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INFINITE DIMENSIONAL STATIONARY SEQUENCES WITH MULTIPLICITY ONE; PART II

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Abstract. The relationship between multiplicity and dimension of the range of spectral measure for J_0-regular stationary sequences is established. As a consequence spectral characterizations for J_0-regular sequences of multplicity one with positive angle are obtained.

1. Introduction

We were inspired to write this note by the paper by Mesropyan [6], where the problem of positivity of the angle between the past and future for two-dimensional stationary sequences of rank one was studied. Our aim is to point out that J_0-regular sequences of finite multiplicity behaves "like" finite dimensional sequences. This behaviour is a consequence of Theorem 2.2 which examines the range of the spectral measure of a J_0-regular sequence and relates its dimension, the dimension of the error space and the multiplicity of the process to each other. Consequently all results concerning the angle problem for sequences with multiplicity one can be easily derived from analytic conditions available for the one-dimensional case. We also include a characterization of J_0-regular sequences with finite multiplicity.

The paper can be considered a continuation of [3]. In the present paper the J_0-regularity plays a fundamental role, whereas in [3] attention was directed toward the extrapolation problem.

Throughout the paper, N, Z and C will denote the sets of positive integers, all integers and complex numbers, respectively; H, K will denote complex separable Hilbert spaces with norm $\| \, \|$ and inner product $(\, , \,)$; $L(H,K)$ will stand for the space of all bounded linear operators from H to K and $| \, |$ will denote the operator norm in $L(H,K)$. An element Γ in $L(H,H)$ is non-negative, $\Gamma \geq 0$, if $(\Gamma x, x) \geq 0$ for all $x \in H$. For $G \subset H$, $\overline{G}$ will denote the closure of G and $\mathrm{sp}\{G\}$ the linear span of G. All functions will be defined over $[-\pi,\pi)$; $\mathscr{B}$ will denote the Borel σ-algebra in $[-\pi,\pi)$ and dt the normalized Lebesgue measure on $[-\pi,\pi)$. Integrals will be over $[-\pi,\pi)$ unless otherwise is stated.

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(1.1) A sequence $X = \{ X_n : n \in Z \} \subset L(H, K)$ is said to be an $L(H, K)$-valued stationary sequence if $X_m^* X_n = X_0^* X_{n-m}$ for all $n, m \in Z$.

If X is an $L(H, K)$-valued stationary sequence then the following notation will be used:

(1.2) $M(X, A) = \overline{\text{sp}}\{ X_n x : x \in H, n \in A \}$, $A \subset Z$; $M(X) = M(X, Z)$;

(1.3) U will denote the *shift* operator of X, i.e., the unitary operator in $M(X)$ defined by $UX_n = X_{n+1}$, $n \in Z$;

(1.4) E will stand for the *spectral measure of* U, i.e. E is a weakly countably additive orthogonal projection valued measure in $L\big(M(X), M(X)\big)$ such that for all $x, y \in M(X)$

$$(U^n x, y) = \int e^{\text{int}} \big(E(dt)x, y\big), \qquad n \in Z;$$

(1.5) $m(E, G) = \overline{\text{sp}}\{ E(\Delta)z : \Delta \in \mathscr{B}, z \in G \} = \overline{\text{sp}}\{ U^n z : n \in Z, z \in G \}$, $G \subset M(X)$;

(1.6) F will denote the *spectral measure of* X defined by $F(\Delta) = X_0^* E(\Delta) X_0$, $\Delta \in \mathscr{B}$;

(1.7) A sequence $X = \{ X_n : n \in Z \} \subset L(H, K)$ will be said to have a spectral density with respect to dt if there exists a function $G(t) \geq 0$, $G(t) \in L(H, H)$ such that for all $x, y \in H$ and $\Delta \in \mathscr{B}$

$$\big(F(\Delta)x, y\big) = \int_\Delta \big(G(t)x, y\big)\, dt.$$

Spectral density is unique dt a.e. if it exists. For the sake of convenience the spectral density will be denoted by the same symbol F when there is no danger of confusion. Note that the spectral density does not have to exist even if the sequence X has multiplicity one (see Example 3.7).

2. J_0-regular sequences of finite multiplicity

Let us recall that a stationary sequence $X = \{ X_n : n \in Z \} \subset L(H, K)$ is said to be:

(i) *minimal*, if $M(X, Z - \{0\}) \neq M(X)$,

(ii) J_0-*regular*, if $\cap_n M(X, Z - \{n\}) = \{0\}$.

