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SEMILINEAR PERTURBATION OF LINEAR HARMONIC SPACES

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Abstract. Until now, non-linear potential theory, the examples as well as the axiomatic theory, has mainly been based on the Perron–Wiener–Brelot method to solve the Dirichlet problem. In a linear context, the PWB-solution of the Dirichlet problem is sufficient to develop notions like specific order, potential kernels, balayage of measures, etc. In non-linear theory however, these notions present great difficulties. We will define specific order and potential kernels in a non-linear context. One of the results is a local version of the Riesz decomposition property. We only consider semilinear perturbations of harmonic spaces. However, our methods are axiomatic in nature and they promise to be very fruitfull in the further development of non-linear potential theory.

1. Introduction

One of the main motivations of potential theory is to describe solutions of differential equations $L(f) = 0$. The potential theory of linear operators L has been studied extensively. For non-linear operators L however, much less is known, especially not in axiomatic theories. Axiomatic non-linear potential theory until now has been mainly based on the Perron–Wiener–Brelot method of solving the Dirichlet problem. Several people have tried to maintain some weak version of linearity (for instance scalar multiplication or addition of constants, see [1], [4], [5], [13], [14], [15] and [16]), but all attempts in that direction have failed, until now, to give essentially more results than described in [8] or [9], where linearity is discarded completely. However, from examples it is clear that a lot more structure, than described in that paper, exists.

In this text, we will use the example of semilinear perturbations of harmonic spaces to introduce extra structure in a non-linear potential theory. Linear perturbations were first considered by Walsh in [18]. After that, several people considered linear perturbations (see [2], [3], [10], [11] and [12]) and non-linear perturbations (see [5] and [17]). We will only consider semilinear perturbations of harmonic spaces, similar to the perturbations defined by Maeda in [17]. However, the tools

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we will use, in particular the idea of biased hyperharmonic functions, are axiomatic in nature and can also be applied in other examples of non-linear potential theory.

Our main motivation is the specific preorder in linear potential theory. In linear theory we say that $f \succsim g$ if there is a hyperharmonic function h such that $f = g + h$. In terms of the operator L this is equivalent to $L(f) \geq L(g)$. This last characterization can also be applied to non-linear operators L and this was the idea behind the specific preorder we will define. We will prove one of the forms of the Riesz-decomposition property for this specific preorder. We will also show that two potentials p and q are equal if $p \sim q$ (i.e. if $Lp = Lq$). This will enable us to define a potential $p \oplus q$ that has the property $L(p \oplus q) = Lp + Lq$. Of course, in general, $p \oplus q$ is not the pointwise sum of p and q. However, the set of potentials, with this addition and a similar scalar multiplication, is a lower complete prevector lattice. Furthermore, similar to the linear case, we can define a specific multiplication.

2. Preliminaries

In this text, X will be a harmonic space with a countable base in the sense of [6], whose notions and notations will be followed unless stated otherwise. Furthermore, for any open set U

1. $\mathscr{P}(U)$ will be the set of continuous bounded potentials on U;
2. $\mathrm{C}(U)$ will be the set of continuous real functions on U;
3. $\mathrm{LSC}(U)$ will be the set of lower finite, lower semicontinuous numerical functions on U;
4. $\mathrm{C}_f(U)$ will be the set of locally bounded fine continuous Borel functions on U;
5. $\mathscr{B}(U)$ will be the set of (numerical) Borel functions on U.

In the next three results, U will be an open set such that $\overline{U}$ is compact and contained in a $\mathscr{P}$-set. Furthermore, $\mathrm{S}_U \equiv I + K_U$ will be a map on $\mathscr{B}(U)_b$, where K_U: $\mathscr{B}(U)_b \to \mathscr{P}(U) - \mathscr{P}(U)$ has the following Lipschitz property:

$$\begin{aligned} &\text{there exists a } q \in \mathscr{P}(U), \|q\| \leq 1, \text{such that for all } M > 0, \\ &\text{there exists a } p \in \mathscr{P}(U), \text{such that for all } f \geq g, |f| \leq M, |g| \leq M, \\ &\qquad \text{we have} : (f-g)\cdot p \succ K_U(f) - K_U(g) \succ (g-f)\cdot q. \end{aligned}$$

Theorem 2.1. *For all f, $g \in \mathscr{B}(U)_b$ with $\mathrm{S}_U(f) - \mathrm{S}_U(g) \in \mathscr{H}^\star(U)$, the following statements are equivalent:*

1. $\mathrm{S}_U(f) - \mathrm{S}_U(g) \geq 0$;
2. $\widehat{\mathrm{S}_U(f) - \mathrm{S}_U(g)} \geq 0$ *on* ∂U;
3. $f - g \geq 0$;
4. $\widehat{f - g} \geq 0$ *on* ∂U.

Furthermore, if one of those conditions is fulfilled, then

$$\{f > g\} = \{\mathrm{S}_U(f) > \mathrm{S}_U(g)\}.$$

Proof. $1 \Leftrightarrow 2$ and $3 \Rightarrow 4$ are evident.

$4 \Rightarrow 1$: There are p, $q \in \mathscr{P}(U)$ such that $\mathrm{S}_U(f) - \mathrm{S}_U(g) + p = f - g + q$. Now since $\mathrm{S}_U(f) - \mathrm{S}_U(g) + p \in \mathscr{H}^\star(U)$ and $\widehat{(\mathrm{S}_U(f) - \mathrm{S}_U(g) + p)} \geq 0$ on ∂U we have $\mathrm{S}_U(f) - \mathrm{S}_U(g) + p \geq 0$. But since p is a potential and $\mathrm{S}_U(f) - \mathrm{S}_U(g) \in \mathscr{H}^\star(U)$, this implies $\mathrm{S}_U(f) - \mathrm{S}_U(g) \geq 0$.

$1 \Rightarrow 3$: First we prove a lemma:

Lemma. *Let p, q, k, $l \in \mathscr{P}(U)$ and let $f, g \in \mathscr{B}(U)^+$ such that $f \wedge g = 0$. If we have $f \cdot p + g \cdot q \succ k$ and $f \cdot q + g \cdot p \succ l$, then we can find p', $q' \in \mathscr{P}$ with $p' \prec p$ and $q' \prec q$ such that $f \cdot p' + g \cdot q' = k$ and $f \cdot q' + g \cdot p' = l$.*

Proof of Lemma. Note that $k \curlywedge (f \cdot p) \prec (f \cdot p + g \cdot q) \curlywedge (f \cdot p) = f \cdot p$. Hence there is a $f' \in \mathscr{B}(U)^+$ with $f' \leq 1$ and $k \curlywedge (f \cdot p) = f' \cdot (f \cdot p) = (f' \cdot f) \cdot p = f \cdot (f' \cdot p)$. Note that we may assume $f' = 0$ on $\{f = 0\}$. Similarly $l \curlywedge (g \cdot p) \prec g \cdot p$ and hence there is a $g' \in \mathscr{B}(U)^+$ with $g' \leq 1$, $g' = 0$ on $\{g = 0\}$ and $l \curlywedge (g \cdot p) = g \cdot (g' \cdot p)$. Now set $p' = (f' + g') \cdot p$, then we have $p' \prec p$, $k \curlywedge (f \cdot p) = f \cdot p'$ and $l \curlywedge (g \cdot p) = g \cdot p'$. In a similar way there is a $q' \in \mathscr{P}(U)$ with $q' \prec q$ and $k \curlywedge (g \cdot q) = g \cdot q'$ and $l = \curlywedge (f \cdot q) = f \cdot q'$. But now we have $k = k \curlywedge (f \cdot p) + k \curlywedge (g \cdot q) = f \cdot p' + g \cdot q'$ and also $l = f \cdot q' + g \cdot p'$. □

Note that

$$(f - g)^+ \cdot p + (g - f)^+ \cdot q \succ K_U(f) - K_U(g) \succ -(g - f)^+ \cdot p - (f - g)^+ \cdot q$$

for some p, $q \in \mathscr{P}(U)$ with $\|q\| < 1$. Now take k, $l \in \mathscr{P}(U)$ such that $K_U(f) - K_U(g) = k - l$ and $k \curlywedge l = 0$. Then we have $(f - g)^+ \cdot p + (g - f)^+ \cdot q \succ k$ and $(f - g)^+ \cdot q + (g - f)^+ \cdot p \succ l$ and hence we can find p', $q' \in \mathscr{P}$ with $p' \prec p$ and $q' \prec q$ such that $(f - g)^+ \cdot p' + (g - f)^+ \cdot q' = k$ and $(f - g)^+ \cdot q' + (g - f)^+ \cdot p' = l$. So we have $K_U(f) - K_U(g) = (f - g) \cdot p' - (f - g) \cdot q'$ with $\|q'\| < 1$. Now we can apply [3, Corollary 2.9] to $(I + K_{p'} - K_{q'})(f - g) = \mathrm{S}_U(f) - \mathrm{S}_U(g) \in \mathscr{H}^\star(U)_b$ and we obtain the desired implication. □

Theorem 2.2. *The map S_U is surjective (and hence bijective).*

Proof. We use the same line of proof as in [17, Theorem 2.1].

Let $f \in \mathscr{B}(U)_b$, then we want to find $u \in \mathscr{B}(U)_b$ such that $\mathrm{S}(u) = f$. First we reduce the problem to a more simple one. Take $p_1 \in \mathscr{P}(U)$ such that $p_1 \succ K_U(f) \succ -p_1$ and set $p_0 = (I - K_q)^{-1}(p_1)$. Then $p_0 \in \mathscr{P}(U)$ and $p_0 - p_0 \cdot q \succ K_U(f) \succ p_0 \cdot q - p_0$. Now define $K'_U(g) = K_U\big((-p_0) \vee (p_0 \wedge g) + f\big)$ for all $g \in \mathscr{P}(U) - \mathscr{P}(U)$. Now suppose we have $v \in \mathscr{P}(U) - \mathscr{P}(U)$ such that $v + K'_U(v) = 0$. Define $v' = v - p_0$, then

$$\begin{aligned} v' = v - p_0 &= -K'_U(v) - p_0 \prec -K'_U(v') + (v - v') \cdot q - p_0 \\ &= -K'_U(v) + p_0 \cdot q - p_0 \prec -K'_U(v') + K_U(f) = -K'_U(v') + K'_U(0). \end{aligned}$$

Now using Theorem 2.1 on K'_U we get that $v' \leq 0$ and hence $v \leq p_0$. Dually we get $v \geq -p_0$ and hence $\mathrm{S}_U(v+f) = v + f + K'_U(v) = f$. So it is sufficient to prove that $\mathrm{S}_U(v) = 0$ has a solution for any $K_U\colon \mathscr{B}(U)_b \to \mathscr{P}(U) - \mathscr{P}(U)$ such that:

1. There exists an $r \in \mathscr{P}(U)$, such that for all f we have: $r \succ K_U(f) \succ -r$;
2. There exists a $p \in \mathscr{P}(U)$, such that for all $f \geq g$ we have:

$$(f-g)\cdot p \succ K_U(f) - K_U(g) \succ (g-f)\cdot p.$$

Define $v_1 = r$ and $v_{n+1} = (I+K_p)^{-1}\big(K_p(v_n) - K_U(v_n)\big)$. Since

$$(I+K_p)(v_2 - v_1) = -K_U(v_1) - v_1 = -K_U(r) - r \prec 0$$

we get $v_2 \prec v_1$. Now suppose $v_n \prec v_{n-1}$, then

$$(I+K_p)(v_{n+1} - v_n) = K_U(v_{n-1}) - K_U(v_n) - (v_{n-1} - v_n)\cdot p \prec 0$$

and hence $v_{n+1} \prec v_n$. So by induction we get that (v_n) is specifically decreasing. Furthermore, note that $v_1 \succ -r$ and suppose we have $v_n \succ -r$, then

$$(I+K_p)(v_{n+1} + r) = r - K_U(v_n) + (v_n + r)\cdot p \succ r - K_U(v_n) \succ 0$$

and hence $v_{n+1} \succ -r$. So again by induction we get that (v_n) is specifically lower bounded by $-r$. Hence $v = \inf_n v_n$ exists and $r \succ v \succ -r$. Since

$$\big|K_U(v) - K_U(v_n)\big| \leq |v - v_n|\cdot p \to 0$$

and since $v_{n+1} + v_{n+1}\cdot p = v_n \cdot p - K_U(v_n)$ we get $v + v\cdot p = v\cdot p - K_U(v)$. Hence v is the function we were looking for. □

Proposition 2.3. *Let $F \subset \mathscr{B}(U)_b$ be upper directed such that $F \prec f$ for some $f \in \mathscr{B}(U)_b$. Then we have $K_U(\vee F) = \lim_{w\in F} K_U(w)$.*

Proof. Note that we may assume that F is bounded. Take $p \in \mathscr{P}(U)$ and take a decreasing sequence $F' \subset F$ such that $\inf\widehat{(f-F)} = \inf\widehat{(f-F')}$. Then we have $\curlywedge\big((f-F')\cdot p\big) = \big(\inf(f-F')\big)\cdot p$ and since $\big\{\wedge(f-F') < \inf(f-F')\big\}$ is semipolar and $\inf(f-F')$ bounded, we get

$$\curlywedge\big((f-F)\cdot p\big) \leq \curlywedge\big((f-F')\cdot p\big) = \big(\wedge(f-F')\big)\cdot p = \big(\wedge(f-F)\big)\cdot p \prec (f-F)\cdot p.$$

Hence $\curlywedge\big((f-F)\cdot p\big) = \big(\wedge(f-F)\big)\cdot p$. Now

$$\lim_{w\in F} w\cdot p = f\cdot p - \lim_{w\in F}(f-w)\cdot p = f\cdot p - \wedge(f-F)\cdot p = \vee F\cdot p.$$

Now for all $w \in F$ we have $(\vee F - w) \cdot p \succ K_U(\vee F) - K_U(w) \succ (w - \vee F) \cdot q$. Since the left and right side of this inequality go to 0 if $w \to \vee F$ we get that $\lim_{w \in F} K_U(w) = K_U(\vee F)$. □

Now let θ be a covering of X with $\mathscr{P}$-sets and let $\mathscr{U}$ be the collection of all U with $\overline{U}$ compact and contained in some $W \in \theta$. By the notation $V \Subset U$ we mean that $V \in \mathscr{U}$ and $\overline{V} \subset U$. A semilinear perturbation is defined as map $U \to \mathrm{S}_U \equiv I + K_U$ on $\mathscr{U}$, where for every $U \in \mathscr{U}$, $K_U\colon \mathscr{B}(U)_b \to \mathscr{P}(U) - \mathscr{P}(U)$ has the previously mentioned Lipschitz property:

(LL) There exists a $q \in \mathscr{P}(U)$, $\|q\| \leq 1$, such that for all $M > 0$, there exists a $p \in \mathscr{P}(U)$, such that for all $f \geq g$, $|f| \leq M$, $|g| \leq M$, we have: $(f - g) \cdot p \succ K_U(f) - K_U(g) \succ (g - f) \cdot q$.

