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APPLICATION OF HOARE'S THEOREM TO SYMMETRIES OF RIEMANN SURFACES

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Abstract. Let X be a compact Riemann surface of genus $g > 1$. A symmetry S of X is an anticonformal involution. We write $|S|$ for the number of connected components of the fixed points set of S. Suppose that X admits two distinct symmetries S_1 and S_2; then we find a bound for $|S_1| + |S_2|$ in terms of the genus of X and the order of S_1S_2. We discuss circumstances in which the bound is attained, showing that this occurs only for hyperelliptic surfaces. In this way we generalize a theorem of S.M. Natanzon.

1. Introduction

Let X be a compact Riemann surface of genus $g > 1$. A *symmetry* S of X is an anticonformal involution $S\colon X \to X$ and a Riemann surface that admits a symmetry is called *symmetric*. By Harnack's theorem [1, 5, 9], the fixed-point set of S is either empty or consists of $k \le g+1$ disjoint simple closed curves or, as we shall call them, *mirrors*. We write $|S|$ for the number of mirrors of S, so that $|S|$ is the number of components of the fixed-point set of S.

Suppose that X admits two distinct symmetries S_1, S_2. If $|S_1| = |S_2| = g+1$ then by a theorem of Natanzon [8], S_1 and S_2 commute and X is hyperelliptic. Usually, however, the total number of mirrors of S_1 and S_2 is much less than $2g+2$. In Theorem 3 we find a sharp upper bound for $|S_1| + |S_2|$ in terms of the genus of X and the order of S_1S_2. In particular, if S_1 and S_2 do not commute then $|S_1| + |S_2| \le g+2$ (Corollary 3).

Theorem 3 follows from a recent theorem of Hoare [3] on subgroups of NEC groups. Using this theorem we find in Section 5 a graphical technique that allows us to determine $|S_1|$ and $|S_2|$ in terms of the signature of a certain NEC group. We then find, in Theorem 2, a formula for $|S_1| + |S_2|$ involving this signature.

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Finally, in Section 7, we discuss circumstances in which the bound of Theorem 3 is attained. We show that this is the case only for hyperelliptic surfaces, a result which extends Natanzon's original theorem.

2. NEC groups

A non-Euclidean crystallographic (NEC) group is a discrete group of isometries of the hyperbolic plane $\mathbf{H}$. As usual, we can choose for our model of $\mathbf{H}$, the upper half-plane with the Poincaré metric. Then every isometry is given by a Möbius or anti-Möbius transformation (the latter being a Möbius transformation composed with $z \to -\bar{z}$). We shall assume that an NEC group has compact quotient space. If Δ is such a group then its algebraic and geometric structure is determined by its *signature*

$$\sigma(\Delta) = \big(h; \pm; [m_1, \dots, m_r]; \big\{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\big\}\big). \tag{1}$$

The quotient space $\mathbf{H}/\Delta$ is then a surface, possibly with boundary, and in the signature h is the genus of $\mathbf{H}/\Delta$, k is the number of its boundary components and $+$ or $-$ is used according to whether the surface is orientable or not. The integers $m_1, \dots, m_r$ are the *proper periods* of Δ and represent the branching over interior points of $\mathbf{H}/\Delta$ in the natural projection $p\colon \mathbf{H} \to \mathbf{H}/\Delta$. The k brackets $(n_{i1}, \dots, n_{is_i})$ are the period cycles and represent the branching over the i^{th} hole. The integers n_{ij} are the *link periods*. The maximal finite subgroups of Δ are either cyclic of order m_i $(i = 1, \dots, r)$ or dihedral of order $2n_{ij}$ $(i = 1, \dots, k$, $j = 1, \dots, s_i)$ and each period represents a conjugacy class of such subgroups.

Associated to the signature (1) we have a presentation of the group Δ and a formula for the area of a fundamental domain for Δ. If $\sigma(\Delta)$ has a $+$ sign then Δ has generators

$$\begin{array}{rl} x_1, \dots, x_r & \text{(elliptic elements)} \\ c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k} & \text{(reflections)} \\ e_1, \dots, e_k & \text{(orientation preserving elements)} \\ a_1, b_1, \dots, a_h, b_h & \text{(hyperbolic elements)} \end{array}$$

and relations

$$\begin{aligned} &x_i^{m_i} = 1 \qquad (i = 1, \dots, r) \\ &c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1, \qquad i = 1, \dots, k,\ j = 1, \dots, s_i \\ &e_i c_{i0} e_i^{-1} = c_{is_i} \qquad (i = 1, \dots, k) \\ &x_1 x_2 \cdots x_r e_1 e_2 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} = 1. \end{aligned}$$

If $\sigma(\Delta)$ has a minus sign then we just replace the hyperbolic generators a_i, b_i by glide reflection generators $a_1, \dots, a_h$ and replace the last relation by

$$x_1 x_2 \cdots x_r e_1 e_2 \cdots e_k a_1^2 a_2^2 \cdots a_h^2 = 1.$$

The hyperbolic area of a fundamental domain for Δ is given by

$$(2) \qquad \mu(\Delta) = 2\pi\Big(\varepsilon h - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + k + \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{ij}}\Big)\Big),$$

(see [11]) where $\varepsilon = 2$ if there is a $+$ sign and $\varepsilon = 1$ if there is a $-$ sign. If $\Lambda < \Delta$ is a subgroup of finite index then

$$(3) \qquad |\Delta : \Lambda| = \mu(\Lambda)/\mu(\Delta).$$

For more details about signatures see ([1, 6, 11]).

