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A UNIQUENESS THEOREM FOR MONOGENIC FUNCTIONS

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Abstract. In this paper it is proved by classical analytical methods that a Borel-monogenic function is completely determined by its derivatives at one point if the absolute value of these derivatives is bounded by M_n where the sum over $(M_n/n!)^{-1/n}$ diverges.

1. Introduction

In 1912 Émile Borel published his paper [1] where he introduced the non-analytic monogenic functions. This paper deals with these functions and with the question under what conditions they will form a class of quasianalytic functions in the sense of J. Hadamard: If a function is infinitely often differentiable at a point z_0 and if the function itself and all its derivatives vanish at z_0, it follows that the function vanishes identically in its domain of definition. Equivalently, if two functions f and g coincide at one point z_0 together with all their derivatives, they are identical.

Up to now a positive answer to Hadamard's question could be given for the monogenic functions introduced by Borel only by applying the theory of real quasi-analytic functions or equivalent results. The assumptions one needs for this are very strong. But Borel's monogenic functions have much more structure than real functions, so that Hadamard's question should get a positive answer for a wider class. In this paper we will provide a positive answer to Hadamard's question for such a wider class. It should be mentioned that this paper is based strongly on Borel's and Carleman's work. We especially use these particular classical methods and avoid the concept of "finely holomorphic and meromorphic functions".

As the domain of definition of a monogenic function as well as the function itself are nowadays not very well known, the definitions will be given in this introduction. In Section 2 of this paper we will establish some further facts about these functions and formulate our result. The proof of the result will be given in

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Section 3. In the final Section 4 we will provide a comment which enables us to apply the result of this paper in special cases under less restrictive assumptions.

So let us define the domain of definition: Let G be a domain in $\mathbf{C}$, a_1, a_2, a_3, ... a sequence of points with $a_\nu \in G$ for all ν, and let r_1, r_2, r_3, ... and ϱ_1, ϱ_2, ϱ_3, ... be two sequences of positive real numbers with

$$1 \geq \varrho_\nu > r_\nu \qquad \text{for all } \nu = 1, 2, \dots$$

and

$$\text{(1.1)} \qquad \sum_{\nu=1}^{\infty} r_\nu < \infty, \qquad \sum_{\nu=1}^{\infty} \varrho_\nu < \infty, \quad \text{and} \quad \sum_{\nu=1}^{\infty} \frac{r_\nu}{\varrho_\nu^n} < \infty \text{ for each integer } n.$$

Sometimes we will additionally assume that $\sum_{\nu=1}^{\infty} \sqrt{\varrho_n} < \infty$. We shall write $D(a, r) = \{z \mid |z - a| < r\}$. For each non-negative integer p we define

$$C_p = G \setminus \bigcup_{\nu=1}^{\infty} D(a_\nu, 2^{-p} r_\nu), \qquad C_p^* = G \setminus \bigcup_{\nu=1}^{\infty} D(a_\nu, 2^{-p} \varrho_\nu),$$

and, if $\sum_{\nu=1}^{\infty} \sqrt{\varrho_\nu} < \infty$,

$$C_p^{**} = G \setminus \bigcup_{\nu=1}^{\infty} D\big(a_\nu, 2^{-p} \sqrt{\varrho_\nu}\,\big).$$

Finally we define

$$C = \bigcup_{p=0}^{\infty} C_p, \qquad C^* = \bigcup_{p=0}^{\infty} C_p^*,$$

and, if $\sum_{\nu=1}^{\infty} \sqrt{\varrho_\nu} < \infty$,

$$C^{**} = \bigcup_{p=0}^{\infty} C_p^{**}.$$

Clearly $C \supset C^* \supset C^{**}$, and for each $p = 0, 1, 2, \dots$, $C_p \supset C_p^* \supset C_p^{**}$, $C \supset C_p$, $C^* \supset C_p^*$, and $C^{**} \supset C_p^{**}$. Here and in what follows any assertion involving C^{**} is made under the hypothesis $\sum_{\nu=1}^{\infty} \sqrt{\varrho_\nu} < \infty$. We further mention that, because of (1.1), C and C^* contain (Lebesgue-measure) almost every point of G. It can happen that the a_ν are dense in G, so that C, respectively C^*, contains no domain of $\mathbf{C}$. We now introduce for any point $z_0 \in C$, respectively $z_0 \in C^*$, ε-neighbourhoods $U_{p,\varepsilon}^{(z_0)}$, respectively $U_{p,\varepsilon}^{*(z_0)}$ in each C_p, respectively C_p^*, as follows $U_{p,\varepsilon}^{(z_0)} = C_p \cap D(z_0, \varepsilon)$, respectively $U_{p,\varepsilon}^{*(z_0)} = C_p^* \cap D(z_0, \varepsilon)$. We are now ready to introduce the notion of a "monogenic function" in C:

