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CYCLIC PARABOLIC QUASICONFORMAL GROUPS THAT ARE NOT QUASICONFORMAL CONJUGATES OF MÖBIUS GROUPS

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Abstract. P. Tukia published in [T2] the first example of a uniformly quasi-isometric and hence quasiconformal group acting on R^n, $n \geq 3$, which is not a quasiconformal conjugate of any Möbius group. We have analyzed this group and have shown that it contains elements which generate cyclic parabolic uniformly quasi-isometric groups that cannot be conjugated by a quasiconformal map to a Möbius group. By an argument of Martin these groups can be chosen smooth.

Since these groups also act on the upper half-space U^n, we can use our result to give a negative answer to a conjecture of Martin and Tukia, where the hope has been that every three-dimensional quasiconformal Fuchsian group, which are groups of quasiconformal homeomorphisms of U^3, was quasiconformally conjugated to a Fuchsian group.

1. Introduction

By a quasiconformal group we mean a group of homeomorphisms of S^n, $n \geq 2$, which is uniformly K-quasiconformal. It was asked by Gehring and Palka [GP] if such a group is always quasiconformally conjugated to a group of Möbius transformations.

If $n = 2$, Sullivan [S] and Tukia [T1] independently gave an affirmative answer to the question. Their proof depends heavily on the Ahlfors–Bers or measurable Riemann mapping theorem, for which no analogue seems to exist in higher dimensions.

Later Pekka Tukia gave in [T2] the first example of a quasiconformal group acting on R^n, $n \geq 3$, isomorphic to $(R^{n-1}, +)$, which is not a quasiconformal conjugate of any Möbius group. Even the discrete subgroups of maximal rank of this group have been shown by Gaven J. Martin [M1] to be counterexamples of the question. In the same paper he indicates how to modify this group to get smooth (except at infinity) discrete counterexamples.

So, what happens with simpler, for example, cyclic groups? Quasiconformal groups generated by an elliptic element are not necessary, not even homeomorphically conjugate to conformal groups (see, for example, Giffen [G] or Martio–Väisälä [MV] for $n \geq 4$). For loxodromic groups there is a quasiconformal conjugacy at

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least for $n \neq 4,5$ [M2], and nothing is known about cyclic quasiconformal parabolic groups.

The goal of this paper is to show that some elements of Tukia's group and of Martin's discrete and smooth versions of it still generate cyclic quasiconformal parabolic groups that are not quasiconformal conjugates of Möbius groups.

Incidentally, this will give a new proof of Tukia's and Martin's theorems, elementary in the sense that it does not appeal to Bieberbach's theorem on cristallographic groups.

Our result means that there is a nontrivial class for the quasiconformal conjugacy of cyclic quasiconformal parabolic groups. Varying the construction of Tukia's group we get in fact uncountably many cyclic quasiconformal (even quasi-isometric) parabolic groups that are all in different quasiconformal conjugacy classes—they are all topologically equivalent.

Since all the considered cyclic groups act on the upper half-space U^n, our main result can be used to give a negative answer to a conjecture, originated by Martin and Tukia [M1], where the hope has been that the so-called three-dimensional quasiconformal Fuchsian groups, quasiconformal groups acting on U^3, are always quasiconformally conjugated to Fuchsian groups.

Using the fact that Tukia's group G is quasi-isometric, Martin has already remarked that $G \times \mathrm{Id}$ is a quasi-isometric Fuchsian group which states that a higher-dimensional version of the conjecture cannot be valid.

The three-dimensional negative answer is more surprising because, with the help of the two-dimensional affirmative answer of the conjugacy problem and the extension theorem for quasiconformal mappings of Tukia and Väisälä, we may assume that the boundary group $G_{|R^2}$ is conformal. It was hoped that G was quasiconformally conjugated to the Poincaré extension of the conformal boundary group.