Now let X be a compact Riemann surface of genus $g > 1$. Then there is a Fuchsian surface group Γ (i.e. an NEC group of signature $(g; +; [\,]\{\ \})$ such that $X = \mathbf{H}/\Gamma$. Let G be a group of automorphisms of X containing a symmetry S. Let Δ be the group generated by all the liftings to $\mathbf{H}$ of the elements of G, then Δ is an NEC group and there is a smooth homomorphism $\theta\colon \Delta \to G$ whose kernel is Γ. (θ is smooth means that θ maps finite subgroups of Δ isomorphically into G.)

Let $L = \langle S\rangle$, (the group generated by S) and $\Lambda = \theta^{-1}(L)$. We then have

$$(4) \qquad X/L = \mathbf{H}/\Gamma/\Lambda/\Gamma = \mathbf{H}/\Lambda.$$

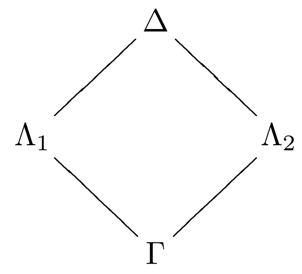
For any NEC group Φ, we let Φ^+ (the *canonical Fuchsian group* of Φ) denote the subgroup of index one or two consisting of the orientation-preserving elements of Φ. As Γ contains only orientation-preserving transformations and is torsion-free, Λ contains orientation-reversing elements. As $|\Lambda : \Gamma| = 2$, $\Lambda^+ = \Gamma$ so that Λ has no elliptic elements. Therefore Λ has a signature of the form

$$(5) \qquad \big(h_0; \pm; [\,]\{(\ )^k\}\big)$$

where the notation signifies that there are k empty period cycles. Thus k is the number of period cycles of Λ which is the number of mirrors of S. Conversely it is clear that if there is an NEC group Λ with signature (5) and containing Γ as subgroup of index two then Λ defines a symmetry on X with exactly k mirrors. This shows that the number of mirrors of a symmetry is an algebraic invariant of an NEC group.

3. Dihedral group actions

We now suppose that X admits two distinct symmetries S_1, S_2. We let $G = \langle S_1, S_2\rangle$ so that G is isomorphic to the dihedral group D_n of order $2n$. We also let Δ be the lift of G to $\mathbf{H}$, $L_i = \langle S_i\rangle$ and $\Lambda_i = \theta^{-1}(L_i)$, $(i = 1, 2)$, where $\theta\colon \Delta \to G$ is the homomorphism of Section 2. We have a subgroup diagram



where Λ_i has signature $\big(h_i; \pm; [\,]\{(\,)^{k_i}\}\big)$.

Now given the signature of an NEC group, Hoare's theorem [3] gives us a procedure for calculating the signature of a subgroup, given the permutation representation of the group on the cosets. We will use the techniques of this theorem to compute k_1, k_2 from the signature of Δ. We find that the algebra in Hoare's results gives us fairly precise information about $k_1 + k_2 = |S_1| + |S_2|$.

We shall find the following simple lemma, also used by Hoare, useful.

Lemma 1. *Let D_n act by right multiplication on the n L_i-cosets $(i = 1, 2)$. If n is even then S_1 fixes exactly two L_1-cosets and no L_2-cosets while if n is odd then S_1 fixes exactly one L_1-coset and one L_2-coset.*

Proof. Let $Q = S_1 S_2$. Then the n L_1-cosets are $L_1 Q^r$ $(r = 0, \dots, n-1)$ and the action of S_1 is $L_1 Q^r \to L_1 Q^r S_1$. If n is even the only fixed cosets correspond to $r = 0, n/2$ while if n is odd the only fixed cosets correspond to $r = 0$. If in the action on the L_2-cosets, S_1 fixes $L_2 Q^r$ then $Q^r S_1 Q^{-r} = S_2$ and thus $Q^{2r+1} = I$. Hence there are no fixed cosets if n is even and one if n is odd.

Another way of stating this lemma is as follows:

Lemma 1′. *Let D_n act by right multiplication on the n L_1-cosets. If n is even then S_1 fixes exactly two L_1-cosets and S_2 fixes no L_1-cosets. If n is odd then both S_1 and S_2 fix exactly one L_1-coset.*

We now explain how Hoare's ideas apply. Let $c \in \Delta$ be a reflection so that θ maps c to a conjugate of S_1 or S_2 in $G \cong D_n$. The action of c on the Λ_i-cosets is the same as the action of $\theta(c)$ on the L_i-cosets. Suppose that d is another reflection in Δ and that cd has finite order μ. Then cd, in its action on the Λ_i-cosets, gives a product of disjoint μ-cycles. (Otherwise some power of cd, not equal to the identity is an elliptic element of Γ which is impossible as Γ is a surface group.) From Lemma 1, each of c and d fixes two cosets or no cosets if μ is even and each fixes 1 coset if μ is odd. By Theorem 1, II(ii) of [3], (or an extension of Lemma 1) each cycle of cd contains either two or none of these fixed cosets.

Suppose that c fixes a coset $\Lambda_1\alpha$. Then $c_\alpha = \alpha c \alpha^{-1}$ is a reflection in Λ_i which is called a reflection induced by c or just an *induced reflection*. Consider a cycle of cd which contains α (we now identify α and $\Lambda_i\alpha$). By the above discussion this cycle contains another fixed point β of c, if μ is even, or a fixed point β of d, if μ is odd. In the first case we say that c_α and c_β are *linked* and in the second case we say that c_α and d_β are linked and write $c_\alpha \sim c_\beta$ in the first case and $c_\alpha \sim d_\beta$ in the second case. Once all the links are known we put them together to form *chains* and then each chain (by Hoare's results) gives a period cycle in Λ_i.