The subspace $N_0(X) = M(X) \ominus M(X, Z - \{0\})$ is called the *error space* of X. The *multiplicity* $m(X)$ of a stationary sequence $X = \{ X_n : n \in Z \}$ is the smallest number $n \in N \cup \{+\infty\}$ such that there exist a sequence $\{ x_k : 1 \leq k < n + 1 \} \subset M(X)$ such that

$$M(X) = \bigoplus_{1 \leq k < n+1} m(E, \{x_k\})$$

where E is the spectral measure of the shift U of X. From [3, Theorem 3.3] it follows that if X has the spectral density $F(t)$, then

$$m(X) = \operatorname{ess\,sup}\bigl[\dim\bigl(F(t)H\bigr)\bigr]. \tag{2.1}$$

It is also known that if X is J_0-regular then $\overline{F(\Delta)H}$, the range of the spectral measure of a stationary sequence $X = \{X_n : n \in Z\}$, is a constant subspace of H for all sets Δ in $\mathscr{B}$ of nonzero Lebesgue measure (see [2, Corollary 47] or [4, 3.1]). The following theorem relates the dimension of the range of the spectral measure and the dimension of the error space to the multiplicity of a J_0-regular stationary sequence.

2.2. Theorem. *Suppose that $X = \{X_n : n \in Z\} \subset L(H,K)$ is a nonzero J_0-regular stationary sequence. Then*
(i) $\overline{F(\Delta)H} = \overline{X_0^* N_0(X)}\, dt(\Delta)$, $\Delta \in \mathscr{B}$,
(ii) $\dim\bigl(F(\Delta)H\bigr) = \dim N_0(X) = m(X)$, *provided* $dt(\Delta) \neq 0$, $\Delta \in \mathscr{B}$.
If, moreover, the spectral density $F(t)$ of X exists, then
(iii) $\overline{F(t)H} = \overline{X_0^* N_0(X)}$ dt *a.e.*,
(iv) $\dim\bigl(F(t)H\bigr) = \dim N_0(X) = m(X)$, dt *a.e.*

Proof. Let $x \in N_0(X)$. Then for every $y \in H$ and $n \in Z$,

$$\int e^{\text{int}}\bigl(y, X_0^* E(dt)x\bigr) = (U^n X_0 y, x) = (X_0 y, x)\delta_{0n} = \int e^{\text{int}}(y, X_0^* x\, dt).$$

Therefore

$$X_0^* E(\Delta)x = (X_0^* x)\, dt(\Delta), \qquad \Delta \in \mathscr{B},\ x \in N_0(X). \tag{2.3}$$

Note that if $X_0^* x = 0$, $x \in N_0(X)$, then for all $y \in M(X)$, $(x, X_n y) = 0$ and so $x = 0$. Thus X_0^* is a one-to-one mapping from $N_0(X)$ into H and from (2.3) it follows that $F(\Delta) \neq 0$ if $dt(\Delta) \neq 0$, $\Delta \in \mathscr{B}$. Let $z = F(\Delta)u = X_0^* E(\Delta) X_0 u$, $u \in H$, $\Delta \in \mathscr{B}$. Since X is J_0-regular, $X_0 u = \lim_k \sum_i^{m_k} E(\Delta_{i,k}) x_{i,k}$ where $x_{i,k} \in N_0(X)$. Therefore

$$z = \lim_k \sum_i X_0^* E(\Delta \cap \Delta_{i,k}) x_{i,k} = \lim_k \sum_i X_0^* x_{i,k}\, dt(\Delta \cap \Delta_{i,k}) \in \overline{X_0^* N_0(X)}\, dt(\Delta).$$

On the other hand, from (2.3) we have that $\overline{X_0^* E(\Delta) N_0(X)} = \overline{X_0^* N_0(X)}$ and so

$$\begin{aligned} F(\Delta)H &= \overline{\bigl(E(\Delta)X_0\bigr)^* \bigl(E(\Delta)X_0\bigr)H} \\ &= \overline{X_0^* E(\Delta) M(X)} \supseteq \overline{X_0^* E(\Delta) N_0(X)} = \overline{X_0^* N_0(X)} \end{aligned}$$

if $dt(\Delta) \neq 0$, because if $T = A^*A$ then $\overline{TH} = \overline{A^*K}$, $A \in L(H,K)$. This proves (i).

