

NEW PARAMETERS OF TEICHMÜLLER SPACES OF RIEMANN SURFACES OF GENUS TWO

Chen Min

Peking University, Department of Mathematics
Beijing 100871, China

Abstract. In this note, new parameters for the Teichmüller space of a compact Riemann surface of genus 2 are presented. The parameter space is \mathbf{R}_+^6 . Given any point in \mathbf{R}_+^6 , we have a concrete way of deciding the generators for the corresponding marked Fuchsian group.

1. Introduction

One important topic in the study of Teichmüller spaces is the parametrization problem. The idea behind it is to shift the focus of research from complicated objects onto easier ones. To be precise, let T_g be the Teichmüller space of a compact Riemann surface of genus g . We want to find out a parameter space P and a correspondence between P and T_g . Naturally, a successful parametrization should satisfy:

- (1) P is simple and $\dim P = \dim T_g$;
- (2) the correspondence is not difficult to handle.

Up to now, there have been several well-known parametrizations with important applications. However, these parametrizations are mainly of theoretic importance, and it is difficult to use them for some other purposes like *numerical calculations*. Recently, B. Maskit gave new parametrizations for Teichmüller spaces of Riemann surfaces which have signatures $(0, 4)$ and $(1, 1)$. His parametrizations have the following features:

- (1) $P =$ half space, $\dim P$ being equal to the dimension of the corresponding Teichmüller space;
- (2) given any point in P , there is a concrete way of describing the generators in the corresponding marked group.

These characteristics provide a clearer model for the considered Teichmüller spaces and also a practical way of doing *numerical calculation*.

In this note, we present a parametrization for T_2 , the Teichmüller space of a compact Riemann surface of genus 2. Our parameter space is \mathbf{R}_+^6 , and given any point in \mathbf{R}_+^6 , we have a concrete way of describing the generators for the

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corresponding marked Fuchsian group. So far the main method for constructing Fuchsian groups has been combination theorems. The method used in this note yields a different way of obtaining all Fuchsian groups which can be geometrically realized by compact Riemann surfaces of genus 2.

Having finished the first draft of this note, the author heard in Montpellier in September 1991 that B. Maskit has also recently studied the same problem by using a different approach. The author thanks B. Maskit for encouragement.

2. Preliminaries

Let H denote the upper-half plane equipped with the hyperbolic metric. Let \mathcal{M} be the set of all hyperbolic Möbius transformations fixing H . Any $g \in \mathcal{M}$ can be written in the form

$$g(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

where a, b, c and d are real numbers. We can associate g with a matrix

$$\tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $GL(2 : \mathbb{R})$. This is not unique. Assume that g is an element in \mathcal{M} . Denote by $a(g), r(g)$ the attracting and repelling fixed points of g . The multiplier $k(g)$ is given by the cross-ratio

$$k(g) = (g(z), z, r(g), a(g)) = \frac{g(z) - r(g)}{g(z) - a(g)} \frac{z - a(g)}{z - r(g)},$$

where z is any point in $\hat{\mathbb{C}}$ not fixed by g . Especially, $k(g) > 1$. The transformation g is completely determined by $a(g), r(g)$ and $k(g)$. In fact,

$$(2.1) \quad g(z) = \frac{(k(g)a(g) - r(g))z - a(g)r(g)(k(g) - 1)}{(k(g) - 1)z + a(g) - k(g)r(g)}.$$

The axis of g , $ax(g)$, is the unique geodesic on H with $a(g)$ and $r(g)$ as its terminal points. Denote the commutator $g_1 \circ g_2^{-1} \circ g_1^{-1} \circ g_2$ of two Möbius transformations g_1 and g_2 by $[g_1, g_2]$.

Let $R = H/G$ be a compact Riemann surface of genus 2, where G is a Fuchsian group acting on H . It is well-known that G has only hyperbolic elements and that there exist four generators h_1^0, g_1^0, h_2^0 , and g_2^0 such that $[h_1^0, g_1^0] \circ [h_2^0, g_2^0] = \text{id}$ and the points below appear on the boundary of H in clockwise order:

$r([h_2^0, g_2^0])$, $a(h_1^0)$, $a(g_1^0)$, $r(h_1^0)$, $r(g_1^0)$, $a([h_2^0, g_2^0])$, $a(h_2^0)$, $a(g_2^0)$, $r(h_2^0)$, $r(g_2^0)$. We can normalize $\{h_j^0, g_j^0\}$ so that $a([h_2^0, g_2^0]) = 0$, $r([h_2^0, g_2^0]) = \infty$, and $a(h_2^0) = 1$. See Figure 1 below:

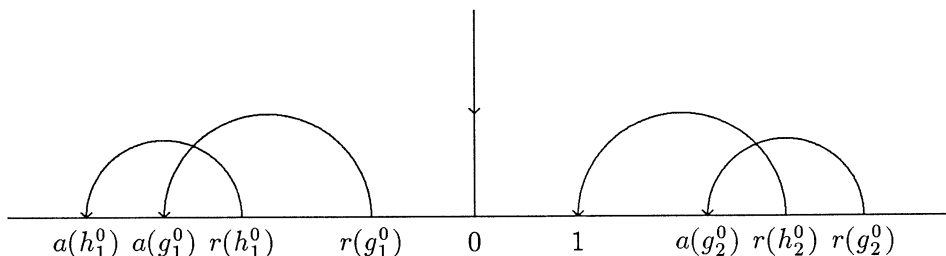


Figure 1.

Denote by Q the set of all quasiconformal mappings which fix $0, 1, \infty$ and H . Let $f_1, f_2 \in Q$; throughout this paper $f_1 \circ (h_j, g_j) \circ f_1^{-1} = f_2 \circ (h_j, g_j) \circ f_2^{-1}$ will mean

$$f_1 \circ h_j \circ f_1^{-1} = f_2 \circ h_j \circ f_2^{-1}, \quad f_1 \circ g_j \circ f_1^{-1} = f_2 \circ g_j \circ f_2^{-1}, \quad j = 1, 2.$$

The Teichmüller space of R is defined as follows:

$$T_2 = \{(h_j, g_j) : (h_j, g_j) = f \circ (h_j^0, g_j^0) \circ f^{-1}, f \in Q\}.$$

Note that if $(h_j, g_j) \in T_2$, then $ax(h_j)$, $ax(g_j)$, and $ax([h_2, g_2])$ will form a figure on H similar to Figure 1.

3. Parametrization of T_2

Lemma 1. Let h, g , and P in \mathcal{M} satisfy:

- (1) $ax(h)$ intersects $ax(g)$,
- (2) $h \circ g^{-1} \circ h^{-1} \circ g = P$,
- (3) $P(z) = \lambda^{-2}z, \lambda > 0$.

Then

$$k(h) = \frac{a(h) - \lambda r(h)}{r(h) - \lambda a(h)}, \quad k(g) = \frac{a(g) - \lambda r(g)}{r(g) - \lambda a(g)}.$$

Proof. By Lemma 3.4 in [7], we have

$$\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}\tilde{g} = \begin{pmatrix} -\lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix},$$

where

$$\tilde{h}^{-1} = \begin{pmatrix} k(h)r(h) - a(h) & -a(h)r(h)(k(h) - 1) \\ k(h) - 1 & r(h) - k(h)a(h) \end{pmatrix},$$

$$\tilde{g}^{-1} = \begin{pmatrix} k(g)r(g) - a(g) & -a(g)r(g)(k(g) - 1) \\ k(g) - 1 & r(g) - k(g)a(g) \end{pmatrix}.$$

The conclusion follows from the trace equalities

$$\text{tr}(\tilde{h}) = \text{tr} \left[\tilde{h}^{-1} \begin{pmatrix} -\lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix} \right] \quad \text{and} \quad \text{tr}(\tilde{g}) = \text{tr} \left[\begin{pmatrix} -\lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix} \tilde{g}^{-1} \right],$$

respectively. Q.E.D.

Denote by \mathcal{D} the set of all pairs (h, g) in $\mathcal{M} \times \mathcal{M}$ with the property

$$a(h) < a(g) < r(h) < r(g).$$

Lemma 2. Let $P(z) = \lambda^{-2}z$, $\lambda > 1$, $z \in \mathbf{C}$. Given three points a , p and r on the real axis satisfying

$$(3.1) \quad 0 < a < p < r < \lambda a,$$

there exists a unique pair $(h, g) \in \mathcal{D}$ such that

$$(3.2) \quad h \circ g^{-1} \circ h^{-1} \circ g = P, \quad \text{and} \quad a(h) = a, \quad r(h) = r, \quad a(g) = p.$$