Note. In general $c_\alpha \sim c_\beta$ (or $c_\alpha \sim d_\beta$) means that $c_\alpha c_\beta$ (or $c_\alpha \sim d_\beta$) has finite order and so gives a link period in a period cycle of Λ_i. In our cases the period cycles of the subgroups Λ_i are empty so that $c_\alpha \sim c_\beta$ means that $(c_\alpha c_\beta)^1 = 1$ or $c_\alpha = c_\beta$ (or $c_\alpha \sim d_\beta$ in the other case).

Before we consider the general situation it might be helpful to look at some examples.

4. Examples

Example 1. Let $n = 12$ and Δ have signature

$$\big(0; +; [\,]; \{(2,4,4,6,3)\}\big),$$

and presentation

$$\big\langle c_i,\ (1 \le i \le 5) \big| c_i^2 = 1\ (1 \le i \le 5),\\ (c_1c_2)^2 = (c_2c_3)^4 = (c_3c_4)^4 = (c_4c_5)^6 = (c_5c_1)^3 = 1\big\rangle.$$

Consider the following homomorphism

$$\theta\colon \Delta \to D_{12} = \big\langle S_1, Q \big| S_1^2 = (S_1Q)^2 = Q^{12} = 1\big\rangle;$$

$$\theta(c_1) = S_1, \quad \theta(c_2) = S_1Q^6, \quad \theta(c_3) = S_1Q^9, \quad \theta(c_4) = S_1Q^6, \quad \theta(c_5) = S_1Q^8.$$

We have the following decomposition of D_{12} as a union of L_1-cosets ($L_1 = \langle S_1\rangle$):

$$D_{12} = \bigcup_{i=0}^{11} L_1Q^i$$

and letting i denote the coset L_1Q^i we obtain the following permutation representation of D_{12} on the right L_1-cosets,

$$\begin{array}{lcccccccccccc}
c_1 \to & (0) & (6) & (1 & 11) & (2 & 10) & (3 & 9) & (4 & 8) & (5 & 7) \\
c_2 \to & (3) & (9) & (1 & 5) & (2 & 4) & (7 & 11) & (8 & 10) & (0 & 6) \\
c_3 \to & (0 & 9) & (1 & 8) & (2 & 7) & (3 & 6) & (4 & 5) & (10 & 11) \\
c_4 \to & (3) & (9) & (1 & 5) & (2 & 4) & (7 & 11) & (8 & 10) & (0 & 6) \\
c_5 \to & (4) & (10) & (1 & 7) & (2 & 6) & (3 & 5) & (0 & 8) & (11 & 9).
\end{array}$$

The induced reflections are

$$c_{1,0}, \quad c_{1,6}, \quad c_{2,3}, \quad c_{2,9}, \quad c_{4,3}, \quad c_{4,9}, \quad c_{5,4}, \quad c_{5,10}.$$

We now form the products

$$\begin{array}{ll}
c_1c_2 \to & (0\ \ 6)\ (1\ \ 7)\ (2\ \ 8)\ (3\ \ 9)\ (4\ \ 10)\ (5\ \ 11) \\
c_2c_3 \to & (0\ \ 3\ \ 6\ \ 9)\ (1\ \ 4\ \ 7\ \ 10)\ (2\ \ 5\ \ 8\ \ 11) \\
c_3c_4 \to & (0\ \ 9\ \ 6\ \ 3)\ (1\ \ 10\ \ 7\ \ 4)\ (2\ \ 11\ \ 8\ \ 5) \\
c_4c_5 \to & (1\ \ 3\ \ 5\ \ 7\ \ 9\ \ 11)\ (2\ \ 4\ \ 6\ \ 8\ \ 10\ \ 0) \\
c_5c_1 \to & (1\ \ 5\ \ 9)\ (2\ \ 6\ \ 10)\ (3\ \ 7\ \ 11)\ (4\ \ 8\ \ 0).
\end{array}$$

As fixed points $0, 6$ of c_1 belong to the same cycle of c_1c_2 we have a link $c_{1,0} \sim c_{1,6}$. Similarly, we have links $c_{2,3} \sim c_{2,9}$, (from c_2c_3), $c_{4,3} \sim c_{4,9}$ (from c_3c_4), $c_{4,3} \sim c_{4,9}$ (from c_4c_5), $c_{5,4} \sim c_{5,10}$ (from c_4c_5), $c_{5,4} \sim c_{1,0}$ (from c_5c_1), $c_{5,10} \sim c_{1,6}$ (from c_5c_1). We then get the chains

$$c_{1,0} \sim c_{1,6} \sim c_{5,10} \sim c_{5,4} \sim c_{1,0}$$
$$c_{2,3} \sim c_{2,9} \sim c_{2,3}$$
$$c_{4,3} \sim c_{4,9} \sim c_{4,3}.$$