Furthermore, we must have:

(SH) $V \Subset U \Rightarrow K_U(f) = \mathrm{H}\big(V, K_U(f)\big) + K_V(f)$.

We will say that a semilinear perturbation is linear if K_U is linear for all $U \in \mathscr{U}$. We will say that a semilinear perturbation is isotone if (LL) is satisfied with $q \equiv 0$ for all $U \in \mathscr{U}$. It is easy to check that the linear perturbations coincide with the perturbations defined in [3] and [12]. In the appendix we will show that the isotone perturbations coincide with the perturbations defined in [17]. Furthermore, we will have by Theorem 3.8, that any semilinear perturbation can be obtained by a (negative) linear perturbation, followed by a (non-linear) isotone perturbation.

Standard example. As an example, consider the classical harmonic space, i.e. the harmonic space describing the solutions of the Laplace equation $\Delta h = 0$ on and let p be a smooth potential on X with $\Delta p = -1$. Now take a Lipschitz continuous function g on X×R and define $K_{\mathrm{X}}(f)(x) = g\big(x, f(x)\big) \cdot p$ and $K_U(f) = K_{\mathrm{X}}(f) - \mathrm{H}\big(U, K_X(f)\big)$. Then we have a semilinear perturbation and the perturbed harmonic space will describe the solutions of the equation $\Delta h - g(\cdot, h) = 0$. At the end of the next section, we will refer to this example to get a better understanding of the definitions and results in that section. Note that all remarks we will make about this standard example are heuristic, so we will not worry about technicalities like differentiability.

3. Biased semilinear hyperharmonic functions

In the rest of this text, S will be a fixed perturbation.

In linear theory, specific order plays an important role. It enables us to get the same results on 'hyperharmonic functions relative to a hyperharmonic function f', as on 'hyperharmonic functions relative to the harmonic function 0'. The proof of such results is usually ad hoc, but an analysis of such proofs reveals that the essential steps are as follows:

1. subtract a hyperharmonic function f;
2. prove the result relative to the harmonic function 0;

3. add the hyperharmonic function f again.

What we will do is formalize this technique. The most essential notion in this formalization is the idea of a bias. This bias should be seen as the function one wants to compare with (the function f above). In this section, we will define hyper- and hypoharmonic functions with respect to a bias, and we will show that these functions have similar properties as hyper- and hypoharmonic functions on the original harmonic space. In the next sections, we will apply these biased hyper- and hypoharmonic functions to prove some nice results on specific order.

An application of these ideas to linear harmonic spaces gives:

1. g is hyperharmonic with respect to a bias f if $g - f \in \mathscr{H}^*$;
2. g is hypoharmonic with respect to a bias f if $g - f \in \mathscr{H}_*$;
3. the harmonic operator with respect to a bias f is $g \to \mathrm{H}(g-f)+f$.

Note that also continuity must be seen relative to the bias f. This implies that the boundary conditions for upper and lower functions with respect to a bias f become more complicated if f is not continuous. Furthermore, for technical reasons, the bias is not always a function, but it is locally a function.

Semilinear biases. A map φ that assigns to every $U \in \mathscr{U}$ a function $\varphi(U) \in \mathrm{C}_f(U)_b$ such that $V \Subset U$ implies $\mathrm{S}_V(\varphi(V)) - \mathrm{S}_U(\varphi(U)) \in \mathscr{H}(V)$ is called a (semilinear) bias. Evidently, every $f \in \mathrm{C}_f(\mathrm{X})$ is a bias. Let φ and φ' be two biases and λ a real number. We define a bias $\varphi \mathbin{\tilde{+}} \varphi'$ by $\mathrm{S}_U(\varphi \mathbin{\tilde{+}} \varphi'(U)) = \mathrm{S}_U(\varphi(U)) + \mathrm{S}_U(\varphi'(U))$. In a similar way we define $\varphi \mathbin{\tilde{-}} \varphi'$ and $\lambda \mathbin{\tilde{\cdot}} \varphi$. Note that this defines a linear structure on the set of biases with a 0-element $\tilde{0}$ defined by $\tilde{0}(U) = \mathrm{S}_U^{-1}(0)$.

Lemma 3.1. *For any bias φ, there is a $\psi \in \mathscr{B}(\mathrm{X})$ such that for all $U \in \mathscr{U}$ we have $\varphi(U) - \psi \in \mathrm{C}(U)$.*

Proof. Since X has a countable base, there is a countable covering $(U_n) \subset \mathscr{U}$ of X. Furthermore, there are $V_n \in \mathscr{U}$ such that $U_n \Subset V_n$ for all n. Define $U^n = \cup_{i=1}^n U_n$ and $V^n = \cup_{i=1}^n V_n$. We will define a sequence ψ_n of bounded functions such that ψ_n is defined on V^n for all n, $\psi_n - \varphi(V_i) \in \mathrm{C}(V^n \cap V_i)$ for all n and all i and $\psi_{n+1} = \psi_n$ on U^n for all n. First set $\psi_1 = \varphi(V_1)$ on $V_1 = V^1$. Now suppose we have ψ_n with the desired properties. Then there is a continuous function f on X such that $f = \psi_n - \varphi(V_{n+1})$ on $\overline{U^n \cap U_{n+1}}$. Now we can define ψ_{n+1} to be $\varphi(V_{n+1}) - f$ on V_{n+1} and to be ψ_n on $V^n \setminus V_{n+1}$. It is easy to check that ψ_{n+1} satisfies the desired properties. Now we define $\psi(x) = \psi_n(x)$ if $x \in U_n$. □

In the rest of this section φ will be a fixed bias and ψ will be a fixed Borel function on X such that for all $U \in \mathscr{U}$ we have $\varphi(U) - \psi \in \mathrm{C}(U)$. Note that for any ψ' with the same property we have $\psi - \psi' \in \mathrm{C}(\mathrm{X})$. It is easy to check that $\psi \in \mathrm{C}_f(\mathrm{X})$ and that all definitions and results stated in this section will be independent of the particular choice of ψ.

Biased semilinear Dirichlet problem. Let $U \in \mathscr{U}$ and g a numerical function on ∂U. If g is upper bounded we define

$$\mathrm{I}^\star(\varphi; U, g) = \inf\big\{f \in \mathscr{B}(U)_b | \mathrm{S}_U(f) - \mathrm{S}_U(\varphi(U)) \in \mathscr{H}^\star(U)_b, \\ \widehat{f - \psi} \geq g - \psi \text{ on } \partial U\big\}.$$

If g is lower bounded, $\mathrm{I}_\star(\varphi; U, g)$ is defined dually and if $\mathrm{I}^\star(\varphi; U, g) = \mathrm{I}_\star(\varphi; U, g)$ for some bounded g, then we denote the common value by $\mathrm{I}(\varphi; U, g)$.

Proposition 3.2. *If $U \Subset W \in \mathscr{U}$ and g is a bounded numerical function on ∂U, then*

$$\mathrm{I}^\star(\varphi; U, g) \leq \mathrm{S}_U^{-1}\big(\overline{\mathrm{H}}(U, g - \mathrm{S}_W(\varphi(W))) + \mathrm{S}_W(\varphi(W)).$$

Proof. Take $V \Subset U$ and $f \in \mathscr{H}^\star\big(U, g - \mathrm{S}_W(\varphi(W))\big)_b$ and define:

$$\begin{aligned} \mathrm{I}(W, \varphi; U, g) &= \mathrm{S}_U^{-1}\big(\overline{\mathrm{H}}(U, g - \mathrm{S}_W(\varphi(W))) + \mathrm{S}_W(\varphi(W))\big) \\ f' &= \mathrm{S}_U^{-1}\big(f + \mathrm{S}_W(\varphi(W))\big) \\ h &= f' + (I - K_q)^{-1}\big((K_U(f') \curlyvee 0)_V\big). \end{aligned}$$

Now, since $\big(K_U(f') \curlyvee 0\big)_V \succ 0$ we get $\mathrm{S}_U(h) \succ f + \mathrm{S}_W\big(\varphi(W)\big)$ and hence $\mathrm{S}_U(h) - \mathrm{S}_W\big(\varphi(W)\big) \in \mathscr{H}^\star(U)_b$. Furthermore, on $U \setminus V$ we have

$$h \geq f' + \big(K_U(f') \curlyvee 0\big)_V \geq f' + K_U(f')_V = f + \mathrm{S}_W\big(\varphi(W)\big).$$

This implies $\widehat{h - \psi} \geq g - \psi$ on ∂U. So we have the following inequalities:

$$\begin{aligned} \mathrm{I}^\star(\varphi; U, g) - h &\leq 0 \\ h - f' &= (I - K_q)^{-1}\big((K_U(f') \curlyvee 0)_V\big) \\ f' - \mathrm{I}(W, \varphi; U, g) &\leq (I - K_q)^{-1}\big(f - \overline{\mathrm{H}}(U, g - \mathrm{S}_W(\varphi(W)))\big). \end{aligned}$$

Now we can use the following lemma and the boundedness of $(I - K_q)^{-1}$, to choose f and V such that the right hand sides of these equations become arbitrary small.

Lemma. *Let $q \in \mathscr{P}(U)$ with $\|q\| < 1$. Then for all $V \Subset U$ and $\varepsilon > 0$ there is a $W \Subset U$ such that $(I - K_q)^{-1}(U \setminus W) < \varepsilon$ on V.*

Proof. Note that for all $V \Subset U$ and all $\varepsilon > 0$ there is a $W \Subset U$ with $K_q(U \setminus W) < \varepsilon$ on V. By induction we get the same property for K_q^n. Now fix $V \Subset U$ and $\varepsilon > 0$. There is a N such that $\sum_{n=N}^{\infty} \|q\|^n < \frac{1}{2}\varepsilon$. For any $n < N$ there is a $W_n \Subset U$ such that $K_q^n(U \setminus W_n) < \varepsilon/2N$ on V. Set $W = \cup_{n=0}^{N-1} W_n$. Then $(I - K_q)^{-1}(U \setminus W) = \sum_{n=0}^{\infty} K_q^n(U \setminus W) < \varepsilon$ on V. □

Hence we get $\mathrm{I}^\star(\varphi; U, g) \leq \mathrm{I}(W, \varphi; U, g)$. □

Proposition 3.3. *Let $U \in \mathscr{U}$, then we have:*

1. *If g is a bounded function on ∂U, then $\mathrm{I}^\star(\varphi; U, g) \geq \mathrm{I}_\star(\varphi; U, g)$;*
2. *If $g \in \mathscr{B}(\partial U)_b$ and $W \in \mathscr{U}$ with $U \Subset W$, then*

$$\mathrm{I}^\star(\varphi; U, g) = \mathrm{I}_\star(\varphi; U, g) = \mathrm{S}_U^{-1}\big(\mathrm{H}(U, g - \mathrm{S}_W(\varphi(W)))\big) + \mathrm{S}_W\big(\varphi(W)\big);$$

3. *If $f, g \in \mathscr{B}(\partial U)_b$ with $f - g \in \mathrm{LSC}(\partial U)^+$ and $x \in U$, then $\mathrm{I}(\varphi; U, f)(x) = \mathrm{I}(\varphi; U, g)(x)$ if and only if $f = g$ on the support of $\mathrm{H}(U, \cdot)(x)$;*
4. *If $F \in \mathscr{B}(\partial U)_b$ is upper directed such that $f = \sup F \in \mathscr{B}(\partial U)_b$ and $\sup \mathrm{H}(U, F) = \mathrm{H}(U, f)$, then $\sup \mathrm{I}(\varphi; U, F) = \mathrm{I}(\varphi; U, f)$;*
5. *If $f \in \mathscr{B}(\mathrm{X})_b$ is fine l.s.c. in x, then $\liminf_{V \to x} \mathrm{I}(\varphi; V, f)(x) \geq f(x)$;*
6. *Let $x \in \partial U$ be regular and $f \in \mathscr{B}(\partial U)_b$ with $f - \psi$ continuous in x, then $\lim_{y \to x}(\mathrm{I}(\varphi; U, f) - \psi)(y) = f(x) - \psi(x)$.*

Proof. Claim 1 follows from 2.1, 2 from 1 and 3.2, 3 and 4 from 2.1.