Definition. A complex valued function f of the complex variable z is monogenic in C if

1. f is defined in each C_p,
2. f is continuous and bounded in each C_p,
3. f is (complex) differentiable in each non-isolated point of C_p.

Clearly, if the a_ν are dense in G, a monogenic function need not be analytic.

Typical examples of monogenic functions are the functions

$$f(z) = \sum_{i=1}^{\infty} \frac{A_{\nu_i}}{z - a_{\nu_i}} \tag{1.2}$$

where for all $\nu = 1, 2, \dots$

$$|A_{\nu_i}|^{1/4} \le r_{\nu_i}.$$

For any $z \in C_p$ we have $|z - a_{\nu_i}| \ge 2^{-p} r_{\nu_i}$ for all $\nu = 1, 2, 3, \dots$ and hence

$$|z - a_{\nu_i}| \ge 2^{-p} r_{\nu_i} \ge 2^{-p} |A_{\nu_i}|^{1/4} = 2^{-p} r_{\nu_i}$$

and

$$\Big| \sum_{\nu=1}^{\infty} \frac{A_{\nu_i}}{z - a_{\nu_i}} \Big| \le 2^p \sum_{\nu=1}^{\infty} |A_\nu|^{3/4} \le 2^p \sum_{i=1}^{\infty} r_{\nu_i}.$$

So $f(z)$ is defined in each C_p, and the series defining $f(z)$ uniformly converges in C_p. Therefore $f(z)$ is continuous and bounded in each C_p. Further, for each $z \in C_p$,

$$\Big| \sum_{\nu=1}^{\infty} \frac{A_{\nu_i}}{(z - a_{\nu_i})^2} \Big| \le 2^p \sum_{i=1}^{\infty} |A_{\nu_i}|^{1/3} \le 2^p \sum_{\nu=1}^{\infty} r_\nu.$$

So it is easy to verify that $f(z)$ is differentiable in each C_p. Hence $f(z)$ is a monogenic function in C. If the a_{ν_i} are dense in C, then $f(z)$ is not analytic at any $z_0 \in C$.

We now want to integrate the functions monogenic in C along closed curves. For this purpose we introduce the following notation: First each closed curve γ is given a counterclockwise orientation. The simply connected region of G bounded by γ is denoted by D. For any closed curve $\gamma \subset G$ with length $|\gamma|$ we denote by γ_p the curve obtained from γ when we replace those parts of γ which are covered by discs $D(a_\nu, 2^{-p} r_\nu)$ by the corresponding parts of the circles $K'_{p,\nu} = \big\{ z \mid |z - a_\nu| = 2^{-p} r_\nu \big\}$ which are contained in $C_p \cap D$. Here the circles $K'_{p,\nu}$ are clockwise oriented. So it is easy to see that γ_p is always an at most countable union of closed curves, all of which are positively oriented, and the sum of the length of these curves is bounded by

$$|\gamma| + 2\pi \sum_{\nu=1}^{\infty} r_\nu.$$

By $K_{p,\nu}$ we denote the union of those parts of the circles $K'_{p,\nu}$ contained in $C_p \cap D$ and not part of γ_p where the circles $K'_{p,\nu}$ are orientatied counterclockwise and the $K_{p,\nu}$ correspondingly. If we sum over all $K_{p,\nu}$, this sum includes by definition only those terms of $K_{p,\nu}$ which really occur (with respect to γ).