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2. Tukia's and analogous groups

Tukia's group is built by conjugating the group of translations $\{x \mapsto x + a; a = (a_1, 0, a_3, \dots, a_n), a_i \in R\}$ by a homeomorphism $F = f \times \mathrm{Id} : R^n \to R^n$, where $f\colon R^2 \to R^2$ is a quasiconformal extension of a quasisymmetric embedding $h\colon R \to R^2$ whose image J is the well-known snowflake J of von Koch. Here we will repeat this construction in a slightly generalized setting in order to get uncountably many groups in different quasiconformal conjugacy classes.

We use Hutchinson's method [H] for self-similar sets to define the snowflake. Thus, let $\alpha \in]\frac{1}{2}, 1[$, τ defined by $\alpha = \log\tau/\log 4$, and let $a = (0,0)$, $b = \left(\frac{1}{2}, \sqrt{(4-\tau)/4\tau}\right)$ and $c = (1,0)$. With these points we associate two contracting similitudes S_1, S_2, mapping $\vec{ac}$ to $\vec{ab}$ for $i = 1$ and to $\vec{bc}$ for $i = 2$, having negative determinant. Then the map

$$S\colon E \longmapsto S_1(E) \cup S_2(E)$$

is a contraction of $\mathscr{I}$, the complete metric space of all nonempty compact subsets of R^n with metric δ the Hausdorff metric

$$\delta(E,F) = \sup\bigl\{\operatorname{dist}(x,F), \operatorname{dist}(y,E)\,;\, x \in E, y \in F\bigr\}; \qquad E,F \in \mathscr{I},$$

and has a unique fixed point $C \in \mathscr{I}$. This is a self-similar arc whose Hausdorff dimension is $\operatorname{Hdim}(C) = 1/\alpha$. By the choice of τ we have $\tau^n C \subset \tau^m C$, for integers $n \le m$, which allows us to set

$$J = \bigcup_{n \ge 0} \tau^n (C \cup -C).$$

Denote, for $p \in J$, the subarc of J joining 0 and p by J_p and the d-dimensional Hausdorff measure of this subarc by $HM^d(J_p)$. So we can define a parametrisation of J as follows:

$$h\colon \begin{cases} R & \to J \subset R^2 \\ x & \mapsto p = \binom{p_1}{p_2} \end{cases} \qquad \text{with } xp_1 \ge 0 \text{ and } HM^{1/\alpha}(J_p) = |x|.$$

This map satisfies, for a constant $M \ge 1$, the Hölder inequality crucial from our point of view,

$$\frac{1}{M}\|x-y\|^\alpha \le \bigl\|h(x)-h(y)\bigr\| \le M\|x-y\|^\alpha \qquad \text{for all } x,y \in R,$$

which is a consequence of a result of Falconer and Marsh [FM].

A consequence of (1) is that h is quasisymmetric. With the aid of the Beurling–Ahlfors extension theorem (see [T3]) h can be extended to a quasiconformal map $f\colon R^2 \to R^2$, and the homeomorphism looked for is $F = f \times \mathrm{Id}\colon R^n \to R^n$.

Let $T = \{x \mapsto x + te_1\ ;\ t \in R\}$. We are interested in the parabolic group $G = \{g_t\ ;\ t \in R\} = F \circ T \circ F^{-1}$ which is quasi-isometric and thus quasiconformal. More explicitly,

$$g_t(x) = F\bigl(F^{-1}(x) + te_1\bigr) = \bigl(f(f^{-1}(x_1,x_2) + (t,0)), x_3, \dots, x_n\bigr).$$

Note that for $\operatorname{Hdim}(J) = \log 4/\log 3$ this exactly corresponds to Tukia's construction and for certain discrete values of $\alpha \in]\frac{1}{2}, 1[$ to that of McKemie [MCK].

3. Conformal cyclic parabolic groups

In this and the next paragraph we consider the group G for an arbitrary fixed $\alpha \in]\frac{1}{2}, 1[$. We will now establish the necessary conditions for parabolic Möbius groups that may be quasiconformal conjugates of G.