5: Take a neighbourhood $W \in \mathscr{U}$ of x and h, g with $\mathrm{S}(h) - \mathrm{S}_W\big(\varphi(W)\big) \in \mathscr{H}^\star(W)$ and $\mathrm{S}_W\big(\varphi(W)\big) - \mathrm{S}(g) \in \mathscr{H}^\star(W)_b$ and $h \geq f \geq g$. Set $M = \|h\| \vee \|g\|$ and let p, $q \in \mathscr{P}(W)$ be as in property (LL). Then for all $V \Subset W$ we have $h \geq \mathrm{I}(\varphi; V, f) \geq g$ and hence

$$(h - g) \cdot \big(p - \mathrm{H}(V, p)\big) \succ K_V\big(\mathrm{I}(\varphi; V, f)\big) - K_V(g) \succ (g - h) \cdot \big(q - \mathrm{H}(V, q)\big).$$

Now if $V \to x$, then $\big((h-g) \cdot (p - \mathrm{H}(V,p))\big)(x) \to 0$, $\big((g-h) \cdot (q - \mathrm{H}(V,q))\big)(x) \to 0$ and $K_V(g)(x) = K_W(g)(x) - \mathrm{H}\big(V, K_W(g)\big)(x) \to 0$. So we must have

$$K_V\big(\mathrm{I}(\varphi; V, f)\big)(x) = \mathrm{H}(V, f - \mathrm{S}_W\big(\varphi(W)\big)(x) + \mathrm{S}_W\big(\varphi(W)\big)(x) - \mathrm{I}(\varphi; V, f) \to 0$$

in x. Now if f is fine l.s.c. in x then so is $f - \mathrm{S}_W\big(\varphi(W)\big)$ and hence

$$\liminf_{V \to x} \mathrm{H}\big(V, f - \mathrm{S}_W\big(\varphi(W)\big)\big)(x) \geq \big(f - \mathrm{S}_W\big(\varphi(W)\big)\big)(x)$$

and hence we must have $\liminf_{V \to x} \mathrm{I}(\varphi; V, f)(x) \geq f(x)$.

6: Take $M = \|\mathrm{I}(\varphi; U, f)\|$, $W \in \mathscr{U}$ with $U \Subset W$ and $p \in \mathscr{P}(W)$ such that $p \succ K_W(h) \succ -p$ for all h with $\|h\| \leq M$. Set $p' = p - \mathrm{H}(U, p)$, then $p' \succ K_U(h) \succ -p'$ for all h with $\|h\| \leq M$ and $\lim_{y \to x} p'(y) = 0$. In particular $\lim_{y \to x} K_U(h)(y) = 0$ for all h with $\|h\| \leq M$. Now

$$K_U\big(\mathrm{I}(\varphi; U, f)\big) = \mathrm{H}\big(U, f - \mathrm{S}_W\big(\varphi(W)\big)\big) + \mathrm{S}_W\big(\varphi(W)\big) - \mathrm{I}(\varphi; U, f)$$

and since

$$\lim_{y \to x} \mathrm{H}\big(U, f - \mathrm{S}_W\big(\varphi(W)\big)\big)(y) = \big(f - \mathrm{S}_W\big(\varphi(W)\big)\big)(x)$$

we also have

$$\lim_{y\to x} \big(\mathrm{I}(\varphi; U, f) - \mathrm{S}_W(\varphi(W))\big)(y) = f(x) - \mathrm{S}_W\big(\varphi(W)\big)(x). \ \square$$

Lemma 3.4. *Let $U \in \mathscr{U}$ and $f, g \in \mathscr{B}(U)$ such that:*

1. $\mathrm{S}_U(g) - \mathrm{S}_U\big(\varphi(U)\big) \in \mathscr{H}_\star(U)_b$;
2. $f - g \in \mathrm{LSC}(U)$;
3. $\widehat{f-g} \geq 0$ *on* ∂U;
4. *For all* $x \in U$, *there is a neighbourhood* $V_x \Subset U$ *of* x *such that* $f(x) \geq \mathrm{I}_\star(\varphi; V_x, f)(x)$.

Then $f \geq g$.

Proof. Take a bounded strict potential p on a neighbourhood of $\overline{U}$ and define $g_\alpha = g - (I - K_q)^{-1}(\alpha p)$. For all $V \Subset U$ we have $\mathrm{H}\big(V, \mathrm{S}_U(g) - \mathrm{S}_U(\varphi(U))\big) \geq \mathrm{S}_U(g) - \mathrm{S}_U\big(\varphi(U)\big)$. Since $\mathrm{S}_U(g) - \mathrm{S}_U(g_\alpha) \succ \alpha p$ and since p is strict, this implies $\mathrm{H}\big(V, \mathrm{S}_U(g_\alpha) - \mathrm{S}_U(\varphi(U))\big) > \mathrm{S}_U(g_\alpha) - \mathrm{S}_U\big(\varphi(U)\big)$. Using (SH) this is equivalent to

$$\mathrm{H}\big(V, g_\alpha - \mathrm{S}_U(\varphi(U))\big) + \mathrm{S}_U\big(\varphi(U)\big) > \mathrm{S}_V(g_\alpha)$$

and using 2.1 we get $\mathrm{I}(\varphi; V, g_\alpha) > g_\alpha$.

Now suppose $f(y) < g(y)$ for some $y \in U$. Now there is an $\alpha > 0$ such that $f \geq g_\alpha$ and $f(x) = g_\alpha(x)$ for some $x \in U$. So we have the contradiction

$$f(x) \geq \mathrm{I}_\star(\varphi; V_x, f)(x) \geq \mathrm{I}(\varphi; V_x, g_\alpha)(x) > g_\alpha(x) = f(x). \ \square$$

Biased semilinear hyperharmonic functions. For each open $U \subset \mathrm{X}$ we define the set of φ-semilinear hyperharmonic functions by

$${}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U) = \big\{f | f - \psi \in \mathrm{LSC}(U), \text{ and for all } V \Subset U : \mathrm{I}_\star(\varphi; V, f) \leq f\big\}.$$

Again ${}^{\mathrm{S}}\mathscr{H}_\star(\varphi; U)$ is defined dually and we set ${}^{\mathrm{S}}\mathscr{H}(\varphi; U) = {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U) \cap {}^{\mathrm{S}}\mathscr{H}_\star(\varphi; U)$. Now for all numerical functions g on ∂U we define

$${}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U, g) = \big\{f \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U) | f \text{ lower bounded and } \widehat{f - \psi} \geq g - \psi \text{ on } \partial U\big\}$$

and ${}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, g) = \inf {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U, g)$. Dually we define ${}^{\mathrm{S}}\mathscr{H}_\star(\varphi; U, g)$ and ${}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; U, g)$. If ${}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, g) = {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; U, g)$, then we denote the common value by ${}^{\mathrm{S}}\mathrm{H}(\varphi; U, g)$.

Proposition 3.5. *Let $U \in \mathscr{U}$, then:*

1. $\mathrm{S}_U(h) - \mathrm{S}_U\big(\varphi(U)\big) \in \mathscr{H}(U)_b$ *if and only if* $h \in {}^{\mathrm{S}}\mathscr{H}(\varphi; U)_b$;
2. *If* $\mathrm{S}_U(h) - \mathrm{S}_U\big(\varphi(U)\big) \in \mathscr{H}^\star(U)_b$, *then* $h \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U)_b$;
3. *For any upper bounded numerical function* g *on* ∂U *we have* ${}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, g) = \mathrm{I}^\star(\varphi; U, g)$;
4. *For any numerical function* g *on* ∂U *we have* ${}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, g) \geq {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; U, g)$;
5. *If* $F \subset {}^{\mathrm{S}}\mathscr{H}(\varphi; U)_b$ *is upper directed and bounded, then* $\sup F \in {}^{\mathrm{S}}\mathscr{H}(\varphi; U)_b$.

Proof. Claim 1: From (SH) and 2.1 we get: $I(V,h) = h$ is equivalent to $\mathrm{H}(V,h) + \mathrm{S}_U(\varphi(U)) - H(V, \mathrm{S}_U(\varphi(U))) = \mathrm{S}_V(h)$ which is equivalent to

$$\mathrm{H}(V, \mathrm{S}_U(h)) + \mathrm{S}_U(\varphi(U)) - H(V, \mathrm{S}_U(\varphi(U))) = \mathrm{S}_U(h).$$

2: From (SH) and 2.1 we get: $\mathrm{H}(V, \mathrm{S}_U(h) - \mathrm{S}_U(\varphi(U))) \le \mathrm{S}_U(h) - \mathrm{S}_U(\varphi(U))$ is equivalent to $\mathrm{H}(V, h - \mathrm{S}_U(\varphi(U))) + \mathrm{S}_U(\varphi(U)) \le \mathrm{S}_V(h)$ and this implies $\mathrm{I}(\varphi; V, h) \le h$.

3: It is clear that ${}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, g) \le \mathrm{I}^\star(\varphi; U, g)$. Now take any $f \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U, g)$. Then there is a $h \in \mathscr{B}(\partial U)_b$, $h \ge g$ such that $\widehat{f - \psi} \ge h - \psi$ on ∂U. By 3.4 we get that $f \ge \mathrm{I}_\star(\varphi; U, h) = \mathrm{I}^\star(\varphi; U, h) \ge \mathrm{I}^\star(\varphi; U, g)$ and hence also ${}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, g) \ge \mathrm{I}^\star(\varphi; U, g)$.

4: Take any $f_1 \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U, g)$ and any $f_2 \in {}^{\mathrm{S}}\mathscr{H}_\star(\varphi; U, g)$. Then there is a $h \in \mathscr{B}(\partial U)_b$ such that also $f_1 \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U, h)$ and $f_2 \in {}^{\mathrm{S}}\mathscr{H}_\star(\varphi; U, h)$. Hence we have $f_1 \ge {}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, h) = \mathrm{I}(\varphi; U, h) = {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; U, h) \ge f_2$ and hence ${}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; U, g) \ge {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; U, g)$.

5: Since $F \subset {}^{\mathrm{S}}\mathscr{H}(\varphi; U)_b$ we have that $\mathrm{S}_U(F) - \mathrm{S}_U(\varphi(U)) \in \mathscr{H}(U)_b$. Furthermore, since F is upper directed, by 2.1 also $\mathrm{S}_U(F) - \mathrm{S}_U(\varphi(U))$ is upper directed and we have that $\mathrm{S}_U(\sup F) - \mathrm{S}_U(\varphi(U)) = \sup(\mathrm{S}_U(F) - \mathrm{S}_U(\varphi(U)))$. So by the Bauer convergence principle $\mathrm{S}_U(\sup F) - \mathrm{S}_U(\varphi(U)) \in \mathscr{H}(U)_b$ and hence $\sup F \in {}^{\mathrm{S}}\mathscr{H}(\varphi; U)_b$. □

We say that a numerical function f on an open set U is nearly φ-semilinear hyperharmonic on U if f is locally lower bounded and for all $x \in U$ there is a neighbourhoodbase $\tau_x \subset \mathscr{U}$ such that for all $V \in \tau_x$ we have $V \Subset U$ and ${}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, f)(x) \le f(x)$.

Theorem 3.6. *Let f be nearly φ-semilinear hyperharmonic on U and let g be the fine l.s.c. regularization of f, then $g \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U)$.*

Proof. It is easy to check that

$${}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; U, g) = \mathrm{I}_\star(\varphi; U, g) = \sup\{\mathrm{I}(\varphi; U, h) \mid g \le h \in \mathscr{B}(\partial U)_b\}$$

and hence ${}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, f) - \psi \in \mathrm{LSC}(V)$. In particular ${}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, f)$ is fine l.s.c., and so also g is nearly φ-semilinear hyperharmonic on U. Note also that from 3.3 we get $\liminf_{V \to x} {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, g)(x) \ge g(x)$. Hence for any neighbourhood $W \in \mathscr{U}$ of x we have

$$\widehat{g - \psi}(x) \ge \lim_{\tau_x \ni V \to x} \left({}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V_x, g) - \psi\right)(x) \ge g(x) - \psi(x) \ge \widehat{g - \psi}(x).$$

So $g - \psi \in \mathrm{LSC}(U)$ and now for any $V \Subset U$ we can apply 3.4 to show $\mathrm{I}_\star(\varphi; V, g) \le g$. □

Note that this result immediately implies that ${}^{\mathrm{S}}\mathscr{H}^\star$ is a sheaf. Note also that X, endowed with ${}^{\mathrm{S}}\mathscr{H}^\star(\varphi) - \psi$ and ${}^{\mathrm{S}}\mathscr{H}_\star(\varphi) - \psi$ satisfies all the assumptions made in [8] and [9]. Hence all the results and notions from those papers can be applied.