As there are three chains, Hoare's theorem implies that Λ_1 has three period cycles and hence $|S_1| = 3$. Similarly, we have the action of the generators on the L_2-cosets,

$$\begin{array}{ll}
c_1 \to & (0\ \ 11)\ (1\ \ 10)\ (2\ \ 9)\ (3\ \ 8)\ (4\ \ 7)\ (5\ \ 6) \\
c_2 \to & (0\ \ 5)\ (1\ \ 4)\ (2\ \ 3)\ (6\ \ 11)\ (7\ \ 10)\ (8\ \ 9) \\
c_3 \to & (4)\ (10)\ (0\ \ 8)\ (1\ \ 7)\ (2\ \ 6)\ (3\ \ 5)\ (9\ \ 11) \\
c_4 \to & (0\ \ 5)\ (1\ \ 4)\ (2\ \ 3)\ (6\ \ 11)\ (7\ \ 10)\ (8\ \ 9) \\
c_5 \to & (0\ \ 7)\ (1\ \ 6)\ (2\ \ 5)\ (3\ \ 4)\ (8\ \ 11)\ (9\ \ 10),
\end{array}$$

and notice how Lemma 1 applies: e.g. c_1 fixes 2 L_1-cosets and 0 L_2-cosets, c_3 fixes 0 L_1-cosets and 2 L_1-cosets. The induced reflections are $c_{3,4}$ and $c_{3,10}$ and from the products c_3c_4 and c_4c_5 we find the single chain $c_{3,4} \sim c_{3,10} \sim c_{3,4}$. Thus $|S_2| = 1$. This example suggests that Lemma 1 should restrict the size of $|S_1| + |S_2|$. As we shall see in Theorem 1 this is indeed the case, but before then we consider another example which illustrates other aspects of Hoare's techniques.

Example 2. Let $n = 3$ and Δ have signature

$$\big(0; +; [3]; \{(3,3,3)\}\big),$$

and presentation

$$\big\langle e, x, c_0, c_1, c_2, c_3 \big| c_i^2 = 1 \ (0 \leq i \leq 3), \\ x^3 = ex = (c_0c_1)^3 = (c_1c_2)^3 = (c_2c_3)^3 = ec_0e^{-1}c_3 = 1 \big\rangle.$$

Consider the following homomorphism

$$\theta\colon \Delta \to D_3 = \langle S_1, Q \,|\, S_1^2 = (S_1Q)^2 = Q^3 = 1\rangle;$$

$$\theta(c_0) = S_1, \quad \theta(c_1) = S_1Q, \quad \theta(c_2) = S_1, \quad \theta(c_3) = S_1Q, \quad \theta(e) = Q.$$

With a similar notation to Example 1,

$$D_3 = \bigcup_{i=0}^{2} L_1Q^i$$

and we have the following permutation representation on the cosets

$$\begin{array}{rlll} c_0 \to & (0) & (1 & 2) \\ c_1 \to & (0 & 1) & (2) \\ c_2 \to & (0) & (1 & 2) \\ c_3 \to & (0 & 1) & (2) \\ e \to & (0 & 1 & 2) \end{array}$$

giving induced reflections $c_{0,0}$, $c_{1,2}$, $c_{2,0}$, $c_{3,2}$ on Λ_1. As

$$\begin{array}{rlll} c_0c_1 \to & (0 & 1 & 2), \\ c_1c_2 \to & (0 & 2 & 1), \\ c_2c_3 \to & (0 & 1 & 2), \\ (ec_0e^{-1})c_3 \to & (0) & (1) & (2), \end{array}$$

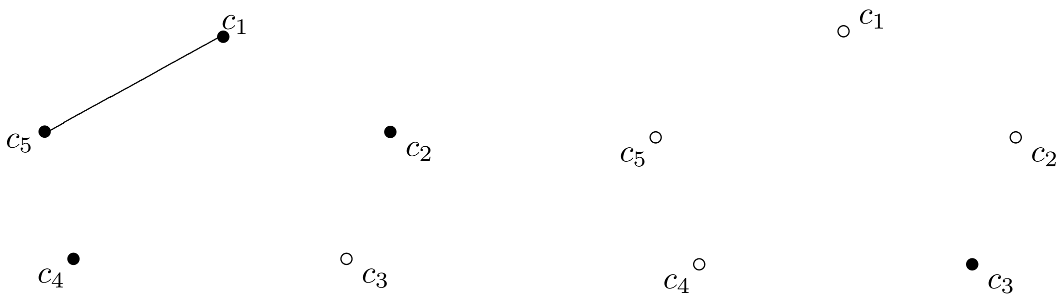
we get links $c_{0,0} \sim c_{1,2}$, $c_{1,2} \sim c_{2,0}$, $c_{2,0} \sim c_{3,2}$, from the first three products. In the final product c_0 fixes 0, ec_0e^{-1} fixes 2 and c_3 also fixes 2. By Theorem 1 (II(ii)) and Example 1 of [3], we have a link $c_{3,2} \sim c_{0,0}$. This gives the single chain $c_{0,0} \sim c_{1,2} \sim c_{2,0} \sim c_{3,2} \sim c_{0,0}$ and thus $|S_1| = 1$. As S_2 is conjugate to S_1 in D_3, $|S_2| = 1$ as well. As we shall see, this case is typical of the cases when n, and hence all the periods of Δ are odd.

5. A graphical technique

We now describe a method that enables us to perform the computations in Section 4 fairly automatically. The examples there show that the number of mirrors depend on the parity of n, the parities of the link periods and the parities of the integers i in S_1Q^i. We suppose that n is even. If $\theta(c_j) = S_1Q^i$ with i even then c_j gives a permutation of the cosets with two fixed points α, β; if i is odd then there are no fixed points. If the link period n_j is even (recall the relation $(c_jc_{j+1})^{n_j} = 1$) then α, β lie in the same cycle of the (permutation induced) by c_jc_{j+1}, giving a link $c_{j,\alpha} \sim c_{j,\beta}$. (All suffices are modulo s the length of the period cycle.) If n_{j-1} is even then we get a link $c_{j,\beta} \sim c_{j,\alpha}$ giving a chain $c_{j,\alpha} \sim c_{j,\beta} \sim c_{j,\alpha}$.