Conversely, if $(h, g) \in \mathcal{D}$ satisfies $h \circ g^{-1} \circ h^{-1} \circ g = P$, $0 < a(h)$, then

$$0 < a(h) < a(g) < r(h) < \lambda a(h).$$

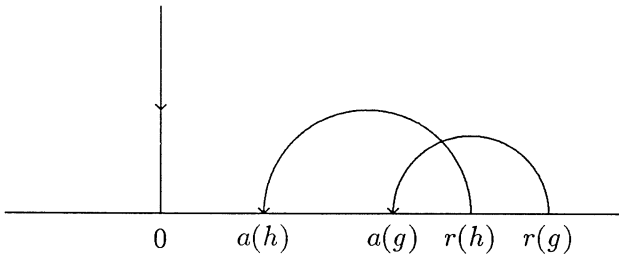


Figure 2.

Proof. The set $S = \{(a, p, r) : 0 < a < p < r < \lambda a\}$ is connected and open in \mathbf{R}^3 . Clearly there are $(a_0, p_0, r_0) \in S$ and $(h_0, g_0) \in \mathcal{D}$ such that (3.2) holds. Assume that (a, p, r) is sufficiently close to (a_0, p_0, r_0) . We get h as follows:

$$a(h) = a, \quad r(h) = r, \quad k(h) = \frac{a - \lambda r}{r - \lambda a}.$$

By the same kind of computation as in Lemma 1 we get

$$(3.3) \quad k(h) = k(h^{-1} \circ P).$$

Since (a, p, r) is close to (a_0, p_0, r_0) and the attracting and repelling fixed points of $h^{-1} \circ P$ are close to the attracting and repelling fixed points of $h_0^{-1} \circ P$, respectively, we see that there exists a unique $g \in \mathcal{M}$, which is not identity, such that

$$(3.4) \quad g^{-1}(a) = r(h^{-1} \circ P), \quad g^{-1}(r) = a(h^{-1} \circ P), \quad g^{-1}(p) = p.$$

This together with (3.3) implies that $g^{-1} \circ h^{-1} \circ g = h^{-1} \circ P$. Clearly the assumption that (a, p, r) is close to (a_0, p_0, r_0) implies that $(a(h), r(h), a(g), r(g))$ is close to $(a(h_0), r(h_0), a(g_0), r(g_0))$, which yields $(h, g) \in \mathcal{D}$. Let S_1 be the subset of S whose elements can be realized by (3.2). From the previous discussion we know that S_1 is a non-empty open set in S . Now we assume that there exist $(a, p, r) \in S$ and $(a_n, p_n, r_n) \in S_1$ ($n = 1, 2, \dots$) such that $(a_n, p_n, r_n) \rightarrow (a, p, r)$, $n \rightarrow \infty$. Consider the corresponding pairs $(h_n, g_n) \in \mathcal{D}$, $n = 1, 2, \dots$. They satisfy

$$\begin{aligned} g_n^{-1} \circ h_n^{-1} \circ g_n &= h_n^{-1} \circ P, \\ 0 < a(h_n) = a_n < r_n = r(h_n) < \lambda a_n, \\ a_n < p_n = a(g_n) < r_n < r(g_n) < \lambda p_n, \end{aligned}$$

where the last inequality follows from Lemma 1 and the fact $k(g_n) > 1$. We may assume that $r(g_n) \rightarrow \tilde{r} \in [r, \lambda p]$, and $k(g_n) \rightarrow k \in [1, +\infty]$. It follows that

$$\lim_{n \rightarrow \infty} k(g_n) = \lim_{n \rightarrow \infty} \frac{p_n - \lambda r(g_n)}{r(g_n) - \lambda p_n} \geq \frac{p - \lambda r}{r - \lambda p} > 1,$$

and $k = +\infty$ if and only if $\tilde{r} = \lambda p$. If $\tilde{r} = r$, we have

$$h_n \rightarrow h, \quad g_n \rightarrow g, \quad g_n^{-1} \circ h_n^{-1} \circ g_n = g_n^{-1} \circ P \rightarrow g^{-1} \circ h^{-1} \circ g = g^{-1} \circ P,$$

where

$$\begin{aligned} a(h) = a, \quad r(h) = r, \quad k(h) &= \frac{a - \lambda r}{r - \lambda a}, \\ p = a(g), \quad r = r(g), \quad k(g) &= \frac{p - \lambda r}{r - \lambda p}. \end{aligned}$$