Such a group can be conjugated by a Möbius transformation to a group H having the form $H = \{h_t(x) = U_t(x) + te_1 \; ; \; t \in R\}$, where

$$U_t = \begin{pmatrix} 1 & 0 \\ 0 & V_t \end{pmatrix} \qquad \text{and} \qquad V_t \in O(n-1).$$

3.1. The non-discrete case. In what follows we make use of the fact that a quasicircle τ is characterized by the Ahlfors three-point condition, which means that there is a constant c with

$$\operatorname{diam}(\tau_{p,q}) \le c\|p-q\| \qquad \text{for all } p, q \in \tau.$$

Here $\tau_{p,q}$ is the (shortest) subarc of τ joining p and q.

We will say that a family of quasicircles is uniformly of bounded turning if it is possible to choose the constant c to be valid for all quasicircles of the family.

Lemma 1. *If $V_t \neq I$ for $t \neq 0$, the trajectories $\tau_x = \{h_t(x) \; ; \; t \in R\}$, $x \in R^n$, are not uniformly of bounded turning.*

Proof. It is sufficient to examine the lemma in dimension $n = 3$. Otherwise one can always find an H-invariant three-dimensional subspace W of R^n. Looking at $H_{|W}$, we turn back to the three-dimensional case.

Let $\theta \in R$ be the angle of rotation of V_1:

$$V_t = \begin{pmatrix} \cos t\theta & -\sin t\theta \\ \sin t\theta & \cos t\theta \end{pmatrix}.$$

For $x = (0, x_2, x_3)$ and $t_0 = \pi/\theta$ we have

$$\|h_{t_0}(x) - x\|^2 = t_0^2 + \big(2\|x\|\big)^2 \qquad \text{and} \qquad \|h_{2t_0}(x) - x\| = 2t_0.$$

Hence

$$\frac{\|h_{t_0}(x) - x\|^2}{\|h_{2t_0}(x) - x\|^2} = \frac{1}{4} + \frac{\|x\|^2}{t_0^2}$$

and, by the Ahlfors three-point condition, we have proved the lemma, since this last quotient cannot be majorized independently of x. □

Consequence 1. *The conformal group H cannot be quasiconformally conjugated to the group $T = \{T_t(x) = x + te_1 \; ; \; t \in R\}$ unless $H = T$.*

Consequence 2. *If there is a conformal group which is a quasiconformal conjugate of G, it must be the translation group T.*

3.2. The discrete case. Let $h(x) = h_1(x) = U_1 x + e_1$ and $g(x) = g_t(x)$, $t > 0$. We want to establish discrete versions of consequences 1 and 2.

Lemma 2. *Let $V_1 \neq I$ and define $E(x) = \{h^k(x)\ ;\ k \in Z\}$. Then it is not possible to find trajectories τ_x, for $x \in 0 \times R^{n-1}$, which are uniformly of bounded turning such that $E(x) \subset \tau_x$.*

Proof. As in the proof of Lemma 1 it suffices to treat $n = 3$. It is handier to write here $h(x) = (x_1 + 1, e^{2\pi i\alpha} z)$ with $z = x_2 + ix_3$ and $\alpha \in R \setminus Z$.

i) α rational: there are $p \in Z^*$; $q \in N \setminus \{1\}$ with $\alpha = p/q$. Put $q' = \frac{1}{2}q$ for even q and $q' = \frac{1}{2}(q-1)$ otherwise. Then

$$\big\|h^q(x) - x\big\|^2 = q^2$$

and

$$\big\|h^{q'}(x) - x\big\|^2 = (q')^2 + |z|^2 \big|e^{(2\pi i q' p/q)} - 1\big|^2 = (q')^2 + c|z|^2,$$

where $c = c(p,q) > 0$. So for every $m \in N$ we can find an $x = (0, z)$ such that

$$m < \frac{\big\|h^{q'}(x) - x\big\|^2}{\big\|h^q(x) - x\big\|^2} = \Big(\frac{q'}{q}\Big)^2 + \frac{c}{q^2}|z|^2,$$

which implies the lemma for rational α.

ii) α irrational: let $\varepsilon > 0$. By Weil's theorem we can find $p \in N$ and $k \in Z$ such that $p\alpha = k + \frac{1}{2} + \delta/2\pi$; $|\delta| < \varepsilon$. Hence

$$\big\|h^p(x) - x\big\|^2 = p^2 + |z|^2 |e^{2\pi i p\alpha} - 1|^2 = p^2 + 4|\cos\tfrac{1}{2}\delta|^2 |z|^2$$

and

$$\big\|h^{2p}(x) - x\big\|^2 = (2p)^2 + |z|^2 |e^{4\pi i p\alpha} - 1|^2 = 4\big(p^2 + 4|\sin\delta|^2 |z|^2\big).$$