If n_j is odd then the cycle $c_j c_{j+1}$ containing α also contains a fixed point γ of c_{j+1} and the cycle containing β contains a fixed point δ of c_{j+1}. We then get links $c_{j,\alpha} \sim c_{j+1,\delta}$ and $c_{j,\beta} \sim c_{j+1,\delta}$. If n_{n+1} is odd we get a link $c_{j+1,\delta} \sim c_{j+2,\varepsilon}$ and so on (and this argument shows that $|S_1| = 1$ if all the link periods of the period cycle are odd). If n_{j+1} is even we get $c_{j+1,\delta} \sim c_{j+1,\gamma} \sim c_{j,\alpha} \sim \cdots$. This process can be illustrated graphically as follows. Suppose that the period cycle has length s. Represent each generator c_j by a vertex of an s-gon labelled c_j. If $\theta(c_j) = S_1 Q^i$, colour the vertex black or white according as to whether i is even or odd respectively. If c_j and c_{j+1} are two black vertices we join them by a black edge if and only if n_j is odd. Then $|S_1|$ is the number of black components. To find $|S_2|$ we just change the colour of the vertices (as $S_2 Q^i = S_1 Q^{i+1}$, changing the parity of the index) and still join black vertices by a black edge if n_j is odd.

Example. From Example 1 of Section 4 we obtain the following pictures.



$|S_1| = 3$ $\qquad$ $|S_2| = 1$

From these graphical ideas we get a simple numerical result.

Notation. We let $t = |S_1| + |S_2|$ the total number of fixed curves of S_1 and S_2.

Theorem 1. *With the previous notation, let n be even and suppose that Δ has a single period cycle and at least one even period. Then t is equal to the number of even link periods in the period cycle.*

Proof. Suppose that the period cycle has length s. In terms of the above graphs we need to show that if the total number of black edges in one graph is u (this is the number of odd link periods) then the number of black components in both graphs is $s - u$. This is by induction on u. If $u = 0$ then the total number of black components is the number of black vertices in both graphs which is the number of vertices in one graph, namely s. Suppose the result is true if there are u black edges. Now increase the number of black edges by one. A black edge cannot join vertices of different colours for if $\theta(c_j) = SQ^{i_1}$ and $\theta(c_{j+1}) = SQ^{i_2}$

then $\theta(c_j c_{j+1}) = Q^{i_2-i_1}$. As n is even and $c_j c_{j+1}$ has odd period, $i_2 \equiv i_1 \bmod 2$. Thus if we introduce another black edge it joins two black vertices in one of the graphs so that the total number of black components is $(s-u)-1 = s-(u+1)$, and the proof follows.

We now extend this result to the case when Δ is a general NEC group. Let us define a non-empty period cycle to be *odd* if it only contains odd link periods. Then, as we saw near the beginning of Section 5, an odd period cycle induces just one mirror on U/Λ_1 and one on U/Λ_2 and so contributes 2 to $t = |S_1| + |S_2|$. Now consider an empty period cycle. The corresponding generators are c_i and e_i. These obey the relation $e_i c_i e_i^{-1} = c_i$. With the previous notation let $\theta(c_i) = S_1$. As the centralizer of S_1 in G is $\{1\}$ if n is odd and $\{1, Q^{n/2}\}$ if n is even, $\theta(e_i) = 1$ if n is odd while $\theta(e_i) = 1$ or $Q^{n/2}$ if n is even. In the case when n is even the permutation on the L_1-cosets induced by c_i is

$$(0)\left(\frac{n}{2}\right)(1\ n-1)(2\ n-2)\ldots$$

and the permutation induced by e_i is either

$$(0)(1)\ldots(n-1) \qquad \text{if } \theta(e_i) = 1$$

or

$$\left(0\ \frac{n}{2}\right)\left(1\ \frac{n}{2}+1\right)\ldots \qquad \text{if } \theta(e_i) = Q^{n/2}.$$

The induced reflections are $c_{i,0}$ and $c_{i,n/2}$. If $\theta(e_2) = 1$ we get two chains $c_{i,0} \sim c_{i,0}$ and $c_{i,n/2} \sim c_{i,n/2}$ and if $\theta(e_i) = Q^{n/2}$ we get a single chain $c_{i,0} \sim c_{i,n/2} \sim c_{i,0}$. (Compare Example 2 in Section 4.) Thus if $e_i \in \Gamma$ then every empty period cycle gives rise to two mirrors of S_1 (and none of S_2 by Lemma 1) and so contributes 2 to t, whilst if $e \in \Gamma$ then every empty period cycle gives rise to one mirror of S_1, and none of S_2, contributing 1 to t.

Notation. We let α be the number of even link periods of Δ, β the number of odd period cycles of Δ, γ the number of empty period cycles of Δ and δ the number of e_i generators associated to the empty period cycles which do not belong to Γ.

We proved the following result.

Theorem 2. *With the above notation:*

(i) *if n is even then $t = \alpha + 2\beta + 2\gamma - \delta$,*

(ii) *if n is odd then $t = 2\beta + 2\gamma$.*

Note. If n is even then we usually let $\theta(e_i) = 1$ so that $\delta = 0$. If n is odd then $|S_1| = |S_2| = \beta + \gamma$.