To prove (ii) let us first note that from (i) it follows that for $dt(\Delta) \neq 0$,

$$\dim\bigl(F(\Delta)H\bigr) = \dim X_0^* N_0(X) = \dim N_0(X) \geq m(X),$$

because X_0^* is a one-to-one mapping from $N_0(X)$ into $F(\Delta)H$ and X is J_0-regular. We shall prove that $m(X) \geq \dim N_0(X)$. Observe that if M, N are two (not necessarily closed) subspaces of $M(X)$ and $K_0 = \overline{(M+N)}$, then

$$K_0 \ominus M = (I - P_M)K_0 = \overline{(I - P_M)N},$$

where P_M denotes the orthogonal projection onto $\overline{M}$ in $M(X)$ and I is the identity operator in $M(X)$. Applying the relation above first to $M = M(X, Z - \{0\})$, $N = X_0H$ and $K_0 = M(X)$ and then to $M = M\bigl(X, (-\infty, -1]\bigr)$, $N = X_0H$, $K_0 = M\bigl(X, (-\infty, 0]\bigr)$ we obtain:

$$\begin{cases} N_0(X) = \overline{(I-P)X_0H} \\ M\bigl(X, (-\infty, 0]\bigr) \ominus M\bigl(X, (-\infty, -1]\bigr) = \overline{(I-Q)X_0H}, \end{cases}$$

where P and Q are orthogonal projections in $M(X)$ onto $M\bigl(X, Z - \{0\}\bigr)$ and $M\bigl(X, (-\infty, -1]\bigr)$, respectively. Since $(I-P) = (I-P)(I-Q)$, $N_0(X)$ is equal to the closure of $(I-P)\bigl(M\bigl(X, (-\infty, 0]\bigr) \ominus M\bigl(X, (-\infty, -1]\bigr)\bigr)$. Hence $\dim N_0(X) \leq \dim\bigl(M\bigl(X, (-\infty, 0]\bigr) \ominus M\bigl(X, (-\infty, -1]\bigr)\bigr)$, and by [3, Lemma 3.4], this last dim is equal to $m(X)$, for X is obviously regular in the sense of [3, p. 140].

Now suppose that X has the spectral density $F(t)$. From [2, Corollary 49] (see also [4, 3.2.2]) it follows that $\overline{F(t)H} = H_1 =$ constant dt a.e. It is clear that $\overline{X_0^* N_0(X)} = \overline{F\bigl([-\pi, \pi)\bigr)H} \subset H_1$. On the other hand, if $y \perp \overline{X_0^* N_0(X)}$ and $y \in H_1$ then for each $\Delta \in \mathscr{B}$ and $i = 1, 2, \ldots$

$$\int_\Delta \bigl(F(t)e_i, y\bigr)\, dt = \bigl(F(\Delta)e_i, y\bigr) = 0$$

where $\{e_i : i = 1, 2, \ldots\}$ is a fixed orthonormal basis in H. Therefore $y \perp \overline{F(t)e_i}$, $i = 1, 2, \ldots$ dt a.e., which proves (iii). (iv) follows immediately from (i), (ii) and (iii). □

If a stationary sequence $X = \{X_n : n \in Z\}$ has the spectral measure F such that $F(\Delta)H \subset H_0$, $\Delta \in \mathscr{B}$, and $\dim H_0 < \infty$, then $F(\Delta) = F(\Delta)P_{H_0}$, $\Delta \in \mathscr{B}$.Hence if we consider the $L(H_0, K)$-valued stationary sequence $Y = \{Y_n : n \in Z\}$ defined by $Y_n x = X_n x, x \in H_0$, then the spectral measure $\tilde{F}$ of $\{Y_n : n \in Z\}$ equals $\tilde{F}(\Delta) = F(\Delta)|_{H_0}$ and $M(X, A) = M(Y, A)$ for any $A \subset Z$. This analysis shows that the process X "behaves" like the finite dimensional process Y. Combining this with Theorem 2.2 we obtain the following characterization of J_0-regular stationary sequences of finite multiplicity.

2.4. Theorem. *Let $X = \{X_n : n \in Z\}$ be a nonzero $L(H, K)$-valued stationary sequence. The following conditions are equivalent:*

(1) *X is J_0-regular and $m(X) = n < \infty$;*

(2) *X has the spectral density $F(t)$ that satisfies*

  (i) *$F(t)H = H_0$ is a constant n-dimensional subspace of H dt a.e. ($n < \infty$), and*

  (ii) *$\int \big(F(t)^{\#}x, x\big)\, dt < \infty$ for all $x \in H$* (or for all x in a linearly dense set in H_0), where $F(t)^{\#} = F(t)^{-1} P_{F(t)H}$.

Proof. (1) ⇒ (2). From Theorem 2.2 it follows that $F(\Delta)H = X_0^* N_0(X)\, dt(\Delta)$, $\Delta \in \mathscr{B}$ and $\dim X_0^* N_0(X) = n$. Therefore $F(\Delta) = \tilde{F}(\Delta) P_{H_0}$, $\Delta \in \mathscr{B}$, where $H_0 = X_0^* N_0(X)$ and $\tilde{F}$ is an $L(H_0, H_0)$-valued measure defined as in the preceding paragraph.