Theorem 3.7. *If the perturbation is linear, then* $(\mathrm{X}, {}^{\mathrm{S}}\mathscr{H}^{\star}(\widetilde{0}))$ *is a harmonic space and we have* $\mathrm{S}_U({}^{\mathrm{S}}\mathscr{H}^{\star}(\widetilde{0};U)_b) = \mathscr{H}^{\star}(U)_b$ *for all* $U \in \mathscr{U}$.

Proof. It is easy to check that ${}^{\mathrm{S}}\mathscr{H}^{\star}(\widetilde{0};U)$ is a convex cone and from the previous results it is clear that X with ${}^{\mathrm{S}}\mathscr{H}^{\star}(\widetilde{0})$ satisfies all axioms of a harmonic space. Since we have $\mathrm{S}_V({}^{\mathrm{S}}\mathscr{H}(\widetilde{0};U)_b) = \mathscr{H}(V)_b$ and $\mathrm{S}_V({}^{\mathrm{S}}\mathscr{H}^{\star}(\widetilde{0};U)_b) \supset \mathscr{H}^{\star}(V)_b$ we can use 2.1 to show that $\mathrm{S}_U^{-1}(p)$ is an $\widetilde{0}$-semilinear potential for any $p \in \mathscr{P}(U)$. Hence $\mathrm{S}_U^{-1} \circ K_U$ is a difference of potential kernels and so $\mathrm{S}_U^{-1} = I - \mathrm{S}_U^{-1} \circ K_U$ defines a perturbation on the $\widetilde{0}$-semilinear harmonic space that returns our original harmonic space. Hence we must have $\mathrm{S}_V({}^{\mathrm{S}}\mathscr{H}^{\star}(\widetilde{0};U)_b) = \mathscr{H}^{\star}(V)_b$. □

Theorem 3.8. *Any semilinear perturbation can be obtained by a (negative) linear perturbation followed by a (non-linear) isotone perturbation.*

Proof. For any $U \in \mathscr{U}$, let $q[U]$ be the smallest $q \in \mathscr{P}(U)$ for which property (LL) holds on the set U. Take $V \Subset U$ and $f, g \in \mathscr{B}(U)_b$ with $f \geq g$. Using (SH) we get that $K_V(f) - K_V(g) \succsim (g-f)\cdot q[U]$. Since $K_q - \mathrm{H}(V, K_q) = K_{q-\mathrm{H}(V,p)}$ this implies $q[V] \leq q[U] - \mathrm{H}(V, q[U])$. Again using (SH) we get $K_U(f) - K_U(g) \succsim (g-f)\cdot q[V]$. Now take $q = q[U]$ on $U \setminus V$ and $q = q[U] \wedge (h + q[V])$ on V, for some $h \in \mathscr{H}^{\star}(V,q) \cap \mathrm{C}(V)$. Then $q \in \mathscr{P}(U)$ and $q \succsim q[U] \curlyvee q[V]$ on V. Hence $K_U(f) - K_U(g) \succsim (g-f)\cdot q$ and so $q \succ q[U]$. Hence $h + q[V] \geq q[U]$ for all $h \in \mathscr{H}^{\star}(V,q)$ and so $q[V] \geq q[U] - \mathrm{H}(V, q[U])$. So we have $q[V] = q[U] + \mathrm{H}(V, q[U])$ and hence the $K_{q[U]}$ form a linear perturbation.

Now define $K'_U = (I - K_{q[U]})^{-1}(K_U + K_{q[U]})$. Then for all $f \geq g$ we have $(I - K_{q[U]})^{-1}((f-g)\cdot(p+q)) \succ K'_U(f) - K'_U(g) \succ 0$. Furthermore, for all f and $V \subset U$ we have that $K'_U(f) - K'_V(f)$ is '(-q)-harmonic' on V. Hence K' defines an isotone perturbation on the '(-q)-harmonic' space. Furthermore, $(I - K_{q[U]}) \circ (I + K'_U) = (I + K_U)$. □

Proposition 3.9. *Let* $U \in \mathscr{U}$, *then:*

1. $\mathrm{S}_U(h) - \mathrm{S}_U(\varphi(U)) \in \mathscr{H}^{\star}(U)_b$ *if and only if* $h \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi;U)_b$;
2. *The fine topology is the coarsest topology, finer than the original topology, for which all* φ*-semilinear hyperharmonic functions are continuous.*

Proof. 1: First suppose the perturbation is isotone. Then from (SH) and 2.1 we get

$$\mathrm{H}(V, \mathrm{S}_U(h) - \mathrm{S}_U(\varphi(U))) \leq \mathrm{S}_U(h) - \mathrm{S}_U(\varphi(U))$$

is equivalent to

$$\mathrm{H}(V, h - \mathrm{S}_U(\varphi(U))) + \mathrm{S}_U(\varphi(U)) \leq \mathrm{S}_V(h)$$

and this is equivalent to $\mathrm{I}(\varphi; V, h) \leq h$. Now suppose the perturbation is linear. Then the statement follows from 3.7. In the general case we can combine these two results using 3.8.

2: Let σ be the fine topology generated by ${}^{\mathrm{S}}\mathscr{H}^{\star}{}_b(\varphi)$. Take U open, $f \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi; U)$ and $x \in U$. If $f(x) = \infty$, then since $f - \psi$ is lower semicontinuous, it is continuous and hence σ-continuous in x. If $f(x) < \infty$, then there is a neighbourhood V of x and a $g \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi; V)_b$ such that $g(x) > f(x)$. Since $g \wedge f \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi; V \cap U)$ we have that $g \wedge f$ is σ-continuous on $V \cap U$. Since also g is σ-continuous on $V \cap U$ and $g(x) > (g \wedge f)(x)$ we have that $g > g \wedge f$ on a σ-neighbourhood W of x. Hence $g \wedge f = f$ on W and so f is σ-continuous on W and in particular in x. So any $f \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi)$ is σ-continuous and hence σ is the fine topology generated by ${}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi)$. Similarly, the fine topologies generated by $\mathscr{H}_b^{\star}$ and $\mathscr{H}^{\star}$ coincide.

Now take $U \in \mathscr{U}$ and $f \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi; U)_b$. Then $g = \mathrm{S}_U(f) - \mathrm{S}_U(\varphi(U)) \in \mathscr{H}^{\star}(U)_b$ and hence $f = g - K_U(f) + \varphi(U) + K_U(\varphi(U)) \in \mathrm{C}_f(U)$. So σ is finer than the fine topology.

Now take $U \in \mathscr{U}$ and $g \in \mathscr{H}^{\star}(U)_b$. Then $f = \mathrm{S}_U^{-1}(g + \mathrm{S}_U(\varphi(U))) \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi; U)_b$ and hence $g = f + K_U(f) - \varphi(U) - K_U(\varphi(U))$ is σ-continuous. ($f, \varphi(U) \in {}^{\mathrm{S}}\mathscr{H}^{\star}(\varphi; U)$ and $K_U(f), K_U(\varphi(U)) \in \mathrm{C}(U)$.) So σ is coarser than the fine topology. □

Standard example. Now let us go back to our standard example. The first thing we should note is that the definition of the semilinear bias φ is such that for all U, $V \in \mathscr{U}$ we have $\Delta\varphi(V) - g(\cdot, \varphi(V)) = \Delta\varphi(U) - g(\cdot, \varphi(U))$ on $U \cap V$. This implies that $\Delta\varphi - g(\cdot, \varphi)$ is a well defined expression. Now we can see the φ-semilinear hyperharmonic functions as those functions f such that $\Delta f - g(\cdot, f) \leq \Delta\varphi - g(\cdot, \varphi)$. Similar expressions hold for the φ-semilinear hypoharmonic and the φ-semilinear harmonic functions. Furthermore, it is easy to check that the specific preorder $f \succsim_{\mathrm{S}} h$ defined in the next section is equivalent to $\Delta f - g(\cdot, f) \leq \Delta h - g(\cdot, h)$.

4. Specific order

For U open and f, $g \in \mathrm{C}_f(U)$ we say $f \succsim_{\mathrm{S}} g$ on U if for all $V \Subset U$ we have $\mathrm{S}_V(f) - \mathrm{S}_V(g) \in \mathscr{H}^{\star}(V)$. If both $f \succsim_{\mathrm{S}} g$ and $f \geq g$ on U then we say $f \succ_{\mathrm{S}} g$ on U. If both $f \succsim_{\mathrm{S}} g$ and $g \succsim_{\mathrm{S}} f$ on U then we say $f \sim g$ on U. Obviously $\succsim_{\mathrm{S}}$ is a preorder, $\succ_{\mathrm{S}}$ is an order and $\sim$ is an equivalence relation on C_f. Note that $f \succsim_{\mathrm{S}} g$ on U implies $f - g \in \mathrm{LSC}(U)$ and that if $U \in \mathscr{U}$ and f, $g \in \mathrm{C}_f(U)_b$, then $f \succsim_{\mathrm{S}} g$ on U if and only if $\mathrm{S}_U(f) - \mathrm{S}_U(g) \in \mathscr{H}^{\star}(U)$. The next theorem is the main reason for introducing biased semilinear hyperharmonic functions. It will enable us to prove the other results on specific order.

Theorem 4.1. *If U is open and f, $g \in \mathrm{C}_f(U)$, then the following statements are equivalent:*

1. $f \succsim_{\mathrm{S}} g$;
2. $f \in {}^{\mathrm{S}}\mathscr{H}^{\star}(g; U)$;
3. $g \in {}^{\mathrm{S}}\mathscr{H}_{\star}(f; U)$.

The proof follows from 3.9. □

Theorem 4.2. *If* $\mathrm{C}_f(U) \ni f \succsim_{\mathrm{S}} F \subset \mathrm{C}_f(U)$, *F locally upper bounded, then the supremum* $\vee F$ *of* F *in* $\mathrm{C}_f(U)$ *exists, it is equal to the fine upper semicontinuous regularization of* $\sup F$ *and we have* $\vee F \precsim_{\mathrm{S}} f$.

Proof. $\sup F$ is nearly f-semilinear hypoharmonic. □

Theorem 4.3. *If* $\mathrm{C}_f(U) \ni f \succ_{\mathrm{S}} F \subset \mathrm{C}_f(U)$, *then the semilinear specific supremum* $\curlyvee_{\mathrm{S}} F$ *of* F *in* $\mathrm{C}_f(U)$ *exists and for all* $V \Subset U$ *and* $h \in \mathrm{C}_f(V)$ *with* $h \succsim_{\mathrm{S}} F$ *on* V, *we have* $h \succsim_{\mathrm{S}} \curlyvee_{\mathrm{S}} F$ *on* V.

Proof. Set $k = \wedge\{g \mid g \succ_{\mathrm{S}} F\}$. From 4.2 we have $k \succ_{\mathrm{S}} F$. Now take $W \Subset V$ and $g \in {}^{\mathrm{S}}\mathscr{H}^\star(h; W, k)$ and define $g' = k \wedge g$ on W and $g' = k$ on $U \setminus W$. It is easy to check that $g \in {}^{\mathrm{S}}\mathscr{H}^\star(f; W, k)$ for all $f \in F$ and hence that $g' \succ_{\mathrm{S}} f$ for all $f \in F$. So $g' \geq k$ and thus $g \geq k$ on W. So now we have ${}^{\mathrm{S}}\overline{\mathrm{H}}(h; W, k) \geq k$ and hence $h \succsim_{\mathrm{S}} k$. This implies $\{g \mid g \succ_{\mathrm{S}} F\} \succ_{\mathrm{S}} k \succ_{\mathrm{S}} F$ and hence $k = \curlyvee_{\mathrm{S}} F$. □

Similar to the linear case, we define reduced functions by ${}^{\mathrm{S}}R^\star(\varphi; U, g) = \inf\{f \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U) \mid f \geq g\}$. Note that ${}^{\mathrm{S}}R^\star(\varphi; U, g)$ is nearly φ-semilinear hyperharmonic. So if $g \in \mathrm{C}_f(U)$, then ${}^{\mathrm{S}}R^\star(\varphi; U, g) \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U)$.

Theorem 4.4. *Suppose* $V \subset U$, φ *a bias,* $g \in \mathrm{C}_f(U)$ *and* $h \in \mathrm{C}_f(V)$ *with* $h \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; V)$ *and* $g \precsim_{\mathrm{S}} h$ *on* V. *If* ${}^{\mathrm{S}}R^\star(\varphi; U, g)$ *is locally bounded on* V, *then* ${}^{\mathrm{S}}R^\star(\varphi; U, g) \precsim_{\mathrm{S}} h$ *on* V.