6. The main application

Theorem 3. *Let S_1S_2 have order n.*

(i) *If n is even then $t \leq (4g/n)+2$.*
(ii) *If n is odd then $t \leq \big((2g-2)/n\big)+4$.*

Proof. (i) Let Δ have signature (1). Then by [11], Δ^+ is a Fuchsian group of signature

$$\big(h^+;+;\big[m_1,m_1,\ldots,m_r,m_r,n_{11},\ldots,n_{1s_1},\ldots,n_{ks_k}\big]\{\ \}\big)$$

where $h^+=\varepsilon h+k-1$, ε being defined in (2). As there is an epimorphism from Δ^+ onto C_n we know by Harvey's theorem [2] that

(i) If m is the least common multiple (lcm) of the periods of Δ^+ then $m|n$ and also these periods obey the *lcm condition* which says that the lcm of all the periods with one period deleted is also equal to m.
(ii) If $h^+=0$, then $m=n$.
(iii) The number of periods of Δ^+ cannot equal 1 and if $h^+=0$ their number if ≥ 3.
(iv) If $2|m$ then the number of periods divisible by the maximum power of 2 dividing m is even.

Besides the notation α, β, γ introduced earlier we also let η equal the number of period cycles with at least one even link period. Then $k=\beta+\gamma+\eta$. Our proof is just to apply the area and index formulae (2) and (3) but we need to break up the calculation into several cases.

(a) $r\geq 1$. From (2) and (3)

$$\begin{aligned}2g-2 &\geq 2n\Big(-2+k+\sum_{i=1}^{n}\Big(1-\frac{1}{m_1}\Big)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1-\frac{1}{n_{ij}}\Big)\Big)\\ &\geq 2n\Big(-2+\beta+\gamma+\eta+\frac{1}{2}+\frac{\alpha}{4}\Big).\end{aligned}$$

If $\alpha\neq 0$, then $\eta\geq 1$ so that

$$2g-2\geq 2n\Big(\beta+\gamma+\frac{\alpha}{4}-\frac{1}{2}\Big)\geq 2n\Big(\frac{t}{4}-\frac{1}{2}\Big).$$

Hence $t\leq 4(g-1)/n+2<(4g/n)+2$. If $\alpha=0$ then $\eta=0$ and now

$$2g-2\geq 2n\Big(\beta+\gamma-\frac{3}{2}\Big)\geq 2n\Big(\frac{t-3}{2}\Big).$$

Therefore

$$t\leq\frac{2(g-1)}{n}+3<\frac{4g}{n}+2,$$

the final inequality being true unless $n \geq 2g+2$; but in this case it follows from the proof of Theorem 4 of [12] that Δ^+ is a triangle group or has signature $\big(0;+;[2,2,m,m]\{\ \}\big)$ with m odd. As Δ^+ has at least one proper period, Δ has at most two even link periods so that $t \leq 2$, by Theorem 2.

We can now assume that $r = 0$.

(b) $\alpha = 0$. We first assume that Δ has no link periods and so has signature $\big(h;\pm;[\ ]\{(\ )^k\}\big)$ with $\varepsilon h - 2 + k \geq 1$ as $\mu(\Delta) > 0$. Then $t = 2k$ by Theorem 2. Also $2g-2 = 2n(\varepsilon h - 2 + k) \geq 2n$ giving $n \leq g-1$. Hence $2g-2 \geq 2n\big((t/2)-2\big)$ giving

$$t \leq \frac{2(g-1)}{n} + 4 < \frac{4g}{n} + 2$$

as $n < g+1$.

If Δ has link periods then they are all odd. If there are β_0 odd link periods then $\beta_0 \geq 2$ by (i) of Harvey's theorem, and clearly $\beta_0 \geq \beta$. Also, by (iii) of Harvey's theorem $h^+ > 0$ as m is odd and n is even. Therefore $\mu(\Delta^+) \geq 2h^+ - 2 + \frac{2}{3}\beta_0 \geq \frac{4}{3}$. As $2g-2 = n\eta(\Delta^+)$, $n < \frac{3}{2}(g-1)$. Now

$$\begin{aligned} 2g-2 &= 2n\Bigg(\varepsilon h - 2 + k + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{ij}}\Big)\Bigg) \\ &\geq 2n\Big(k - 2 + \frac{2}{3}\Big) = 2n\Big(\frac{t}{2} - \frac{4}{3}\Big). \end{aligned}$$

Therefore $t \leq \big(2(g-1)/n\big) + 8/3 < (4g/n) + 2$ as $n < \frac{3}{2}(g+1)$.

Thus from now on we can assume that $r = 0$, $\alpha \geq 1$ and thus $\eta \geq 1$.

(c) We now assume that $h^+ > 0$. Then

$$\begin{aligned} 2g-2 &\geq 2n\Big(\varepsilon h + \beta + \gamma + \eta - 2 + \frac{\alpha}{4}\Big) \\ &= 2n\Big(\varepsilon h + \frac{t}{4} + \frac{\beta+\gamma}{2} + \eta - 2\Big). \end{aligned}$$

If $\beta = \gamma = 0$, then $h^+ = \varepsilon h + \eta - 1 \geq 1$ so that $2g-2 \geq 2nt/4$ and $t \leq (4g-4)/n$. If $\beta + \gamma > 0$, then $\varepsilon h + \eta + (\beta+\gamma)/2 \geq \frac{3}{2}$, and $2g-2 \geq 2n\big((t/4) - \frac{1}{2}\big)$ giving

$$t \leq \frac{4(g-1)}{n} + 2 < \frac{4g}{n} + 2.$$

(d) We now assume that $h^+ = 0$ so that $k = 1$. If the unique period cycle is odd or empty then $t = 2$ so that we can assume that it contains even link periods.