Applying once again Ahlfors' three-point condition and proceeding by contradiction, there must be a constant $c \geq 1$ such that

$$c \geq \frac{\big\|h^p(x) - x\big\|^2}{\big\|h^{2p}(x) - x\big\|^2} = \frac{p^2 + 4|\cos\frac{1}{2}\delta|^2 |z|^2}{4\big(p^2 + 4|\sin\delta|^2 |z|^2\big)},$$

which implies, for ε sufficiently small,

$$(c - \tfrac{1}{4})p^2 \geq (\cos^2\tfrac{1}{2}\delta - c\sin^2\delta)|z|^2 \geq (\cos^2\varepsilon - c\sin^2\varepsilon)|z|^2.$$

Because this inequality cannot be valid for all $z = x_2 + ix_3$, we have the contradiction we looked for. □

Consequence 3. *Let $h(x) = Ux + e_1$ be a parabolic Möbius transformation with non-trivial rotation matrix $U \in O(n) \setminus \{I\}$. Then the discrete conformal group generated by h cannot be quasiconformally conjugated to the group generated by $T_1(x) = x + e_1$. (But the two groups are topologically equivalent; see for example [GM].)*

Consequence 4. *If there is a conformal group which is a quasiconformal conjugate of a discrete subgroup of G it must be a group of translations, for example $\langle T_1 \rangle$.*

4. The main results

We are now able to prove

Theorem 1. *There is no quasiconformal homeomorphism conjugating G to a Möbius group.*

Theorem 2. *No discrete subgroup of G or its smooth Martin version is a quasiconformal conjugate of a Möbius group.*

We only need to prove the second theorem.

Proof of Theorem 2. Let $g = g_t$, $t > 0$, and $\langle g \rangle$ be one of the groups of Theorem 2. We suppose that there is a K-quasiconformal map $\Phi\colon R^n \to R^n$ normalized by $\Phi(0) = 0$ and conjugating the group to a conformal group. Consequence 4 says that this conformal group must be a translation group, for example $\langle T_1 \rangle$. Then Φ must verify $\Phi(x + ke_1) = g^k\bigl(\Phi(x)\bigr)$ for every $x \in R^n$, $k \in Z$. Define $\gamma = \Phi^{-1}(Re_3)$. Inequality (1) and the definition of g give for every $p \in \gamma$ and $k \in Z$

$$\frac{(tk)^\alpha}{M} \le \bigl\|\Phi(p+ke_1) - \Phi(p)\bigr\| = \bigl\|(f(tk,0), q, 0, \dots, 0) - (0, 0, q, 0, \dots, 0)\bigr\| \le M(tk)^\alpha$$

where $\Phi(p) = (0, 0, q, 0, \dots, 0)$. This, together with the fact that quasiconformal maps of R^n are quasisymmetric (see [V]), becomes

$$\frac{(tk)^\alpha}{C} \le \bigl\|\Phi(p') - \Phi(p)\bigr\| \le C(tk)^\alpha \tag{2}$$

for $C = MK$ and for every $p \in \gamma$; $p' \in R^n$ whenever $\|p - p'\| = k \in Z$.

Below we show that (2) and "$\Phi(\gamma)$ is a line" are not compatible. To do this, let γ_N be the maximal subarc of γ contained in $B(0, N)$, and consider a subdivision $p_0, \dots, p_m$ of γ_N such that $\|p_{j+1} - p_j\| = 1$, $0 \le j \le m - 2$, which is maximal. Then $m \ge 2N$. (2) and $\Phi(\gamma) = Re_3$ allow us to establish

$$\operatorname{diam} \Phi(\gamma_N) \le 2C(tN)^\alpha, \tag{3}$$

$$\operatorname{diam}\Phi(\gamma_{p_{j+1}p_j}) \geq \frac{t^\alpha}{C} \qquad (0 \leq j \leq m-2), \tag{4}$$