Let $\tilde{F}(t) = d\tilde{F}/dt$ and $F(t) = \tilde{F}(t) P_{H_0}$, $t \in [-\pi, \pi)$. Then $F(t)$ is the spectral density of F and by Theorem 2.2, $F(t) = X_0^* N_0(X)$ is a constant n-dimensional subspace of H dt a.e. Moreover, the stationary sequence $Y_n = X_n|_{H_0}$ is J_0-regular and by [4, 3.2.5], $\int \tilde{F}(t)^{-1}\, dt$ exists. Therefore (ii).

(2) ⇒ (1). From [4, 3.2.5], it follows that the n-dimensional stationary sequence $Y_n = X_n | H_0$ is J_0-regular. Since $M(X, A) = M(Y, A)$, $A \subset Z$, X is also J_0-regular. Obviously, by Theorem 2.2 (iv), $m(X) = n$. □

3. Sequences with multiplicity one

In this section we discuss the interpolation and the angle problem for processes with multiplicity one. Note that if $F(t) \in L(H, H)$, $F(t) \geq 0$, is a weakly integrable function with $F(t)H = H_0$ being a constant one-dimensional space dt a.e., then

$$F(t)x = |F(t)|(x, e)e, \qquad x \in H, \text{ and} \tag{3.1}$$

$$F(t)^{\#}x = |F(t)|^{-1}(x, e)e, \qquad x \in H, \tag{3.2}$$

where e is a fixed unit vector in H_0 and $|F(t)|$ denotes the operator norm of $F(t)$. From (3.2) and Theorem 2.4 with $n = 1$ we obtain the following result.

3.3. Theorem. *Let $X = \{X_n : n \in Z\}$ be a nonzero $L(H, K)$-valued stationary sequence. Then X is J_0-regular and $m(X) = 1$ if and only if X has a spectral density $F(t)$ of constant one dimensional range dt a.e. which satisfies one of the following three conditions:*

(i) *for every $x \in H$ either $\big(F(\cdot)x, x\big) = 0$ a.e. or else $\int \big(F(t)x, x\big)^{-1}\, dt < \infty$,*

(ii) *there exists $x \in H$ such that $\int \big(F(t)x, x\big)^{-1}\, dt < \infty$,*

(iii) *$\int |F(t)|^{-1}\, dt < \infty$.*

Now we turn to the problem of angle between the past and future. Suppose that $X = \{X_n : n \in Z\}$ is an $L(H, K)$-valued stationary sequence. The quantity

$$\varrho(X) = \sup\big\{\, |(u,v)| : u \in M\big(X;(-\infty,-1]\big), v \in M\big(X;[0,+\infty)\big), \|u\| = \|v\| = 1 \,\big\}$$

represents the cosine of the angle between past and future. A sequence is said to be of positive angle if $\varrho(X) < 1$. It is known ([4, 5.2.7]) that if $\varrho(X) < 1$, then X is J_0-regular. The remarkable results of Hunt, Muckenhoupt and Wheeden [1] show that:

(3.4) a one dimensional stationary sequence $\{X_n : n \in Z\}$ is of positive angle if and only if $\{X_n : n \in Z\}$ has the spectral density $f(t)$ which satisfies the condition (A_2), namely

(A_2) there exists a constant $c < \infty$ such that for every interval or complement of an interval $I \subset [-\pi, \pi)$

$$\Big(\int_I f(t)\,dt\Big)\Big(\int_I f(t)^{-1}\,dt\Big) \le c\big(dt(I)\big)^2.$$

If a function f satisfies (A_2) we write $f \in (\mathrm{A}_2)$.

Below is a characterization of stationary sequences of multiplicity one with positive angle. The characterization is a consequence of the fact that a J_0-regular process with multiplicity one resembles a one-dimensional sequence with spectral density $|F(t)|$ (see (3.1)). The characterization provides extensions and entirely different simple proofs of certain results in [6], where only the case of $\dim H = 2$ was considered.

3.5. Theorem. *Suppose that $X = \{X_n : n \in Z\} \subset L(H, K)$ is a nonzero stationary sequence. Then the following conditions are equivalent:*

(A) $\varrho(X) < 1$ *and* $m(X) = 1$,

(B) X *has the spectral density* $F(t)$ *of constant one-dimensional range* dt *a.e. which satisfies one of the following three conditions:*

(i) *for every* $x \in H$ *either* $\big(F(\cdot)x, x\big) = 0$ a.e. *or else* $\big(F(\cdot)x, x\big) \in (\mathrm{A}_2)$,

(ii) *there exists* $x \in H$ *such that* $\big(F(\cdot)x, x\big) \in (\mathrm{A}_2)$,

(iii) $|F(\cdot)| \in (A_2)$.