By the above arrangement it is easy to see that for the function $f(z)$ of the type (1.2) we have

$$\int_{\gamma_p} f(z)\,dz - \sum_\nu \int_{K_{p,\nu}} f(z)\,dz = 0. \tag{1.3}$$

In [1] Borel also stated that (1.3) holds for each monogenic function. For the function of type (1.2) follows also directly for any point $z_0 \in C_p^*$ surrounded by γ_p

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma_p} \frac{f(z)}{z-z_0}\,dz - \sum_{\nu=1}^{\infty} \frac{1}{2\pi i}\int_{K_{p,\nu}} \frac{f(z)}{z-z_0}\,dz, \tag{1.4}$$

where the series converges absolutely and uniformly (in z_0) because of (1.1). We mention without proof that (1.4) also follows from (1.3) for each monogenic function. (This can be proved just like Cauchy's formula if one observes that the value $f(z)$ is independent of p and that $f(z)$ is continuous in each C_p; and one constructs circles

$$C^R = \{z \mid |z - z_0| = R\} \subset C_{p+k(R)},$$

where $k(R)$ is a positive integer and R can be chosen arbitrarily small. These circles C^R can be constructed by methods similar to those to be described in 2.)

2. Results and further preliminaries

First we get the following theorem from the integral formula.

Theorem B. *If f is monogenic in C, the restriction of f to C_p^* is arbitrarily often differentiable with*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma_p} \frac{f(z)}{(z-z_0)^{n+1}}\,dz - \sum_\nu \frac{n!}{2\pi i}\int_{K_{p,\nu}} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \tag{2.0}$$

and

$$\left|f^{(n)}(z_0)\right| \le n! M_0\left(\frac{|\gamma|}{\inf\{|z-z_0| \mid z \in \gamma\}^{n+1}} + 2^{np+n+1}\sum_{\nu=1}^{\infty} \frac{r_\nu}{\varrho_\nu^{n+1}}\right). \tag{2.1}$$

Proof. The proof of the theorem is essentially the same as the proof for analytic functions; we just have to prove in addition that the series of the derivatives of the integrals will converge uniformly. To prove this we have to estimate

$$\left|\int_{K_{p,\nu}} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \le \int_{K_{p,\nu}} \frac{\bigl|f(z)\bigr|}{(z-z_0)^{n+1}}\,|dz| \le 2\pi \frac{M_0}{(\varrho_\nu 2^{-p} - r_\nu 2^{-p})^{n+1}} \frac{r_\nu}{2^p}, \tag{2.2}$$

where

$$M_{p,0} = M_0 = \sup\bigl\{\bigl|f(z)\bigr| \mid z \in C_p\bigr\}.$$

From this it follows that the series in (2.0) converges absolutely and uniformly for $z_0 \in C_p^*$. We only need to prove the estimate (2.1).

We have

$$\begin{aligned} 2\pi \frac{M_0}{(\varrho_\nu 2^{-p} - r_\nu 2^{-p})^{n+1}} \frac{r_\nu}{2^p} &= 2\pi M_0 \Bigl(\frac{r_\nu}{\varrho_\nu^{n+1}} 2^{np}\Bigr) \Bigm/ \Bigl(1 - \frac{r_\nu}{\varrho_\nu}\Bigr)^{n+1} \\ &\le 2\pi M_0 2^{np} \Bigl(\frac{r_\nu}{\varrho_\nu^{n+1}}\Bigr) \Bigm/ \Bigl(1 - \frac{r_\nu}{\varrho_\nu}\Bigr)^{n+1}. \end{aligned}$$

For all sufficiently large ν we have $r_\nu/\varrho_\nu \le \frac{1}{2}$, and so from (2.2) it follows that

$$\begin{aligned} \bigl|f^{(n)}(z_0)\bigr| &= \frac{n!}{2\pi i} \int_{\gamma_p} \Bigl|\frac{f(z)}{(z-z_0)^{n+1}}\,dz\Bigr| + \frac{n!}{2\pi} \sum_\nu \int_{K_{p,\nu}} \Bigl|\frac{f(z)}{(z-z_0)^{n+1}}\,dz\Bigr| \\ &\le |\gamma| n! \frac{M_0}{\inf\bigl\{|z-z_0| \mid z \in \gamma\bigr\}^{n+1}} + n! M_0 \sum_{\nu=1}^{\infty} \frac{r_\nu}{\varrho_\nu^{n+1}} 2^{n+1} 2^{np}, \end{aligned}$$

but this just gives (2.1).

Next we want to discuss some construction methods also developed by Borel when he introduced the monogene functions. We will not formulate these as theorems so as to be able to explain the ideas better. We first give a construction applicable in C and C^*. Suppose we have an interval s of length $|s|$ on a straight line, and we suppose $s \subset G$. Then there exists an infinity of points ζ on s such that the straight line g perpendicular to s through ζ satisfies $g \cap G \subset C_p$, respectively C_p^*, for each $p \ge p_0 = p_0\bigl(|s|\bigr)$.