Proof. Take $W \Subset V$ and $u \in {}^{\mathrm{S}}\mathscr{H}^\star\big(h; W, {}^{\mathrm{S}}R^\star(\varphi; U, g)\big)$ and define $u' = {}^{\mathrm{S}}R^\star(\varphi; U, g)$ on $U \setminus W$ and $u' = {}^{\mathrm{S}}R^\star(\varphi; U, g) \wedge u$ on W. Now we have $\widehat{u - \psi} \geq (\widehat{u - h}) + (\widehat{h - \psi}) \geq ({}^{\mathrm{S}}R^\star(\varphi; U, g) - h) + (h - \psi) = {}^{\mathrm{S}}R^\star(\varphi; U, g) - \psi$ on ∂W and hence $u' - \psi \in \mathrm{LSC}(U)$. So we get $u' \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U)$. Furthermore, we have $u \in {}^{\mathrm{S}}\mathscr{H}^\star(h; W, g)$ and hence $u \geq {}^{\mathrm{S}}\mathrm{H}(h; W, g) \geq g$. So $g \leq u' \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U)$ and hence $u' \geq {}^{\mathrm{S}}R^\star(\varphi; U, g)$. Since u was arbitrary we get ${}^{\mathrm{S}}\mathrm{H}\big(h; W, {}^{\mathrm{S}}R^\star(\varphi; U, g)\big) \geq {}^{\mathrm{S}}R^\star(\varphi; U, g)$. Now, since W was arbitrary we get ${}^{\mathrm{S}}R^\star(\varphi; U, g) \in {}^{\mathrm{S}}\mathscr{H}_\star(h; V)$. □

Note that this result implies that, for $h \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U)$ with $h \succ_{\mathrm{S}} g$ on U, we have $h \succ_{\mathrm{S}} {}^{\mathrm{S}}R^\star(\varphi; U, g)$ on U. In linear theories, this is equivalent to the Riesz decomposition property.

Lemma 4.5. *Let* $U \in \mathscr{U}$ *and let* $F \subset \mathrm{C}_f(U)$ *be bounded. Then the following statements are equivalent:*

1. *There is a* $g_1 \in \mathscr{H}^\star(U)_b$ *with* $g_1 \succ F$;
2. *There is a* $g_2 \in \mathscr{H}^\star(U)_b$ *with* $g_2 \succsim F$;
3. *There is a* $g_3 \in {}^{\mathrm{S}}\mathscr{H}^\star(\widetilde{0}; U)_b$ *with* $g_3 \succ_{\mathrm{S}} F$;
4. *There is a* $g_4 \in {}^{\mathrm{S}}\mathscr{H}^\star(\widetilde{0}; U)_b$ *with* $g_4 \succsim_{\mathrm{S}} F$.

Furthermore, if we can choose one of the g *to be continuous, then we can also choose the other* g *continuous.*

Proof. $1 \Rightarrow 3$: Take $k \in \mathscr{P}(U)$ with $k \succ K_U(F)$ and set $g_3 = \mathrm{S}_U^{-1}(g_1 + k)$.
$3 \Rightarrow 4$: Set $g_4 = g_3$.
$4 \Rightarrow 2$: Take $k \in \mathscr{P}(U)$ with $K_U(F) \succ -k$ and set $g_2 = \mathrm{S}_U(g_4) + k$.
$2 \Rightarrow 1$: Take $k \in \mathscr{H}^\star(U)_b \cap \mathrm{C}(U)$ with $k \geq F - g_2$ and set $g_1 = g_2 + k$. □

For all open U, define $\mathscr{R}(U) = \{f \in \mathrm{C}_f(U) \mid$ for all $V \Subset U$ there exists a $g \in {}^{\mathrm{S}}\mathscr{H}^\star(\tilde{0}; V)_b \cap \mathrm{C}(U), h \in {}^{\mathrm{S}}\mathscr{H}_\star(\tilde{0}; V)_b \cap \mathrm{C}(U) : g \succsim_{\mathrm{S}} f \succsim_{\mathrm{S}} h$ on$V\}$. From 4.5 it is clear that $\mathscr{R}$ is the sheaf of functions that can locally be expressed as difference of bounded continuous hyperharmonic functions. So $\mathscr{R}$ is independent of the perturbation.

5. Potentials and their linear structure

Let U be an open set and φ a bias. We define the set of φ-semilinear superharmonic functions on U by ${}^{\mathrm{S}}\mathscr{S}^\star(\varphi; U) = \{f \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U) \mid$ for all $V \subset\subset U : {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, f) \in {}^{\mathrm{S}}\mathscr{H}(\varphi; V)\}$. Now for any $h \in \mathrm{C}_f(U)$ we say that f is an upper h-semilinear potential on U if $f \in {}^{\mathrm{S}}\mathscr{S}^\star(h; U)$ and h is the greatest h-semilinear hypoharmonic minorant of f. We denote the set of h-semilinear potentials on U by ${}^{\mathrm{S}}\mathscr{P}^\star(h; U)$. It is easy to check that if $p \in {}^{\mathrm{S}}\mathscr{P}^\star(h, U)$ and $p \geq q \succ_{\mathrm{S}} h$ on U, then we have $q \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)$. Furthermore, if $f \in {}^{\mathrm{S}}\mathscr{P}^\star(g; U)$ and $g \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)$, then $f \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)$. The set ${}^{\mathrm{S}}\mathscr{P}_\star(h; U)$ of lower h-semilinear potentials on U is defined dually.

In the rest of this section we will assume $U \in \mathscr{U}$ and $h \in \mathrm{C}_f(U)_b$ and we will only consider bounded potentials. These restrictions are necessary in 5.1, 5.2 and 5.5. All other results can, except for their dependence on Theorems 5.2 and 5.5, be proved for U open, $h \in \mathrm{C}_f(U)$ and potentials in $\mathrm{C}_f(U)$.

Proposition 5.1. *We have* $\mathrm{S}_U({}^{\mathrm{S}}\mathscr{P}^\star(h; U)_b) - \mathrm{S}_U(h) = \mathscr{P}^\star(U)_b$.

Proof. Take $p \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)_b$, then evidently $\mathrm{S}_U(p) - \mathrm{S}_U(h) \in \mathscr{H}^\star(U)_b^+$. Now if $g \in \mathscr{H}_\star(U)^+$, $g \leq \mathrm{S}_U(p) - \mathrm{S}_U(h)$, then we have $g + \mathrm{S}_U(h) \prec \mathrm{S}_U(p)$ and $g + \mathrm{S}_U(h) \precsim \mathrm{S}_U(h)$. Hence $\mathrm{S}_U^{-1}(g + \mathrm{S}_U(h)) \prec_{\mathrm{S}} p$ and $\mathrm{S}_U^{-1}(g + \mathrm{S}_U(h)) \precsim_{\mathrm{S}} h$. Now since we had $p \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)_b$ this implies $\mathrm{S}_U^{-1}(g + \mathrm{S}_U(h)) \prec_{\mathrm{S}} h$ and hence $g + \mathrm{S}_U(h) \prec \mathrm{S}_U(h)$. So $g \prec 0$ and hence $\mathrm{S}_U(p) - \mathrm{S}_U(h) \in \mathscr{P}^\star(U)_b$.

Now take a $p \in \mathscr{P}^\star(U)_b$. Then evidently $p' = \mathrm{S}_U^{-1}(p + \mathrm{S}_U(h)) \succ_{\mathrm{S}} h$ and p' is bounded. Now if $g \precsim_{\mathrm{S}} h$ and $g \leq p'$, then we have $g \prec_{\mathrm{S}} p'$ and hence $\mathrm{S}_U(g) \precsim \mathrm{S}_U(h)$ and $\mathrm{S}_U(g) \prec p + \mathrm{S}_U(h)$. So $\mathrm{S}_U(g) - \mathrm{S}_U(h) \in \mathscr{H}_\star(h)$ and $\mathrm{S}_U(g) - \mathrm{S}_U(h) \leq p$. Hence $\mathrm{S}_U(g) - \mathrm{S}_U(h) \prec 0$ and so $\mathrm{S}_U(g) \prec \mathrm{S}_U(h)$. So $g \prec_{\mathrm{S}} h$ and hence $p' \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)_b$. □

Theorem 5.2. *If* $p, q \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)_b$ *and* $p \sim q$, *then* $p = q$.

Proof. Take f the smallest p-semilinear hyperharmonic majorant of h. Then $f \sim p \sim q$ and $f \leq p, q$ and hence also $f \in {}^{\mathrm{S}}\mathscr{P}^\star(h; U)_b$. So without loss of generality we may suppose $p \geq q$.

Now set $h' = \mathrm{S}_U^{-1}(\mathrm{S}_U(h) + \mathrm{S}_U(p) - \mathrm{S}_U(q))$. Then $h' \sim h$ and by 2.1 we have $h \leq h' \leq p$. So since $p \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ we get $h' = h$. But this implies $\mathrm{S}_U(p) = \mathrm{S}_U(q)$ and hence, again by 2.1, we get $p = q$. □

Let p, $q \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ and $\lambda > 0$.

1. If there is a $f \in {}^{\mathrm{S}}\mathscr{H}(p \tilde{+} q \tilde{-} h;U) \cap {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$, then we write $f = p \oplus q$.
2. If there is a $f \in {}^{\mathrm{S}}\mathscr{H}(\lambda \tilde{\cdot} (p \tilde{-} h) \tilde{+} h) \cap {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$, then we write $f = \lambda \odot p$.

Note that by 5.2, these definitions are unique and that it is not evident that $p \oplus q$ and $\lambda \odot p$ always exist. Note also that these definitions depend on U and h. It is easy to check that $p \oplus q = q \oplus p$, $(p \oplus q) \oplus k = p \oplus (q \oplus k)$, $p \oplus h = p$, $\lambda \odot (p \oplus q) = (\lambda \odot p) \oplus (\lambda \odot q)$, $(\lambda + \gamma) \odot p = (\lambda \odot p) \oplus (\gamma \odot p)$, $(\lambda\gamma) \odot p = \lambda \odot (\gamma \odot p)$ and $1 \odot p = p$. Furthermore, in all these equations, the left part exists if and only if the right part exists.

Proposition 5.3. *Let p, $q \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$, then the following statements are equivalent:*

1. $p \succ_{\mathrm{S}} q$;
2. $p \succsim_{\mathrm{S}} q$;
3. *There is a $k \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ with $p = q \oplus k$;*
4. $q \in {}^{\mathrm{S}}\mathscr{P}_\star(p;U)_b$;
5. $p \in {}^{\mathrm{S}}\mathscr{P}^\star(q;U)_b$.

If either of these is true, then the k mentioned in the third statement is unique and will be denoted by $k = p \ominus q$.

Proof. $4, 5 \Rightarrow 1 \Rightarrow 2$ is evident.

$2 \Rightarrow 3$: It is easy to check that $p \in {}^{\mathrm{S}}\mathscr{H}^\star(p \tilde{+} h \tilde{-} q;U)$ and $h \in {}^{\mathrm{S}}\mathscr{H}_\star(p \tilde{+} h \tilde{-} q;U)$. Now let k be the greatest $(p \tilde{+} h \tilde{-} q)$-semilinear hypoharmonic minorant of p. Then $k \in {}^{\mathrm{S}}\mathscr{H}(p \tilde{+} h \tilde{-} q;U)$ and $p \succ_{\mathrm{S}} k \succ_{\mathrm{S}} h$, hence $k \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$. Since $k \in {}^{\mathrm{S}}\mathscr{H}(p \tilde{+} h \tilde{-} q;U)$ is equivalent to $p \in {}^{\mathrm{S}}\mathscr{H}(q \tilde{+} k \tilde{-} h;U)$ we get $p = q \oplus k$. Now take a general $k' \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ with $p = q \oplus k'$. Then we must have $k' \in {}^{\mathrm{S}}\mathscr{H}(p \tilde{+} h \tilde{-} q;U)$ and hence $k' \sim k$ and so $k' = k$.

$3 \Rightarrow 2$: Since $p \in {}^{\mathrm{S}}\mathscr{H}(q \tilde{+} k \tilde{-} h;U)$ we have for all $V \Subset U$ that $\mathrm{S}_V(p) - \mathrm{S}_V(q) - \mathrm{S}_V(k) + \mathrm{S}_V(h) \in \mathscr{H}(V)$. Since also for all $V \Subset U$ we have $\mathrm{S}_V(k) - \mathrm{S}_V(h) \in \mathscr{H}^\star(V)$ we get $\mathrm{S}_V(p) - \mathrm{S}_V(q) \in \mathscr{H}^\star(V)$ for all $V \Subset U$ and hence $p \succsim_{\mathrm{S}} q$.

$2 \Rightarrow 1$: Let f be the greatest q-semilinear hypoharmonic minorant of p. Then $p \geq f \geq h$ and $f \sim q \succ_{\mathrm{S}} h$. So $f \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ and hence $f = q$. Hence $q \leq p$.

$1 \Rightarrow 4$: Let f be the smallest p-semilinear hyperharmonic majorant of q. Then $p \geq f \geq h$ and $f \sim p \succ_{\mathrm{S}} h$. So $f \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ and hence $f = p$. Hence $q \in {}^{\mathrm{S}}\mathscr{P}_\star(p;U)_b$.

$1 \Rightarrow 5$: Let f be the greatest q-semilinear hypoharmonic minorant of p. Then $p \geq f \geq h$ and $f \sim q \succ_{\mathrm{S}} h$. So $f \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ and hence $f = q$. Hence $p \in {}^{\mathrm{S}}\mathscr{P}^\star(q;U)_b$. □

Now it is easy to check that $u \oplus v = u \oplus w$ implies $v = w$ and that $p \oplus q = h$ implies $p = q = h$.