Proof. (A) $\Rightarrow$ (B). By [4, 5.2.7] and Corollary 2.4 X has spectral density $F(t)$ which has the form (3.1). Let $Y_n x = e^{\mathrm{in}\cdot}\sqrt{|F(\cdot)|}\,(x, e)$, where $n \in Z$, $x \in H$, and e is as in (3.1). Then the sequence $Y = \{Y_n : n \in Z\}$ is an $L\big(H, L^2(dt)\big)$-valued stationary sequence with the spectral density $F(t)$ and $\varrho(X) = \varrho(Y)$. Since $M(Y, A) = \overline{\mathrm{sp}}\{e^{\mathrm{in}\cdot}\sqrt{|F(\cdot)|} : n \in A\}$, $A \subset Z$, the sequence Y has positive angle if and only if the $L^2(dt)$-valued sequence $e^{\mathrm{in}\cdot}\sqrt{|F(\cdot)|}$ has a positive angle, which in view of (3.4), holds if and only if $|F(\cdot)| \in (\mathrm{A}_2)$.

(B) $\Rightarrow$ (A). From Theorem 3.3 it follows that $m(X)=1$. The same argument as above shows that $\varrho(X)=\varrho(Y)<1$.

The equivalence of (i), (ii) and (iii) follows from (3.1) provided $F(t)H$ is a constant one-dimensional space dt a.e.

3.6. Example. Suppose that $H=\mathbf{C}^2$, $|q(t)|=1$ and

$$F(t)=\begin{bmatrix} 1 & q(t) \\ \bar{q}(t) & 1 \end{bmatrix}.$$

If $\{X_n : n\in Z\}$ has a positive angle then it is J_0-regular and $F(t)\mathbf{C}^2 = \overline{\mathrm{sp}}\{1,\bar{q}(t)\}$ = constant, which holds if and only if $q(t)$ = constant (so we obtain Theorem 1 in [6]). In fact, from Theorem 3.5 it follows that $\varrho(X)<1$ if and only if $q(t)$ = constant. Since $|F(t)|=2$, this example also shows that condition $|F(t)|\in(A_2)$ in general is not sufficient for stationary sequences with multiplicity one to be of positive angle.

3.7. Example. There exist a stationary sequence $X=\{X_n : n\in Z\}$ with multiplicity one and spectral measure equivalent to the Lebesgue measure which fails to have the spectral density. To see this consider the $L\big(L^2(dt),L^2(dt)\big)$-valued stationary sequence defined by

$$(X_n f)(\cdot)=e^{\mathrm{in}\cdot}f(\cdot), \qquad f\in L^2(dt), n\in Z.$$

Then $m(X)=1$ and the spectral measure F of X is given by

$$\big(F(\Delta)f\big)(\cdot)=1_\Delta(\cdot)f(\cdot), \qquad \Delta\in\mathscr{B}, f\in L^2(dt).$$

Suppose that $F(t)\in L\big(L^2(dt),L^2(dt)\big)$ is a weakly integrable function such that $F(t)\geq 0$ and $\big(F(\Delta)f,g\big)=\int_\Delta\big(F(t)f,g\big)\,dt$, $\Delta\in\mathscr{B}$, $f,g\in L^2(dt)$. Setting $f\equiv 1$, $g=1_{(a,b)}$, where a, b are rationals we obtain

$$dt\big(\Delta\cap(a,b)\big)=\int_\Delta\big(F(t)1,1_{(a,b)}\big)\,dt, \qquad \Delta\in\mathscr{B}$$

and so there exists $\Delta_0\in\mathscr{B}$, $dt(\Delta_0)=0$ such that for all rationals a, b, $-\pi\leq a<b<\pi$ and $t\notin\Delta_0$

$$\big(F(t)1,1_{(a,b)}\big)_{L^2}=1_{(a,b)}(t).$$

Hence if $t\notin\Delta_0$ then $F(t)1\perp\overline{\mathrm{sp}}\{1_{(a,b)} : (a,b)\text{ not containing } t\}=L^2(dt)$, which leads to contradiction.

3.8. Remark. In [5], Masani showed that an orthogonally scattered measure with a nonatomic control measure does not possess a Bochner density with respect to any control measure. Our example 3.7 shows that an orthogonally scattered measure need not even have a density in the sense of Pettis. This point should be of independent interest.

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