To see this we just project the discs $D(a_\nu, 2^{-p} r_\nu)$ on the line through s. The total projection will have measure less than $2^{1-p} \sum r_\nu$. So if $p \ge p_0$ with $2^{1-p_0} < |s|$, we always have infinitely many points with the stated property. Obviously the same method can be applied to get circles which have prescribed centers $z_0 \in G$ and intersections with G contained in C_p (or in C_p^*) for large enough p, with radii at certain intervals which can be chosen arbitrarily. Another construction enables us to construct for any point $z_0 \in C^{**}$ infinitely many straight lines g containing z_0 with $g \cap G \subset C_p^*$ for all $p \geq p_0$, and there is even an infinity of such lines in each angle with vertex z_0. To achieve this construction we first observe that $z_0 \in C_{p_1}^{**}$ for some integer p_1. If we now project from z_0 each of the discs $D(a_\nu, 2^{-p_1-d}\varrho_\nu)$ for some integer $d \geq 2$ on $K = \{z \mid |z - z_0| = 1\}$, this projection can cover at most a point set of linear measure

$$\arcsin \frac{2^{-p_1-d}\varrho_\nu}{|z_0 - a_\nu|} \leq \arcsin 2^{-d}\frac{\varrho_\nu}{\sqrt{\varrho_\nu}} \leq 2^{-d}\sqrt{\varrho}.$$

Now $\sum_{\nu=1}^\infty \sqrt{\varrho_n} < \infty$ gives the assertion if we choose $p_0 = p_1 + d$ sufficiently large so that the projection of these discs cannot cover any prescribed arc on K.

This last construction was given for a particular reason: In the hypotheses of the present paper we start with a certain property of the function at one point, z_0. To prove the result we will need a straight line s through z_0 with $z_0 \in s$ such that $s \cap G \subset C_p^*$ for some p, and we have just seen that this assumption is fulfilled for any point z_0 of C^{**}.

With the preceding hints in mind we now state the result of this paper in the following theorem and its corollary.

Theorem. *Let there be given a domain $G \subset \mathbf{C}$, a sequence of points $a_\nu \in G$, $n = 1, 2, \dots$ and a sequence of positive real numbers r_ν with*

$$\sum_{\nu=1}^\infty r_\nu < \infty$$

and

$$\sum_{\substack{\nu=1 \\ r_\nu \neq 1}}^\infty \left(\log\frac{1}{r_\nu}\right)^{-1} < \infty.$$

Define $\varrho_\nu = \bigl(\log(1/r_\nu)\bigr)^{-1}$ and for each positive integer p

$$C_p = G \setminus \bigcup_{\nu=1}^\infty D(a_\nu, 2^{-p} r_\nu),$$

$$C_p^* = G \setminus \bigcup_{\nu=1}^\infty D(a_\nu, 2^{-p}\varrho_\nu).$$

Let $C = \cup_{p=1}^\infty C_p$ and $C^* = \cup_{p=1}^\infty C_p^*$. Let $f(z)$ be a monogenic function in C. Then f is arbitrarily often differentiable on each C_p^*. If now $z_0 \in C^*$ is a point such that there exists a straight line s through z_0 with $z_0 \in s \cap G \subset C_p^*$ for some p and $f(z_0) = f^{(n)}(z_0) = 0$ for $n = 1, 2, \dots$, then $f(z) \equiv 0$ for $z \in C^*$.

Corollary. *If in the theorem we replace the supposition*

$$\sum_{\substack{\nu=1\\ r_\nu\neq 1}}^{\infty}\Big(\log\frac{1}{r_\nu}\Big)^{-1}<\infty$$

with

$$\sum_{\substack{\nu=1\\ r_\nu\neq 1}}^{\infty}\Big(\log\frac{1}{r_\nu}\Big)^{-1/2}<\infty,$$

the proposition of the theorem holds for each z_0 *with*

$$z_0\in C^{**}=\bigcup_{p=1}^{\infty}\Big\{G\setminus\bigcup_{\nu=1}^{\infty}D\big(a_\nu,2^{-p}\sqrt{\varrho_\nu}\,\big)\Big\}.$$