Thus Δ^+ has signature $\big(0;+;[n_1,\ldots,n_t,\nu_1,\ldots,\nu_\tau]\big)$ where the n_i are even and the ν_i are odd. Then

$$\begin{aligned}\frac{2g-2}{n} &= -2+\sum_{i=1}^{t}\Big(1-\frac{1}{n_i}\Big)+\sum_{i=1}^{\tau}\Big(1-\frac{1}{\nu_i}\Big)\\ &\ge -2+\frac{t}{2}+\sum_{i=1}^{\tau}\Big(1-\frac{1}{\nu_i}\Big).\end{aligned}$$

Now if $\tau>0$ then $\tau\ge 2$ by Harvey's theorem, part (i). Therefore

$$\frac{2g-2}{n}\ge -2+\frac{t}{2}+\frac{4}{3}$$

and thus

$$t\le\frac{4g-4}{n}+\frac{4}{3}<\frac{4g}{n}+2.$$

Thus we can assume that $\tau=0$, i.e. that all the link periods in the period cycle are even. We assume that u of them are equal to 2 and v are greater than 2. We denote the latter by $n_1,\ldots,n_v$. Then

$$\begin{aligned}2g-2 &= n\bigg(-2+\frac{u}{2}+\sum_{i=1}^{v}\Big(1-\frac{1}{n_i}\Big)\bigg)\ge n\Big(-2+\frac{u}{2}+\frac{3v}{4}\Big)\\ &= n\Big(-2+\frac{u+v}{2}+\frac{v}{4}\Big)\ge n\Big(-1+\frac{t}{2}\Big)\end{aligned}$$

if $v\ge 4$, by Theorem 2. Now $t\le\big(4(g-1)/n\big)+2<(4g/n)+2$. If $v=3$, then $\{n_1,n_2,n_3\}\ne\{4,4,4\}$, $\{4,4,6\}$ or $\{4,6,6\}$ by conditions (i) and (iv) of Harvey's theorem and thus $\sum_1^3 1/n_i\le\frac{1}{2}$, so that

$$2g-2=n\bigg(1+\frac{u}{2}-\sum_{1}^{3}\frac{1}{n_i}\bigg)\ge n\Big(\frac{u+1}{2}\Big).$$

Thus

$$t=u+3\le\frac{4(g-1)}{n}+2<\frac{4g}{n}+2.$$

If $v=2$ then by the lcm condition Δ^+ has signature $\big(0;+;[2,2,\ldots,n,n];\{\ \}\big)$ and as n is even $t=(4g/n)+2$. We cannot have $v=1$ by the lcm condition and if $v=0$ then $n=2$ and $t=2g+2=(4g/2)+2$.

(ii) This is much easier! We have

$$2g-2\ge 2n(-2+k)=2n(-2+\beta+\gamma)=2n\Big(\frac{t}{2}-2\Big)$$

by Theorem 2. Thus $t\le\big(2(g-1)/n\big)+4$.

Notes. The bound in Theorem 3 (i) is only obtained if and only if Δ has signature $\big(0;+;[\,];\{2,\ldots,2,n,n\}\big)$ with n even. A similar bound, but with topological hypotheses, has been found by Natanzon in [9].

7. Attainment of the bounds

We show that the bounds in Theorem 3 are attained and then study some of the Riemann surfaces which attain these bounds. First we prove the following lemma which is of independent interest.

Lemma 2. *Let* Λ *be a Fuchsian group of signature* $\big(0;+;[m_1,\dots,m_r];\{\ \}\big)$ *and suppose that there exists a homomorphism* $\varphi\colon \Lambda\to C_n$. *If* Λ^* *is an NEC group of signature* $\big(0;+;[\ ]\{(m_1,\dots,m_r);\}\big)$ *with* $(\Lambda^*)^+=\Lambda$ *then* φ *extends to a homomorphism* $\varphi^*\colon\Lambda^*\to D_n$. *This extension is essentially unique.*

Proof. Suppose that Λ has presentation $\big\langle x_1,\dots,x_r \,|\, x_1^{m_1}=\cdots=x_r^{m_r}=x_1x_2\cdots x_r=1\big\rangle$ and suppose that C_n is generated by q and that $\varphi(x_i)=q^{\sigma_i}$. Then $n\,|\,\sum_{i=1}^r\sigma_i$. Let

$$D_n=\big\langle y,q\,|\,y^2=q^n=(yq)^2=1\big\rangle$$

and

$$\Lambda^*=\big\langle c_1,\dots,c_r\,|\,c_1^2=\cdots=c_r^2=(c_1c_2)^{m_1}=\cdots=(c_{r-1}c_r)^{m_{r-1}}=(c_rc_1)^{m_r}=1\big\rangle,$$

with $c_ic_{i+1}=x_i$.

If we let $\varphi^*(c_1)=y$ then we must have $\varphi^*(c_2)=yq^{\sigma_1}$, $\varphi^*(c_3)=yq^{\sigma_1+\sigma_2}$, $\dots$, $\varphi^*(c_r)=yq^{\sigma_1+\cdots+\sigma_{r-1}}$. $\varphi^*\colon\Lambda^*\to D_n$ is a homomorphism that extends φ, and it is unique once the image of c_1 is known. As the image of c_1 must be a reflection any two extensions differ by an automorphism of D_n.

Theorem 4. (i) *The bound in Theorem* 3 (i) *is attained for every even* n *and every* $g>1$ *such that* $n|4g$.

(ii) *The bound in Theorem* 3 (ii) *is attained for every odd* n *and every* $g>1$ *such that* $n|g-1$.