Hence $g^{-1} \circ h^{-1} \circ g(r) = r$, or $h^{-1} \circ P(r) = r$. This is impossible. If $\tilde{r} = \lambda p$, since $k(g_n) \rightarrow +\infty$ it follows that

$$g_n^{-1} \circ h_n \circ g_n(r_n) \rightarrow \tilde{r} = \lambda p.$$

On the other hand,

$$P^{-1} \circ h_n(r_n) = P^{-1}(r_n) = \lambda^2 r_n \rightarrow \lambda^2 r.$$

This is a contradiction. Hence $(a, p, r) \in S_1$, and S_1 is both open and closed in S . We deduce that $S = S_1$. Now assume that (a, p, r) and (h, g) satisfy (3.1) and (3.2). By Lemma 1 we see that h is determined uniquely by a and r . The uniqueness of g follows from (3.4). The other assertion of the lemma follows from Lemma 1. Q.E.D.

Similarly, we have the following assertion:

Lemma 2'. Let $P'(z) = \lambda^2 z$, $\lambda > 1$, $z \in \mathbf{C}$. Given three points a , p and r on the real axis satisfying

$$a < p < r < \lambda^{-1}a < 0,$$

there exists a unique pair $(h, g) \in \mathcal{D}$ such that

$$h \circ g^{-1} \circ h^{-1} \circ g = P', \quad \text{and} \quad a(h) = a, \quad r(h) = r, \quad a(g) = p.$$

Conversely, if $(h, g) \in \mathcal{D}$ satisfies $h \circ g^{-1} \circ h^{-1} \circ g = P'$, $r(g) < 0$, then

$$a(h) < a(g) < r(h) < \lambda^{-1}a(h).$$

Basing on Lemma 2 and Lemma 2', we are now able to give T_2 a new parametrization. Let $p = (h_i, g_i)$ be a point in T_2 . By Lemma 2 and Lemma 2', we can associate with it a point $L(p) = (t_1, \dots, t_6)$ in \mathbf{R}_+^6 , where

$$(3.5) \quad \begin{aligned} t_1 &= -a(h_1), & \frac{a(h_1)(1+t_2)}{1+\lambda t_2} &= r(h_1), & \frac{a(h_1)+t_3 r(h_1)}{t_3+1} &= a(g_1), \\ \lambda &= \sqrt{k(h_1 \circ g_1^{-1} \circ h_1^{-1} \circ g_1)} = t_4 + 1, \\ \frac{1+t_5 \lambda}{t_5+1} &= r(h_2), & \frac{1+t_6 r(h_2)}{t_6+1} &= a(g_2). \end{aligned}$$

Theorem 1. The mapping $L: T_2 \rightarrow \mathbf{R}_+^6$ is a real analytic diffeomorphism.

Proof. The mapping L is obviously real analytic. To see it is injective, we only have to verify that $L(p) = (t_1, \dots, t_6)$ determines p uniquely for any $p = (h_i, g_i)$ in T_2 . But this is true by Lemma 2 and Lemma 2'. Assume that (t_1, \dots, t_6) as above is the image of $p = (h_i, g_i) \in T_2$ under L . For any $t > 0$, we want to show that (tt_1, t_2, \dots, t_6) is the image of a point in T_2 . The set $T = \{t > 0 : (tt_1, t_2, \dots, t_6) \in T_2\}$ is open in \mathbf{R}_+ by the invariance of domain. Without loss of generality, we may assume that $t > 1$ and for any $1 < t' < t$, $(t't_1, t_2, \dots, t_6) = L(p(t'))$, where $p(t') = (h_i(t'), g_i(t')) \in T_2$. If $(tt_1, t_2, \dots, t_6) \notin L(T_2)$, then $p(t') \rightarrow \partial T_2$. Since the geodesics on R "covered" by $h_1^0, g_1^0, h_2^0, g_2^0$, and $h_1^0 \circ h_2^0{}^{-1}$ fill up the surface R , the sum $s(t') = k(h_1(t')) + k(g_1(t')) + k(h_2(t')) + k(g_2(t')) + k(h_1(t') \circ h_2^{-1}(t'))$ must approach infinity when t' tends to t according to Lemma 3.1 in [4]. On the other hand, we see from (3.5), Lemma 1 and the proof in Lemma 2 that $k(h_1(t'))$, $k(g_1(t'))$, $k(h_2(t'))$, and $k(g_2(t'))$ are all bounded. So $k(h_1(t') \circ h_2^{-1}(t'))$ must approach infinity when t' tends to t . Since $a(h_j(t'))$, $r(h_j(t'))$ and $k(h_j(t'))$ are all bounded, it follows that the determinant of $\tilde{h}_1(t') \circ \tilde{h}_2^{-1}(t')$ approaches 0 when t' approaches t , where