3. Proof of the theorem and its corollary

If the r_ν and ϱ_ν do not fulfill the condition $1>\varrho_\nu>r_\nu$, we just replace such ϱ_ν with $\varrho_\nu=3/4$ and the corresponding r_ν with $r_\nu=1/2$. Because of the convergence of the series of the r_ν and ϱ_ν there can be only finitely many such r_ν and ϱ_ν. Hence, in the theorem and the corollary we can assume, without loss of generality, the first condition of Section 1 with respect to C, C^* (and in the case of our corollary also with respect to C^{**}). In the case of the corollary we need the definition of C^{**} only to be sure that a straight line s through the given point $z_0\in C^*$ exists. So in the remaining part of the proof of our theorem we only have to replace

$$\sum_{\nu=1}^{\infty}\Big(\log\frac{1}{r_\nu}\Big)^{-1}$$

in the assumptions with

$$\sum_{\nu=1}^{\infty}\Big(\log\frac{1}{r_\nu}\Big)^{-1/2}<\infty$$

in order to obtain the corollary. To prove the theorem we can assume without loss of generality that after a linear transformation of $\mathbf{C}$, s is an interval of the imaginary axis and $z_0=0$. Further, we can replace in our assumption the sequence a_1, a_2, $\dots$ by a_1, $-a_1$, $\overline{a}_1$, $-\overline{a}_1$, a_2, $-a_2$, $\overline{a}_2$, $-\overline{a}_2$, a_3, $\dots$ and the sequence r_1, r_2, r_3, $\dots$ by r_1, r_1, r_1, r_1, r_2, r_2, r_2, r_2, r_3, $\dots$.

After these substitutions we have $C_p=\overline{C}_p=-C_p=-\overline{C}_p$ for any integer $p\geq 0$, and similarly with C_p replaced by C, C_p^*, or C^*. Or, instead, the function $F_1(z)=f(z)\overline{f}(z)$ and $F_2=F_1(z)F_1(-z)$ $F_2(z)$ is monogenic in C, with $F_2(0)=F_2^{(n)}(0)=0$ for all n.

Hence we can also assume without any loss of generality (by replacing again $F_2(z)$ by the notation $f(z)$) that

$$f(z) = \overline{f}(\overline{z}) = f(-z) = \overline{f}(-\overline{z}).$$

The first consequence of this is that $f(z)$ is real on the imaginary axis and at all points of the same C_p belonging to the real axis. It causes no further restriction if we suppose $\gamma = \{z \mid |z| = 1\} \subset G$.

Now we apply the integral formula of Theorem B for the derivative of f. Hence, for each $z \in s$ and $n = 0, 1, 2, \ldots$ we have

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i} \int_{\gamma_p} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta - \frac{n!}{2\pi i} \sum_{\nu} \int_{K_{p,\nu}} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \\ &= \frac{n!}{2\pi i} \int_{\gamma_p^1} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta + \frac{n!}{2\pi i} \int_{\gamma_p^2} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \\ &\quad - \frac{n!}{2\pi i} \sum \nu \int_{K_{p,\nu}^1} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta - \frac{n!}{2\pi i} \sum_{\nu} \int_{K_{p,\nu}^2} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \end{aligned}$$

where $\gamma_p^1 = \gamma_p \cap \{z \mid \operatorname{Re} z > 0\}$, $K_{p,\nu}^1 = K_{p,\nu} \cap \{z \mid \operatorname{Re} z > 0\}$ and $\gamma_p^2 = \gamma_p \cap \{z \mid \operatorname{Re} z < 0\}$, $K_{p,\nu}^2 = K_{p,\nu} \cap \{z \mid \operatorname{Re} z < 0\}$. Now

$$g(\zeta) = \frac{1}{2\pi i} \int_{\gamma_p^1} \frac{f(\zeta)}{(\zeta - z)}\, dz - \frac{1}{2\pi i} \sum_{\nu} \int_{K_{p,\nu}^1} \frac{f(\zeta)}{(\zeta - z)}\, dz$$

is obviously analytic in $\operatorname{Re} z < 0$, continuous and arbitrarily often differentiable in

$$\{z \mid |z| \leq \tfrac{1}{2}, \operatorname{Re} z \leq 0\} = H$$

with

$$g^{(n)}(\zeta) = \frac{n!}{2\pi i} \int_{\gamma_p^1} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, dz - \frac{n!}{2\pi i} \sum_{\nu} \int_{K_{p,\nu}^1} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, dz.$$