where $\gamma_{p_{j+1}p_j}$ is the subarc of γ_N limited by p_j and p_{j+1}. Putting (3) and (4) together and appreciating that the $\Phi(\gamma_{p_{j+1}p_j}) \subset Re_3$ are disjoint intervals we get

$$\frac{2N-1}{C} \leq \frac{m-1}{C} \leq t^{-\alpha}\sum_{j=0}^{m-2}\operatorname{diam}\Phi(\gamma_{p_{j+1}p_j}) \leq t^{-\alpha}\operatorname{diam}\Phi(\gamma_N) \leq 2CN^\alpha,$$

which is impossible for N large enough ($\alpha < 1$). □

5. Different quasiconformal conjugacy classes

We now distinguish the different groups generated by different snowflakes $J = J_\alpha$ and note them by $G_\alpha = \{g_{t,\alpha}\ ;\ t \in R\}$ as well as the corresponding map by f_α and the constant of (1) by M_α. We will establish

Theorem 3. *Let $\alpha, \beta \in]\frac{1}{2}, 1[$, $\alpha \neq \beta$. Then the groups G_α and G_β are not quasiconformally equivalent, which means that there is no quasiconformal map $\Phi\colon R^n \to R^n$ with $\Phi \circ G_\alpha \circ \Phi^{-1} = G_\beta$.*

Proof. Assume that there is a K-quasiconformal map $\Phi\colon R^n \to R^n$ normalized by $\Phi(0) = 0$ satisfying $\Phi \circ g_{t,\alpha} = g_{t,\beta} \circ \Phi$ for all $t \in R$. Let $\gamma = \Phi^{-1}(Re_3)$ and $p \in \gamma$.

i) $p \in J_\alpha \times R^{n-2}$: it suffices to apply inequality (1) twice in the same way as in the previous proof to obtain, for all $p' \in R^n$ and a constant $C_p \geq 1$,

$$\frac{1}{C_p}\|p-p'\|^{\beta/\alpha} \leq \big\|\Phi(p)-\Phi(p')\big\| \leq C_p\|p-p'\|^{\beta/\alpha}. \tag{5}$$

ii) $p \notin J_\alpha \times R^{n-2}$: $f = f_\alpha\colon R^2 \to R^2$ is, in a neighbourhood of p, a diffeomorphism because it is the composition of a conformal map with the Ahlfors–Beurling extension of a quasisymmetric map which has this property. Thus we can calculate

$$\frac{1}{t}\big\|g_{t,\alpha}(p)-p\big\| = \Big\|\big(f'_\alpha\big(f_\alpha^{-1}(p_1,p_2)+(t,0)\big),0,\dots,0\big)e_1\Big\| + o(1) = \big\|v(p)\big\| + o(1)$$

for $t \to 0$. Let us take $p' \in R^n$ with $\|p-p'\| = \big\|g_{t,\alpha}(p)-p\big\| = t\big\|v(p)\big\| + o(t)$. Again, equation (1) permits us to get, for a constant $c \geq 1$,

$$\frac{1}{c}t^\beta \leq \big\|\Phi(p)-\Phi(p')\big\| \leq K\big\|\Phi(p)-g_{t,\beta}\big(\Phi(p)\big)\big\| \leq ct^\beta.$$

Hence we can choose constants $C_p \geq 1$ and $t_0 = t_0(p) > 0$ such that for all $t \leq t_0$

$$\frac{1}{C_p}\|p-p'\|^\beta \leq \big\|\Phi(p)-\Phi(p')\big\| \leq C_p\|p-p'\|^\beta. \tag{6}$$

For every $q \in Re_3$ we now have because of (5) and (6)

$$\big\|\Phi^{-1}(q)-\Phi^{-1}(q')\big\| = o\big(\|q-q'\|^{1+\varepsilon}\big)$$

for $q' \to q$ and $1 < 1+\varepsilon < 1/\beta$. Consequently, Φ^{-1} restricted to the line Re_3 is constant, which is impossible. □

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