$$\begin{aligned} \tilde{h}_1(t') &= \begin{pmatrix} a(h_1(t'))k(h_1(t')) - r(h_1(t')) & -a(h_1(t'))r(h_1(t'))(k(h_1(t')) - 1) \\ k(h_1(t')) - 1 & a(h_1(t')) - k(h_1(t'))r(h_1(t')) \end{pmatrix}, \\ \tilde{h}_2^{-1}(t') &= \begin{pmatrix} r(h_2(t'))k(h_2(t')) - a(h_2(t')) & -r(h_2(t'))a(h_2(t'))(k(h_2(t')) - 1) \\ k(h_2(t')) - 1 & r(h_2(t')) - k(h_2(t'))a(h_2(t')) \end{pmatrix}. \end{aligned}$$

By a straightforward computation we get

$$\det(\tilde{h}_1(t')\tilde{h}_2^{-1}(t')) = k(h_1(t'))k(h_2(t')) \cdot [r(h_1(t')) - a(h_1(t'))]^2 [r(h_2(t')) - a(h_2(t'))]^2.$$

This is a contradiction. Keeping to the same analysis on t_2, \dots , and t_6 , we can prove that any point in \mathbf{R}_+^6 is the image of a point in T_2 , i.e., L is surjective. Q.E.D.

Corollary 1. Assume that $g, h \in \mathcal{D}$ satisfy (3.2). Then the group generated by g and h is discrete.

Corollary 2. Let G be a group generated by four hyperbolic transformations h_1, g_1, h_2 , and g_2 , and assume that $[h_1, g_1] = [h_2, g_2]^{-1}$ is also hyperbolic. If the axes of h_j, g_j , and $[h_1, g_1]$ form a picture conjugate to Figure 1, then G is discrete.

Let \mathbf{R}_+^6 be our parameter space of T_2 . Given a point (t_1, \dots, t_6) in \mathbf{R}_+^6 , we can get the generators for the corresponding marked group in the following way. First of all, h_1 and h_2 can easily be defined by (3.5), Lemma 1 and (2.1). For obtaining g_1 and g_2 we need more calculations. From $h_2 \circ g_2^{-1} \circ h_2^{-1} \circ g_2 = P$ we have

$$(3.6) \quad g_2(P_a) = r(h_2), \quad g_2(P_r) = a(h_2),$$

where

$$P_a = a(h_2^{-1} \circ P) = [2(1 - k(h_2))]^{-1} \left(a(h_2) - k(h_2)r(h_2) + \lambda^2(r(h_2) - k(h_2)a(h_2)) + \sqrt{(a(h_2) - k(h_2)r(h_2) + \lambda^2(r(h_2) - k(h_2)a(h_2)))^2 - 4\lambda(k(h_2) - 1)^2 a(h_2)r(h_2)} \right)$$

$$P_r = r(h_2^{-1} \circ P) = [2(1 - k(h_2))]^{-1} \left(a(h_2) - k(h_2)r(h_2) + \lambda^2(r(h_2) - k(h_2)a(h_2)) - \sqrt{(a(h_2) - k(h_2)r(h_2) + \lambda^2(r(h_2) - k(h_2)a(h_2)))^2 - 4\lambda(k(h_2) - 1)^2 a(h_2)r(h_2)} \right).$$

Let

$$a_2 = a(g_2), \quad r_2 = r(g_2), \quad k_2 = k(g_2),$$

$$\tilde{a}_2 = a(h_2), \quad \tilde{r}_2 = r(h_2).$$

By (3.6) and Lemma 1, we get

$$\begin{cases} (a_2 - \tilde{a}_2)(P_r - \lambda a_2)k_2 + (a_2 + \lambda a_2 - \lambda \tilde{a}_2 - P_r)r_2 = a_2^2 - \tilde{a}_2 P_r \\ (a_2 - \tilde{r}_2)(P_a - \lambda a_2)k_2 + (a_2 + \lambda a_2 - \lambda \tilde{r}_2 - P_a)r_2 = a_2^2 - \tilde{r}_2 P_a. \end{cases}$$

Solving the system of equations we get k_2 and r_2 , which together with a_2 give g_2 . The transformation g_1 can be obtained in a similar way.

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