Further, by the construction of $g(z)$ and C, respectively C_p,

$$\operatorname{Im} g^{(n)}(-x) = 0 \qquad \text{for} \qquad x > 0 \text{ and } n = 0, 1, 2, \ldots$$

holds and

$$\operatorname{Re} g(iy) = \tfrac{1}{2} f(iy).$$

So we will now first prove

$$f(z) \equiv 0 \qquad \text{for} \qquad z \in \{z \mid z \in s, |z - z_0| \leq \tfrac{1}{2}\} = s^1. \tag{3.1}$$

To do this we will apply the following theorem of Carleman [2], with the mention that we have unified two theorems of [2] in Theorem C :

Theorem C. *Let $f(z)$ be analytic in $|z|<1$, continuous in $|z|\le 1$, and let $f(z)$ have the asymptotic development zero in $z=1$, that is,*

$$\left|\frac{f(z)}{(z-1)^n}\right| \le A_n \qquad \text{for } n=0,1,2,\dots, |z|<1, \tag{3.2}$$

with $A_n \le A_{n+1}$ for all n. Then the necessary and sufficient condition that $f(z)\equiv 0$ in $|z|\le 1$ follows from (3.2) is the divergence of

$$\sum_{\nu=0}^{\infty} \frac{1}{\sqrt[\nu]{A_\nu}}.$$

To apply Theorem C for the proof of (3.1) we have to estimate in $|z+1|\le 1$ for all $n=0,1,2,\dots$ $\left|g(z)/z^n\right|$. To do this we first observe that g is bounded in $|z+1|\le 1$ so that with $M=\max\left\{|g(z)| \;\middle|\; |z+1|\le 1\right\}$

$$\left|\frac{g(z)}{z^n}\right| \le 2^n M \qquad \text{for} \quad z\in\left\{\zeta \;\middle|\; |\zeta+1|\le 1\right\}\cap\left\{\zeta \;\middle|\; |\zeta|\ge \tfrac{1}{2}\right\}. \tag{3.3}$$

If we apply the maximum principle to $g(z)/z^n$, in view of (3.3) we only need to estimate $\left|g(z)/z^n\right|$ for $z\in s^1$. To do this we consider $\operatorname{Re} g(z)$ and $\operatorname{Im} g(z)$ separately. Now by the extended mean value theorem, if we observe $f^{(n)}(0)=0$, $\left|\operatorname{Re} g^{(n)}(z)\right| = \frac{1}{2}\left|f^{(n)}(z)\right|$ and $\operatorname{Im} g^{(n)}(0)=0$ for all n, we get for $z\in s^1$

$$\left|\frac{\operatorname{Re} g(z)}{(-iz)^n}\right| \le \left|\frac{g^{(n)}(\zeta^1)}{n!}\right|$$

and

$$\left|\frac{\operatorname{Im} g(z)}{(-iz)^n}\right| \le \left|\frac{g^{(n)}(\zeta^2)}{n!}\right|,$$

where $\zeta^1,\zeta^2\in s^1$. Hence for $z\in s^1$ and $n=0,1,2,\dots$

$$\left|\frac{g(z)}{z^n}\right| \le 2\sup\left\{\left|\frac{g^{(n)}(z)}{n!}\right| \; z\in s_1\right\}. \tag{3.4}$$

Now we apply the estimate (2.1) of Theorem B. With some real constant k and some real constant k_1 this gives

$$\sup\left\{\left|g^{(n)}(z)\right| \; z\in s^1\right\} \le n! k_1 k^n M + k_1 n! M \sum_{\nu=0}^{\infty} \frac{r_\nu}{\varrho_\nu^n} 2^{np}.$$

We now take $\varrho_\nu = \big(\log(1/r_\nu)\big)^{-1}$ and use the assumption $\sum_{\nu=1}^{\infty}\big(\log(1/r_\nu)\big)^{-1} < \infty$. This gives (here with the positive integer $t = 1$)

$$\sum_{\nu=0}^{\infty} \frac{r_\nu}{\varrho_\nu^n} \le (n+1)! \sum_{\nu=1}^{\infty} \Big(\log \frac{1}{r_\nu}\Big)^{-1}.$$