Lemma 5.4. *Let $\lambda > 0$ and $p \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$. Now if $\lambda \odot p$ exists, then for all γ with $0 < \gamma < \lambda$ we have that $\gamma \odot p$ exists.*

Proof. Take γ with $0 < \gamma < \lambda$ and define the bias $\varphi = \gamma \tilde{\cdot} (p \tilde{-} h) \tilde{+} h$. It is easy to check that $\lambda \odot p \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi;U)$ and $h \in {}^{\mathrm{S}}\mathscr{H}_\star(\varphi;U)$. Now let k be the greatest φ-semilinear hypoharmonic minorant of $\lambda \odot p$. Then $k \in {}^{\mathrm{S}}\mathscr{H}(\varphi;U)$ and $\lambda \odot p \succ_{\mathrm{S}} k \succ_{\mathrm{S}} h$, hence $k \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$. □

Theorem 5.5. *For all p, $q \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$, $p \oplus q$ exists.*

Proof. $p \oplus q = \mathrm{S}_U^{-1}(\mathrm{S}_U(p) + \mathrm{S}_U(q) - \mathrm{S}_U(h))$. □

Theorem 5.6. *The set ${}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$, equipped with $\oplus$, $\odot$ is a lower complete prevector lattice with 0-element h and specific order $\succ_{\mathrm{S}}$.*

Proof. From 5.4 and 5.5 we get that ${}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ is a convex cone with 0-element h. From 5.3 we get that the specific order on ${}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ is $\succ_{\mathrm{S}}$. From 4.3 we get that ${}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ is lower complete. The other properties are easy to check. □

For any numerical function f and any bias φ, ${}^{\mathrm{S}}\mathrm{C}(\varphi;f)$ denotes the smallest closed set K such that f is φ-semilinear harmonic outside K. Since ${}^{\mathrm{S}}\mathscr{H}(\varphi)$ is a sheaf, it is clear that ${}^{\mathrm{S}}\mathrm{C}(\varphi;f)$ always exists.

Theorem 5.7. *The map ${}^{\mathrm{S}}\mathrm{C}(h;.)$ is an abstract carrier on $\left({}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b, U\right)$.*

Proof. Evidently ${}^{\mathrm{S}}\mathrm{C}(h;p) = \emptyset$ if and only if $p = h$. Also evidently $p \prec_{\mathrm{S}} q$ implies ${}^{\mathrm{S}}\mathrm{C}(h;p) \subset {}^{\mathrm{S}}\mathrm{C}(h;q)$. Now let $p \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ and let F_1 and F_2 be two closed subsets of U with $F_1 \cup F_2 = U$. Define

$$p_1 = \curlyvee\left\{f \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b \mid f \prec_{\mathrm{S}} p, {}^{\mathrm{S}}\mathrm{C}(h;f) \subset F_1\right\}.$$

By 4.3 we have ${}^{\mathrm{S}}\mathrm{C}(h;p_1) \subset F_1$. Furthermore, $p \succ_{\mathrm{S}} p_1 \succ_{\mathrm{S}} h$ and hence $p_1 \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ and there is a $p_2 \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$ with $p = p_1 \oplus p_2$. Define $q = \inf\{f \mid f \succ_{\mathrm{S}} h$ and $f \geq p_2$ on $\mathrm{X} \setminus F_1\}$. Then $h \prec_{\mathrm{S}} q \leq p_2$ and hence $q \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$. Define $f = {}^{\mathrm{S}}R^\star\big(h;U, p_2 \ominus (p_2 \curlywedge_{\mathrm{S}} q)\big)$. Since $p_2 = q$ on $\mathrm{X} \setminus F_1$ we have $p_2 \ominus (p_2 \curlywedge_{\mathrm{S}} q) \sim h$ on $\mathrm{X} \setminus F_1$ and hence $f \sim h$ on $\mathrm{X} \setminus F_1$. Furthermore, since $p_2 \ominus (p_2 \curlywedge_{\mathrm{S}} q) \prec_{\mathrm{S}} p_2$ we have $h \prec_{\mathrm{S}} f \prec_{\mathrm{S}} p_2$. So now we have $p_1 \oplus f \prec_{\mathrm{S}} p_1 \oplus p_2 = p$ and ${}^{\mathrm{S}}\mathrm{C}(h;p_1 \oplus f) \subset F_1$. Hence by definition of p_1 we get $p_1 \oplus f \prec_{\mathrm{S}} p_1$ and hence $f = h$. Hence $h \prec_{\mathrm{S}} p_2 \ominus (p_2 \curlywedge_{\mathrm{S}} q) \leq h$ and so $p_2 = q$. Hence ${}^{\mathrm{S}}\mathrm{C}(h;p_2) \subset \overline{\mathrm{X} \setminus F_1} \subset F_2$. □

So, as in [6, p. 189], we can define the specific multiplication $f \odot p$ of a positive bounded continuous function f with $p \in {}^{\mathrm{S}}\mathscr{P}^\star(h;U)_b$.

6. Polar and semipolar sets

For simplicity, we will prove the results in this section only in the isotone case. However, using 3.8, it is easy to check that all results concerning polar sets, thinness and semipolar sets also hold in the general case.

Balayage. Let $U \subset \mathrm{X}$ be an open set. For any $A \subset \mathrm{X}$, any $f \in \mathrm{C}_f(U)$ and any $g \in {}^{\mathrm{S}}\mathscr{H}^\star(f;U)$ with $g \geq f$ we define ${}^{\mathrm{S}}\mathrm{B}^A(U,f;g) = \wedge\{h \in {}^{\mathrm{S}}\mathscr{H}^\star(f;U) \mid h \geq g \text{ on } A, h \geq f\}$. Furthermore, for any $A \subset \mathrm{X}$ and any $g \in \mathscr{H}^\star(U)^+$ we define $\mathrm{B}^A(U;g) = \wedge\{h \in \mathscr{H}^\star(U)^+ \mid h \geq g \text{ on } A\}$.

Lemma 6.1. *Let $U \subset \mathrm{X}$ be open, φ a bias, $f \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi,U)$ and $g \in \mathscr{H}^\star(U)^+$, then $f+g \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi;U)$. Furthermore, if $f \in {}^{\mathrm{S}}\mathscr{S}^\star(\varphi,U)$ and $g \in \mathscr{S}^\star(U)^+$, then $f+g \in {}^{\mathrm{S}}\mathscr{S}^\star(f;U)$.*

Proof. Take $V \Subset W \Subset U$, $f_n = (f-\psi)\wedge n + \psi$ and $g_n = g \wedge n$. Then

$$\begin{aligned}
{}^{\mathrm{S}}\mathrm{H}(\varphi;V,f_n+g_n) &= \mathrm{H}\big(V, f_n+g_n - \mathrm{S}_W(\varphi(W))\big)\\
&\quad + \mathrm{S}_W\big(\varphi(W)\big) - K_V\big({}^{\mathrm{S}}\mathrm{H}(\varphi;V,f_n+g_n)\big)\\
&\leq \mathrm{H}(V,g_n) + \mathrm{H}\big(V, g_n - \mathrm{S}_W(\varphi(W))\big)\\
&\quad + \mathrm{S}_W\big(\varphi(W)\big) - K_V\big({}^{\mathrm{S}}\mathrm{H}(\varphi;V,f_n+g_n)\big)\\
&= \mathrm{H}(V,g_n) + {}^{\mathrm{S}}\mathrm{H}(\varphi;V,f_n) \leq \underline{\mathrm{H}}(V,g) + {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi;V,f).
\end{aligned}$$

Since $f_n + g_n - \psi \in \mathrm{LSC}(\partial V)$ we get

$${}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi;V,f+g) = \sup_n {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi;V,f_n+g_n) \leq \underline{\mathrm{H}}(V,g) + {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi;V,f). \ \square$$

Lemma 6.2. *If $U \subset \mathrm{X}$ is open, $f \in \mathrm{C}_f(U)$ and $g \in \mathscr{H}^\star(U)^+$, then $\mathrm{B}^A(U;g) + f \geq {}^{\mathrm{S}}\mathrm{B}^A(U,f;f+g)$.*

Proof. We have by 6.1 that

$$\begin{aligned}
\mathrm{B}^A(U;g) + f &= \wedge\{h+f \mid h \in \mathscr{H}^\star(U)^+, h \geq g \text{ on } A\}\\
&\geq \wedge\{h \in {}^{\mathrm{S}}\mathscr{H}^\star(f;U) \mid h \geq f+g \text{ on } A, h \geq f\}\\
&= {}^{\mathrm{S}}\mathrm{B}^A(U,f;f+g). \ \square
\end{aligned}$$

Lemma 6.3. *Let $U \in \mathscr{U}$, $f \in \mathrm{C}_f(U)_b$ and $g \in \mathscr{H}^\star(U)^+_b$, then*

$$\mathrm{S}_U\big({}^{\mathrm{S}}\mathrm{B}^A(U,f;f+g)\big) \geq \mathrm{B}^A(U;g) + \mathrm{S}_U(f).$$

Proof. Define $F = \{h \in {}^{\mathrm{S}}\mathscr{H}^\star(f;U)_b \mid h \geq f+g \text{ on } A, h \geq f\}$. Now for any $h \in F$ we have both $\mathrm{S}_U(h) \succ \mathrm{S}_U(f)$ and $\mathrm{S}_U(h) = h + K_U(h) \geq h + K_U(f) \geq g + f + K_U(f) = g + \mathrm{S}_U(f)$ on A. Hence $\mathrm{S}_U(h) - \mathrm{S}_U(f) \geq \mathrm{B}^A(U;g)$ and hence $\mathrm{S}_U(F) \geq \mathrm{B}^A(U;g) + \mathrm{S}_U(f)$. Now by 2.3 we get $\mathrm{S}_U\big({}^{\mathrm{S}}\mathrm{B}(U,f;f+g)\big) = \mathrm{S}_U(\wedge F) = \wedge \mathrm{S}_U(F) \geq \mathrm{B}^A(U;g) + \mathrm{S}_U(f)$. $\square$

Lemma 6.4. *Let* $U \in \mathscr{U}$ *and* $f, g \in \mathrm{C}_f(U)_b$ *with* $f \prec_{\mathrm{S}} g$, *then*

$$\mathrm{B}^A\big(U; \mathrm{S}_U(g) - \mathrm{S}_U(f)\big) \leq {}^{\mathrm{S}}\mathrm{B}^A(U, f; g) + K_U(g) - \mathrm{S}_U(f).$$

Proof. Take $h \in \mathrm{C}_f(U)$ with $f \prec_{\mathrm{S}} h \leq g$ and $h = g$ on A. Then

$$h + K_U(g) - \mathrm{S}_U(f) = \mathrm{S}_U(h) - \mathrm{S}_U(f) + K_U(g) - K_U(h) \succ \mathrm{S}_U(h) - \mathrm{S}_U(f) \succ 0$$

and $h + K_U(g) - \mathrm{S}_U(f) = \mathrm{S}_U(g) - \mathrm{S}_U(f)$ on A. Hence $\mathrm{B}^A\big(U; \mathrm{S}_U(g) - \mathrm{S}_U(f)\big) \leq h + K_U(g) - \mathrm{S}_U(f)$ and since h was arbitrary we get $\mathrm{B}^A\big(U; \mathrm{S}_U(g) - \mathrm{S}_U(f)\big) \leq {}^{\mathrm{S}}\mathrm{B}^A(U, f; g) + K_U(g) - \mathrm{S}_U(f)$. □

Polar sets. As in [6], we say that a set $A \subset \mathrm{X}$ is polar if there is a covering of X with open sets U such that $\mathrm{B}^A(U; \infty) = 0$. Note that this is only one of several equivalent definitions of polarity. As we will see, many of these equivalent definitions have their semilinear counterpart.

Proposition 6.5. *The following statements are equivalent:*

1. *A is a polar set;*
2. *For all $U \in \mathscr{U}$ and $f \in \mathrm{C}_f(U)$ we have ${}^{\mathrm{S}}\mathrm{B}^A(U, f; \infty) = f$;*
3. *There is a covering of sets U such that for all U, there are $f \in \mathrm{C}_f(U)$ and $g \in {}^{\mathrm{S}}\mathscr{H}^\star(f; U)$ with $g > f$ and ${}^{\mathrm{S}}\mathrm{B}^A(U, f; g) = f$.*

Proof. $1 \Rightarrow 2$: By 6.2 we get $f = \mathrm{B}^A(U; \infty) + f \geq {}^{\mathrm{S}}\mathrm{B}^A(U, f; \infty) \geq f$.

$2 \Rightarrow 3$ is evident.