Proof. (i) Let Δ be an NEC group of signature $\big(0;+;[\ ]\{(2^{(4g/n)},n,n);\}\big)$ ($4g/n$ periods equal to 2). Then Δ^+ has signature $\big(0;+;[2^{(4g/n)},n,n];\{\ \}\big)$ and as n is even there is a homomorphism $\theta\colon\Delta^+\to C_n=\langle Q\rangle$. This takes the involution generators of Δ^+ to $Q^{n/2}$ and the generators of order n to Q and Q^{-1}. By Lemma 2, θ extends to a homomorphism $\theta^*\colon\Delta\to D_n$. If Γ is the kernel of θ^* then $X=\mathbf{H}/\Gamma$ is the surface we require.

(ii) We let Δ_1 have signature $\big(0;+;[\ ]\{(\ )\big((g-1)/n\big)+2\}\big)$. There is clearly a smooth homomorphism $\theta_1\colon\Delta_1\to D_n$ and if Γ_1 is the kernel of θ_1 then $X_1=\mathbf{H}/\Gamma_1$ as the required surface.

Theorem 5. *Let* X *be a Riemann surface for which the bound of Theorem* 3 (i) *is attained. Then* X *is hyperelliptic with* $(S_1S_2)^{n/2}$ *being the hyperelliptic involution.*

Proof. The proof of Theorem 3 (i) shows that if $X = \mathbf{H}/\Gamma$ is a surface for which the bound is attained then Γ must be a normal subgroup of an NEC group Δ of signature $\{0;+;[\ ]\{(2^{(4g/n)}n,n)\}\}$ and Theorem 4 shows that Γ must be the kernel of $\theta^+\colon \Delta^+ \to C_n$. By [10] we calculate that $(\theta^+)^{-1}(\langle(S_1S_2)^{n/2}\rangle)$ has signature $(0;+;[2^{(2g+2)}];\{\ \})$ so by [7], $(S_1S_2)^{n/2}$ is the hyperelliptic involution.

Finally, we investigate the number of mirrors of each class of symmetries of the surfaces for which the bound of Theorem 3 (i) is attained. If Δ, Δ^+ are as in the proof of Theorem 4 (i), then with the notation of the proof of Lemma 2, the homomorphism $\theta^+\colon \Delta^+ \to C_n$ is given by $x_1 \to Q^{n/2},\dots,x_{4g/n} \to Q^{n/2}$, $x_{(4g/n)+1} \to Q$, $x_{(4g/n)+2} \to Q^{-1}$, (this is unique up to automorphisms of C_n) and then the homomorphism $\theta\colon \Delta \to D_n$ is given, in the case $4g/n$ even, by $c_1 \to S_1$, $c_2 \to S_1Q^{n/2}$, $c_3 \to S_1$, …, $c_{4g/n} \to S_1/Q^{n/2}$, $c_{(4g/n)+1} \to S_1$, $c_{(4g/n)+2} \to S_1Q$ and in the case $4g/n$ odd by

$$c_1 \to S_1, \quad c_2 \to S_1Q^{n/2},\dots,c_{4g/n} \to S_1,$$
$$c_{(4g/n)+1} \to S_1Q^{n/2}, \quad c_{(4g/n)+2} \to S_1Q^{n/2+1}.$$

As there are no odd link periods, the graphs of Section 5 contain no black edges and the number of fixed curves of S_1 is the number of black vertices, i.e. the number of even exponents of Q. We immediately obtain

Theorem 6. *If $4|n$ then the surfaces X for which the bounds of Theorem 3 (i) are attained admit two classes of symmetries, one with $(4g/n)+1$ mirrors and the other with 1 mirror. If $n \equiv 2 \bmod 4$ then both classes of symmetries have $(2g/n)+1$ mirrors.*

We consider some corollaries concerning "M-symmetries" [8]. These are symmetries with $g+1$ mirrors, the maximum possible number by Harnack's bound, and they are the symmetries to which most attention has been paid in the literature. From Theorems 3 (i) and 5 we obtain Natanzon's theorem [8].

Corollary 1. *If a Riemann surface X admits two M-symmetries S_1, S_2 then S_1S_2 has order 2. Also, X is hyperelliptic and S_1S_2 is the hyperelliptic involution.*

Also from Theorem 3 we deduce

Corollary 2. *If X admits an M-symmetry S_1 and another symmetry S_2 then S_1S_2 has order 2 or 4.*

Corollary 3. *If S_1 and S_2 do not commute then $|S_1|+|S_2| \le g+2$ and this bound is attained for every $g \ge 2$.*

Proof. The order of S_1S_2 is not equal to 2. We see from Theorem 3 that $|S_1|+|S_2|\le g+2$ if n is even and, by Theorem 4, this bound is attained for every g with $n=4$. If n is odd then $|S_1|+|S_2|\le \frac{2}{3}(g-1)+4$ and this bound is attained only if $3|g-1$. Thus for n odd, $|S_1|+|S_2|\le g+2$ with equality only for $g=4$, $n=3$.

Another important question about a symmetry S of a Riemann surface X is to decide whether its mirrors separate X, (e.g. see [1]). This is equivalent to whether $X/\langle S\rangle$ is orientable or not, or the NEC group Λ of (5) has orientable quotient space or not. The techniques of [4] allow us to do this. We note that the mirrors of an M-symmetry always separate. We do not go into the details here but it can be shown that in Theorem 6 that when $n>4$, the mirrors of all symmetries do not separate. If $n=4$, then one class of symmetries consists of M-symmetries and by the above remark the mirrors of each such symmetry do separate. The other class of symmetries have only one mirror which can be shown to be non-separating.

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