In the case of the Corollary we now have to add the factors $(n+t+1)$, $(n+t+2)$, $\ldots$, $(n+2t)$. Hence with some constants τ and T we finally get

$$\sup\big\{\big|g^{(n)}(z)\big|\ z \in s^1\big\} \le (n!)^2(n+1)(n+2)\cdots(n+t)\tau^n T,$$

and therefore, by (3.4),

$$\Big|\frac{g(z)}{z^n}\Big| \le 2n!(n+1)(n+2)\cdots(n+t)\tau^n T$$

for $z \in s^1$. Hence by this estimation and (3.3), where we choose $\tau \ge 2$, it follows in $|z+1| \le 1$ that

$$\Big|\frac{g(z)}{z^n}\Big| \le 2n!(n+1)(n+2)\cdots(n+t)\tau^n T = A_n.$$

Therefore all suppositions of Theorem C are fulfilled wherever

$$\sum_{\nu=0}^{\infty} \frac{1}{\sqrt[\nu]{A_\nu}}$$

diverges. Hence $g(z) \equiv 0$ for $z \in s^1$. Here we mention that this remains true if we take the supposition of the corollary.

Because of $\operatorname{Re} g(z) = \frac{1}{2} f(z)$ for $z \in s_1$ the assertion (3.1) is proved. To get from (3.1) the proposition of the theorem we again apply a theorem of Carleman from [2], which we need slightly to extend. (As a hint on how this extension can be done note that in Carleman's proof we have to apply the subharmonic function $(\log 1/|r-a_\gamma|)^t$ instead of the harmonic function $(\log 1/|r-a_\gamma|)$).

Theorem D. *Let the supposition of the theorem hold where we replace the supposition* $f^{(n)}(z_0) = 0$ $(n = 1, 2, \ldots)$ *with the supposition that* $f(z) \equiv 0$ *on some analytic arc contained in* C^*. *Then* $f(z) \equiv 0$ *in* C^*.

So, as from (3.1), Theorem C immediately gives the proposition of the theorem. Hence the theorem is proved.

In conclusion, we want to mention that in principle the application of Theorem D can be omitted. Then one has to repeat proving (3.1), where the straight lines are constructed by the methods given at the beginning of this section. Clearly in this case the proposition of the theorem can only be related to those points which are accumulation points of such lines contained in C_p^* for some p.

4. Extension of the theorem

In the theorem, and likewise in its corollary, we need the assumption

$$\sum_{\substack{\gamma=1\\ \tau_\gamma\neq 1}}^{\infty}\Big(\log\frac{1}{r_\gamma}\Big)^{-1}<\infty,$$

respectively

$$\sum_{\substack{\gamma=1\\ \tau_\gamma\neq 1}}^{\infty}\Big(\log\frac{1}{r_\gamma}\Big)^{-1/2}<\infty,$$

only to be sure that C^* (or C^{**}), contains many straight lines, so that we can reach almost all points of C^* (or C^{**}). Further, this assumption means that C^*, respectively C^{**}, contains all points of G with the exception of a set of measure zero at most. If we replace the assumption

$$\sum_{\substack{\gamma=1\\ \tau_\gamma\neq 1}}^{\infty}\Big(\log\frac{1}{r_\gamma}\Big)^{-1}<\infty,$$

respectively

$$\sum_{\substack{\gamma=1\\ \tau_\gamma\neq 1}}^{\infty}\Big(\log\frac{1}{r_\gamma}\Big)^{-1/2}<\infty,$$

with

$$\sum_{\substack{\gamma=1\\ \tau_\gamma\neq 1}}^{\infty}\Big(\log\frac{1}{r_\gamma}\Big)^{-t}<\infty,$$

for some integer $t\geq 1$ the conclusion $f(z^*)=0$ of the theorem remains true for all points $z^*\in C^*$ on polygons $P\subset C_p^*$ for some p. The complement of C with respect to G is in this case not necessarily of measure zero, the existence of such polygons is depending on the distribution of the points a_ν (the centres of the excluded discs). The proof works with the same definition of $\varrho_\nu=\big(\log(1/r_\nu)\big)^{-1}$. In the estimation of all integrals one only has to observe the estimate

$$\frac{r_\nu}{\varrho_\gamma^n}\leq\frac{(n+t)!}{\big(\log(1/r_\nu)\big)^t}\leq\frac{n!}{\big(\log(1/r_\nu)\big)^t}(n+t)^t,$$

where $\lim_{n\to\infty}\sqrt[n]{(n+t)^t}=1$.

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