$3 \Rightarrow 1$: Take $x \in \mathrm{X}$ and a neighbourhood V of x with $V \Subset U$ for some U in the covering. Now $g \geq f + \varepsilon$ on V for some $\varepsilon > 0$ and we can take a $h \in \mathscr{H}^\star(V)_b^+$ with $\varepsilon > h > 0$. Furthermore, f is bounded on V. Now evidently $f = {}^{\mathrm{S}}\mathrm{B}^A(U, f; g) \geq {}^{\mathrm{S}}\mathrm{B}^A(V, f; g) \geq {}^{\mathrm{S}}\mathrm{B}(V, f; f + h) \geq f$. So by 6.3 we get

$$\mathrm{S}_V(f) = \mathrm{S}_V\big({}^{\mathrm{S}}\mathrm{B}(V, f; f + h)\big) \geq \mathrm{B}^A(V; h) + \mathrm{S}_V(f)$$

and hence $\mathrm{B}^A(V; h) = 0$. □

Proposition 6.6. *If A is polar and $f \in \mathrm{C}_f(U)$, then there is a $g \in {}^{\mathrm{S}}\mathscr{S}^\star(f; U)$ with $g \geq f$ and $A \subset \{g = \infty\}$.*

Proof. From 6.1 we get that for all $g \in \mathscr{S}^\star(U)^+$ there is a $h \in {}^{\mathrm{S}}\mathscr{S}^\star(f; U)$ with $\{g = \infty\} = \{h = \infty\}$. □

Proposition 6.7. *Let A be a closed polar set, $U \subset \mathrm{X}$ open, φ a bias and $f \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U \setminus A)$ such that $\hat{f}$ is lower finite on U. Then there is a $g \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; U)$ with $g = f$ on $U \setminus A$.*

Proof. Define $g = \widehat{(f - \psi)} + \psi$ and take $V \Subset U$ and $v \in \mathscr{H}^\star(V)^+$ with $v = \infty$ on A. Define $h = \infty$ on A and $h = \infty \wedge (f + v) = f + v$ on $V \setminus A$. Then $h - \psi \in \mathrm{LSC}(V)$ and hence $h \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; V)$. Furthermore $\widehat{h - \psi} \geq \widehat{g - \psi} \geq g - \psi$ on ∂V and hence $h \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi; V, g)$. Since $\inf\{v \in \mathscr{H}^\star(V)^+ \mid v = \infty \text{ on } A\} = 0$ on $V \setminus A$ we get $f \geq {}^{\mathrm{S}}\overline{\mathrm{H}}(\varphi; V, g) \geq {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, g)$ on $V \setminus A$. Since ${}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, g) - \psi \in \mathrm{LSC}(V)$ this implies $g \geq {}^{\mathrm{S}}\underline{\mathrm{H}}(\varphi; V, g)$ on V. □

Proposition 6.8. *Let A be a closed polar set, $U \subset \mathrm{X}$ open, φ a bias and $f \in {}^{\mathrm{S}}\mathscr{H}(\varphi; U \setminus A)$ such that $\hat{f}$ is lower finite and $\check{f}$ is upper finite on U. Then there is a $g \in {}^{\mathrm{S}}\mathscr{H}(\varphi; U)$ with $g = f$ on $U \setminus A$.*

Proof. Define $g^\star = \widehat{(f - \psi)} + \psi$ and $g_\star$ dually. For any $V \Subset U$ we have by 6.7,

$$ {}^{\mathrm{S}}\mathrm{H}(\varphi; V, g^\star) \leq g^\star \leq g_\star \leq {}^{\mathrm{S}}\mathrm{H}(\varphi; V, g_\star). $$

But since $g^\star$ and $g_\star$ are bounded functions on ∂V, equal on $(\partial V) \setminus A$ we have $\mathrm{H}(V, g^\star) = \mathrm{H}(V, g_\star)$ and hence ${}^{\mathrm{S}}\mathrm{H}(\varphi; V, g^\star) = {}^{\mathrm{S}}\mathrm{H}(\varphi; V, g_\star)$. □

Thinness and semipolar sets. As in [6], we say that a set $A \subset \mathrm{X}$ is thin at $x \in \mathrm{X}$ if there are two open neighbourhoods U, V of x and a $u \in \mathscr{H}^\star(U)^+$ such that $V \subset U$ and $\mathrm{B}^{A \cap U}(U; u)(x) < u(x)$. Furthermore, A is called totally thin if it is thin at every $x \in \mathrm{X}$ and it is called semipolar if it is a countable union of totally thin sets. As with polarity, there are several equivalent definitions of thinness and semipolarity.

Proposition 6.9. *The following statements are equivalent:*

1. *A is thin at x;*
2. *For all neighbourhoods $U \in \mathscr{U}$ of x, all $f \in \mathrm{C}_f(U)$ and all $g \in {}^{\mathrm{S}}\mathscr{H}^\star(f; U)$ with $f \leq g$ and $g - f$ continuous and strictly positive in x, there is a neighbourhood $V \subset U$ of x such that ${}^{\mathrm{S}}\mathrm{B}^{A \cap V}(U, f; g)(x) < g(x)$;*
3. *For all neighbourhoods $U \in \mathscr{U}$ of x and all $f \in \mathrm{C}_f(U)$, there is a finite $g \in {}^{\mathrm{S}}\mathscr{P}^\star(f; U)$ with $g - f$ continuous and ${}^{\mathrm{S}}\mathrm{C}(f; g)$ compact, such that ${}^{\mathrm{S}}\mathrm{B}^{A}(U, f; g)(x) < g(x)$;*
4. *There are neighbourhoods U, V of x, $V \subset U$, $f \in \mathrm{C}_f(U)$ and $g \in {}^{\mathrm{S}}\mathscr{H}^\star(f; U)$ with $f \leq g$, such that ${}^{\mathrm{S}}\mathrm{B}^{A \cap V}(U, f; g)(x) < g(x)$.*

Proof. $1 \Rightarrow 2$: There is a $v \in \mathscr{H}^\star(U)^+ \cap \mathrm{C}(U)$ with $\mathrm{B}^A(U; v)(x) < v(x)$. Take $\beta > 0$ such that $\beta \mathrm{B}^A(U; v)(x) < (f - g)(x) < \beta v(x)$ and take a neighbourhood $V \subset U$ of x such that $f - g < \beta v$ on V. Then by 6.2 we get

$$ {}^{\mathrm{S}}\mathrm{B}^{A \cap V}(U, f; g)(x) \leq {}^{\mathrm{S}}\mathrm{B}^{A \cap V}(U, f; f + \beta v) \leq f(x) + \beta \mathrm{B}^{A \cap V}(U; v)(x) < g(x). $$

$1 \Rightarrow 3$: There is a $v \in \mathscr{H}^\star(U)^+ \cap \mathrm{C}(U)$ with $\mathrm{B}^A(U; v)(x) < v(x)$. Hence

$$ {}^{\mathrm{S}}\mathrm{B}^{A}(U, f; f + v)(x) \leq \mathrm{B}^A(U; v)(x) + f < (f + v)(x). $$

Take $g \in {}^{\mathrm{S}}\mathscr{P}^\star(f;U)$ such that $g-f$ is continuous, ${}^{\mathrm{S}}\mathrm{C}(f;g)$ is compact, $g \leq f+v$ and ${}^{\mathrm{S}}\mathrm{B}^A(U,f;f+v)(x) < g(x)$. Now we have

$$ {}^{\mathrm{S}}\mathrm{B}^A(U,f;g)(x) \leq {}^{\mathrm{S}}\mathrm{B}^A(u,f;f+v)(x) < g(x). $$

$2,3 \Rightarrow 4$ is evident.

$4 \Rightarrow 1$: Take a neighbourhood $W \Subset U \cap V$ of x and note

$$ {}^{\mathrm{S}}\mathrm{B}^A(W,f;g)(x) \leq {}^{\mathrm{S}}\mathrm{B}^{A\cap V}(U,f,g)(x) < g(x). $$

Furthermore f bounded on W and there is a $g' \in {}^{\mathrm{S}}\mathscr{H}^\star(f;W)_b$ with $g \geq g' \geq f$ and $g(x) > g'(x) > {}^{\mathrm{S}}\mathrm{B}^{A\cap V}(U,f;g)(x)$. Hence $g'(x) > {}^{\mathrm{S}}\mathrm{B}^A(W,f;g')(x)$ and so by 6.4 we get

$$ \begin{aligned} \mathrm{B}^A\big(W;\mathrm{S}_W(g')-\mathrm{S}_W(f)\big)(x) &\leq {}^{\mathrm{S}}\mathrm{B}^A(W,f;g')(x) + K_W(g')(x) - \mathrm{S}_W(f)(x) \\ &< \mathrm{S}_W(g')(x) - \mathrm{S}_W(f)(x). \ \square \end{aligned} $$

Proposition 6.10. *The following statements are equivalent:*

1. *A is a semipolar set;*
2. *For all $U \in \mathscr{U}$ and all $f \in \mathrm{C}_f(U)$, there is a locally lower bounded $F \subset {}^{\mathrm{S}}\mathscr{H}^\star(f;U)$ with $A \cap U = \{\wedge F < \inf F\}$;*
3. *There is a covering of open sets U such that for all U there is a bias φ and a $F \subset {}^{\mathrm{S}}\mathscr{H}^\star(\varphi;U)$ locally lower bounded with $A \cap U = \{\wedge F < \inf F\}$.*

Proof. $1 \Rightarrow 2$: Let $U \in \mathscr{U}$ and $f \in \mathrm{C}_f(U)$. There is a $F \subset \mathscr{H}^\star(U)^+$ with $A \cap U = \{\wedge F < \inf F\}$. By 6.1 we have that $F + f \in {}^{\mathrm{S}}\mathscr{H}^\star(f;U)$.

$2 \Rightarrow 3$ is evident.

$3 \Rightarrow 1$: Take $x \in \mathrm{X}$ and a neighbourhood $V \Subset U$ for some U in the covering. Let φ be the bias corresponding to U. Take $V \Subset U$ and $h \in \mathscr{H}^\star(V)_b$ with $h \geq 1$. Define $f_n = \mathrm{S}_V^{-1}\big(nh + \mathrm{S}_v(\varphi(V))\big)$ and $F_n = F \wedge f_n$ on V. Then $G_n = \mathrm{S}_V(F_n) - \mathrm{S}_V(g) \subset \mathscr{H}^\star(V)$ is and hence $\{\wedge G_n < \inf G_n\}$ is semipolar. But by 2.3 we have $\wedge G_n = \wedge F_n + K_U(\wedge F_n)$ and $\inf G_n = inf F_n + K_U(\wedge F_n)$ and hence $\{\wedge G_n < \inf G_n\} = \{\wedge F_n < \inf F_n\}$. So $\{\wedge F < \inf F_n\}$ and hence also $A \cap V = \{\wedge F < \inf F\} = \cup_n\{\wedge F < \inf F_n\}$ is semipolar. So for every $x \in \mathrm{X}$, there is a neighbourhood $V_x \in \mathscr{U}$ of x with $A \cap V_x$ semipolar. But there is a countable covering of such V_x and hence A is semipolar. □

Proposition 6.11. *Let φ be a bias and $f, g \in {}^{\mathrm{S}}\mathscr{H}^\star(\varphi;\mathrm{X})$. If f and g coincide outside a semipolar set, then $f = g$.*

Proof. A semipolar set is of the first category in the fine topology, the fine topology is a Baire space and f and g are fine continuous. □

7. Appendix: Isotone perturbations

In this section $\mathscr{U}$ is the collection of all relatively compact sets for which the closure is contained in a $\mathscr{P}$-set.

Lemma 7.1. *Let $U, V \in \mathscr{U}$ and let $(f_n) \subset \mathscr{B}(U \cup V)_b$ be a bounded increasing sequence with $\sup_n f_n = f$. For $W \in \{U, V\}$, suppose that:*

1. *$\mathscr{R}(W)$ is a sublattice of $\mathscr{B}(W)_b$ containing the constants;*
2. *$(f_n) \subset \mathscr{R}(W)$;*
3. *$K_W\colon \mathscr{R}(W) \to \mathscr{P}(W) - \mathscr{P}(W)$ is a map with the following property:*
   $(*)$ for all $M > 0$, there exist $p, q \in \mathscr{P}(W)$,
   such that for all $f \geq g$, $|f| \leq M$, $|g| \leq M$,
   we have: $(f - g) \cdot p \succ K_W(h) - K_W(g) \succ (g - h) \cdot q$.

Then there exist unique extensions of K_W that have the same property on the sublattice of $\mathscr{B}(W)_b$ generated by $\mathscr{R}(W)$ and f. The p and q needed for the extensions are the same as for the original map. Furthermore, if

$$(**) \qquad K_U(h) - K_V(h) \in \mathscr{H}(U \cap V) \text{ for all } h \in \mathscr{R}(U) \cap \mathscr{R}(V),$$

then the extensions have the same property.

Proof. Let $W \in \{U, V\}$. Let h be in the sublattice of $\mathscr{B}(W)_b$ generated by $\mathscr{R}(W)$ and f.

If $h = f$, then set $h_n = f_n \vee (-\|h\|)$.

If $h \in \mathscr{R}(W)$, then set $h_n = h$.

If $h = f \vee g$ for some $g \in \mathscr{R}(U)$, then set $h_n = (f_n \vee g) \vee (-\|h\|)$.

If $h = f \wedge g$ for some $g \in \mathscr{R}(U)$, then set $h_n = (f_n \wedge g) \vee (-\|h\|)$.

Since $\mathscr{B}(W)_b$ is distributive, one of these four cases is always true. Note that $h_n \uparrow h$ and $\|h_n\| \leq \|h\|$.

(Uniqueness) Take any extension of K_W to $\mathscr{R}(W) \cup \{h\}$ with $(*)$ and denote it again by K_W. Let p, q as in $(*)$ with $M = \|h\|$. Then for all m we have $(h - h_m) \cdot p \succ K_W(h) - K_W(h_n) \succ (h_m - h) \cdot q$. Since $(h - h_m) \cdot p \to 0$ and $(h_m - h) \cdot q \to 0$ we get $K_W(h_n) \to K_W(h)$. So $K_W(h)$ is uniquely determined.

(Existence) Let p, q as in $(*)$ with $M = \|h\|$. For all $n \geq m$ we have

$$(h - h_m) \cdot p \succ K_W(h_n) - K_W(h_m) \succ (h_m - h) \cdot q.$$

Since $(h - h_m) \cdot p \downarrow 0$ and $(h - h_m) \cdot q \downarrow 0$ we get that $K_W(h_n)$ is a Cauchy-sequence. So we can define $K_W(h)$ as the limit of this sequence.

