 3\mu<\varepsilon.\end{aligned}$$

Note that the above inequality follows since we can choose $|\lambda|$ so small that the coefficients in the series expansion have modulus less than $|\lambda|$ as in the proof of Theorem 1. Hence $f(z)$ satisfies Becker's criterion and so it must be conformal yet it fails to be a starlike function. □

Remark. Note that $E=z/(1-z)^\lambda$ can be chosen instead of $E=z/(1-z^2)^\lambda$ in the above proof and it is still sufficient to construct, by the same argument as in the proof of Theorem 1, a counter-example $f(z)$ for Theorem 2. But it fails to become a counter-example for Theorem 1.

Note added after the proof. It can be shown that the domain $A(\varepsilon)$ constructed in Theorem 1 is not starlike with respect to any of its interior points, provided ε is chosen sufficiently small.

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