(∗): Since

$$K_W(h) - K_W(h_m) = \sum_{n=m}^{\infty} \big(K_W(h_{n+1}) - K_W(h_n)\big)$$
$$= \sum_{n=m}^{\infty} \big(K_W(h_{n+1}) - K_W(h_n)\big) \curlyvee 0$$
$$+ \sum_{n=m}^{\infty} \big(K_W(h_{n+1}) - K_W(h_n)\big) \curlywedge 0$$

and since for every n we have (with p and q as in (∗) with $M = \|h\|$)

$$(h_{n+1} - h_n) \cdot p \succ \big(K_W(h_{n+1}) - K_W(h_n)\big) \curlyvee 0$$
$$(h_n - h_{n+1}) \cdot q \prec \big(K_W(h_{n+1}) - K_W(h_n)\big) \curlywedge 0$$

we get that

$$(h - h_m) \cdot p \succ K_W(h) - K_W(h_m) \succ (h_m - h) \cdot q.$$

Now suppose $h^1 \geq h^2$ and let p, q as in (∗) (on $\mathscr{R}(W)$) with $M = \|h^1\| \vee \|h^2\|$. For all n we have

$$(h^1 - h^1_n) \cdot p \succ \quad K_W(h^1) - K_W(h^1_n) \quad \succ (h^1_n - h^1) \cdot q$$
$$(h^1_n \vee h^2_n - h^1_n) \cdot q \succ K_W(h^1_n) - K_W(h^1_n \vee h^2_n) \succ (h^1_n - h^1_n \vee h^2_n) \cdot p$$
$$(h^1_n \vee h^2_n - h^2_n) \cdot p \succ K_W(h^1_n \vee h^2_n) - K_W(h^2_n) \succ (h^2_n - h^1_n \vee h^2_n) \cdot q$$
$$(h^2 - h^2_n) \cdot q \succ \quad K_W(h^2_n) - K_W(h^2) \quad \succ (h^2_n - h^2) \cdot p.$$

By adding these inequalities and taking the limit $n \to \infty$ we get

$$(h^1 - h^2) \cdot p \succ K_W(h^1) - K_W(h^2) \succ (h^2 - h^1) \cdot q.$$

(∗∗): Take W resolutive in $U \cap V$. Since $K_U(h_n) \to K_U(h)$ uniformly on ∂W, we have that $\mathrm{H}\big(W, K_U(h_n)\big) \to \mathrm{H}\big(W, K_U(h)\big)$. Similarly $K_V(h_n) \to K_V(h)$ and $\mathrm{H}\big(W, K_V(h_n)\big) \to \mathrm{H}\big(W, K_V(h)\big)$. Hence

$$\mathrm{H}\big(W, K_U(h) - K_V(h)\big) = \lim_n \mathrm{H}\big(W, K_U(h_n) - K_V(h_n)\big)$$
$$= \lim_n (K_U(h_n) - K_V(h_n)) = K_U(h) - K_V(h). \quad \square$$

Proposition 7.2. *Let U, $V \in \mathscr{U}$. For $W \in \{U, V\}$, suppose that:*

1. *$\mathscr{R}(W)$ is a linear sublattice of $\mathscr{B}(W)_b$ containing the constants and dense in $\mathrm{C}(W)_b$;*
2. *$K_W \colon \mathscr{R}(W) \to \mathscr{P}(W) - \mathscr{P}(W)$ is a map with (∗).*

Then there exist unique extensions of K_W to $\mathscr{B}(W)_b$ that have the same property. The p and q needed for the extensions are the same as for the original map. Furthermore, if K_U and K_V have property (∗∗), then so do the extensions.

Proof. (Existence) Let $W \in \{U, V\}$. Consider the space of pairs $(\mathscr{R}, K)$, where K is a map with the desired properties on a linear sublattice $\mathscr{R}$ of $\mathscr{B}(W)_b$. Define an order on this space by $(\mathscr{R}, K) \geq (\mathscr{R}', K')$ if and only if $\mathscr{R} \supset \mathscr{R}'$ and $K = K'$ on $\mathscr{R}'$.

(Uniqueness) Let $W \in \{U, V\}$ and K_1 and K_2 be two extensions of K_W to $\mathscr{B}(W)_b$. Consider the space of linear sublattices $\mathscr{R}$ of $\mathscr{B}(W)_b$ on which K_1 and K_2 coincide. Define an order on this space by inverse inclusion.

(**): Let $W = U \cup V$. Consider the space of linear sublattices $\mathscr{R}$ of $\mathscr{B}(W)_b$ on which the extensions satisfy (**). Define an order on this space by inverse inclusion.

In each of these cases, it is easy to check that the space is inductively ordered. Hence, by Zorn's lemma, there exists a maximal element. Using 7.1 we show that the $\mathscr{R}$ of this maximal element is closed for increasing sequences. Hence $\mathscr{R} = \mathscr{B}(W)_b$. □

Now we must introduce some notions and notations of [17]. $\mathscr{R}$ is the same sheaf as we defined at the end of Section 4, i.e. it is the sheaf of local differences of bounded continuous superharmonic functions. Furthermore, it is assumed that $1 \in \mathscr{R}$ and that there is a (linear) sheaf homomorphism σ of $\mathscr{R}$ into the sheaf $\mathscr{M}$ of signed Radon-measures on X such that $\sigma(f) \geq 0$ if and only if $f \in \mathscr{S}^* \cap \mathrm{C}$. σ is called a measure representation. Furthermore, $\mathscr{M}_\sigma$ is the sheaf Radon-measures that are local images of σ and for any $U \in \mathscr{U}$ we set $\mathscr{M}_{BC}(U) = \sigma_U(\mathscr{P}(U) - \mathscr{P}(U))$.

A semilinear perturbation in the sense of Maeda is defined as a sheaf morphism F of $\mathscr{R}$ to $\mathscr{M}_\sigma$ such that for all $U \in \mathscr{U}$ we have

1. $F(0) \in \mathscr{M}_{BC}(U)$;
2. for all $M > 0$, there exists a $\mu \in \mathscr{M}_{BC}(U)$
   such that for all $f \geq g$, $|f| \leq M$, $|g| \leq M$
   we have: $(f - g)\mu \geq F_U(f) - F_U(g) \geq (g - f)\mu$.

Lemma 7.3. *Let F be a semilinear perturbation in the sense of Maeda. For $U \in \mathscr{U}$ there is a unique map K_U: $\mathscr{R}(U)_b \to \mathscr{P}(U) - \mathscr{P}(U)$ such that for all $f \in \mathscr{R}(U)_b$ we have: $K_U(f) = q \Leftrightarrow \sigma(q) = F(f)$. This map has the following properties:*

1. *For all $M > 0$, there exist a $p \in \mathscr{P}(U)$,*
   such that for all $f \geq g$, $|f| \leq M$, $|g| \leq M$,
   we have: $(f - g) \cdot p \succ K_U(f) - K_U(g) \succ 0$.
2. *If $V \Subset U$, then $K_U(f) - K_V(f) \in \mathscr{H}(V)$.*

(See [7, p. 18] for first appearance of K_U in this context.)

Proof. Take $f \in \mathscr{R}(U)_b$. From [17, Lemma 2.1] we know there is a $q \in \mathscr{P}(U) - \mathscr{P}(U)$ such that $F(f) = \sigma(q)$ on U. Now suppose there is a $q' \in \mathscr{P}(U) - \mathscr{P}(U)$ with the same property. Then $\sigma(q - q') = 0$ and hence $q - q' \in \mathscr{H}(U) \cap \mathscr{P}(U) - \mathscr{P}(U)$. Hence $q = q'$ and K_U is well defined on $\mathscr{R}(U)$.

1: Take $M > 0$. Then there is a $p \in \mathscr{P}(U)$ such that for all $f, g \in \mathscr{R}(U)$ with $-M \leq f \leq g \leq M$ we have $0 \leq F(f) - F(g) \leq (f-g)\sigma(p)$. Hence

$$\sigma(0) \leq \sigma\big(K_U(f) - K_U(g)\big) \leq \sigma\big((f-g)\cdot p\big)$$

and so we get $0 \prec K_U(f) - K_U(g) \prec (f-g)\cdot p$.

2: $\sigma_V\big(K_U(f)\big) = \sigma_U\big(K_U(f)\big) = F_U(f) = F_V(f) = \sigma_V\big(K_V(f)\big)$. □

Lemma 7.4. *Suppose for all $U \in \mathscr{U}$ there is a map $K_U\colon \mathscr{R}(U)_b \to \mathscr{P}(U) - \mathscr{P}(U)$ such that*

1. *For all $M > 0$, there exist a $p \in \mathscr{P}(U)$,*
   such that for all $f \geq g$, $|f| \leq M$, $|g| \leq M$,
   we have: $(f-g)\cdot p \succ K_U(f) - K_U(g) \succ 0$.
2. *If $V \Subset U$, then $K_U(f) - K_V(f) \in \mathscr{H}(V)$.*

Then $F_U(f) = \sigma_U\big(K_U(f)\big)$ defines a semilinear perturbation in the sense of Maeda.

Proof. Let $V \Subset U \in \mathscr{U}$ and $f \in \mathscr{R}(U)_b$. Then since $K_U(f) - K_V(f) \in \mathscr{H}(V)$ we have

$$F_V(f) = \sigma_V\big(K_V(f)\big) = \sigma_V\big(K_U(f)\big) = \sigma_U\big(K_U(f)\big) = F_U(f)$$

on V. So F is a sheaf morphism on $\mathscr{R}$.

Let $U \in \mathscr{U}$. Then evidently $F(0) \in \mathscr{M}_{BC}(U)$. Now take $f, g \in \mathscr{R}(U)$ with $f \geq g$ and $|f|, |g| \leq M$. Then

$$\sigma_U\big((f-g)\cdot p\big) \geq \sigma_U\big(K_U(f) - K_U(g)\big) \geq \sigma_U(0).$$

Hence $(f-g)\sigma_U(p) \geq F_U(f) - F_U(g) \geq 0$. Since p only depends on M (and U), F is a semilinear perturbation in the sense of Maeda. □

Theorem 7.5. *Every positive semilinear perturbation is a perturbation in the sense of Maeda. Every perturbation in the sense of Maeda can be uniquely extended to a positive semilinear perturbation.*

The proof follows from 7.3, 7.4 and 7.2. □

Now it is easy to check that the sheaf of hyperharmonic functions considered in [17] is just ${}^{S}\mathscr{H}^{\star}(\widetilde{0}) \cap \mathrm{C}$.

References

[1] Bertin, E.M.J.: Axiomatic convex potential theory. - Preprint 607, University of Utrecht, 1990.

[2] Boukricha, A.: Das Picard-Prinzip und verwandte Fragen bei Störung von harmonischen Räumen. - Math. Ann. 239, 1979, 247–270.

[3] Boukricha, A., W. Hansen, and H. Hueber: Continuous solutions of the generalized Schrödinger equation and perturbation of harmonic spaces. - Exposition. Math. 5, 1987, 97–135.

[4] Constantinescu, C.: An axiomatic theory for the non-linear Dirichlet problem. - Rev. Roumaine Math. Pures Appl. 10, 1965, 755–764.

[5] Constantinescu, C.: Le problème de Dirichlet pour des systèmes d'équations non-linéaires. - In: Séminaire de théorie du potentiel. Paris, 1972–1974. Lecture Notes in Math. 518, 1976, 54–73.

[6] Constantinescu, C., and A. Cornea: Potential theory on harmonic spaces. - Springer-Verlag, Berlin, 1972.

[7] Donker, J.: Semi-linear potential theory. - Graduate thesis, Utrecht, 1990.

[8] van Gool, F.: Non-linear potential theory. - Preprint 606, University of Utrecht, 1990.

[9] van Gool, F.: On axiomatic non-linear potential theory. - In: Potential theory. Proceedings. Nagoya, 1990. Walter de Gruyter & Co., Berlin, 1991, 332–343.

[10] Hansen, W.: Cohomology in harmonic spaces. - In: Seminar on potential theory II. Lecture Notes in Math. 226, 1971, 63–101.

[11] Hansen, W.: Perturbation of harmonic spaces and construction of semigroups. - Invent. Math. 19, 1973, 149–164.

[12] Keuntje, J-M.: Störung lokaler Familien harmonischer Randkerne und Anwendung der Störung von Bauer-Räumen auf die verallgemeinerte Schrödinger-Gleichung. - Diplomarbeit, Bielefeld, 1988.

[13] Laine, I.: Introduction to a quasi-linear potential theory. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 339–348.

[14] Laine, I.: Axiomatic non-linear potential theories. - In: Potential theory. Surveys and problems. Prague, 1987. Lecture Notes in Math. 1344, 1988, 118–132.

[15] Lehtola, P.: An axiomatic approach to non-linear potential theory. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 62, 1986, 1–40.

[16] Lehtola, P.: The obstacle problem in a non-linear potential theory. - In: Potential theory, Proceedings, Prague, 1987. Plenum Press, New York, 1988, 209–213.

[17] Maeda, F.-Y.: Semi-linear perturbation of harmonic spaces. - Hokkaido Math. J. 10, 1981, 464–493.

[18] Walsh, B.: Perturbation of harmonic structures and an index-zero theorem. - Ann. Inst. Fourier 20, 1970, 317–359.

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