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NON-UNIQUENESS OF GEODESICS IN INFINITE-DIMENSIONAL TEICHMÜLLER SPACES (II)

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Abstract. The non-uniqueness of geodesics joining two given points in universal Teichmüller space is proved in the previous paper [7]. The purpose of the present paper is to discuss the non-uniqueness of geodesics in any infinite-dimensional Teichmüller space. It is proved that if μ_1 and μ_2 are two extremal Beltrami differentials belonging to a point $[\mu]$ of a Teichmüller space and $\mu_1 - \mu_2$ does not belong to N-class, the paths $[t\mu_1]$ and $[t\mu_2]$ $(0 \le t \le 1)$ are different geodesics joining $[0]$ and $[\mu]$. Making use of this theorem, the result of [7] is generalized to hold for any infinite-dimensional Teichmüller space. This is a complete answer to a problem posed by F.P. Gardiner [1].

1. Introduction

There are some essential differences in the geometry between the finite-dimensional Teichmüller spaces and the infinite-dimensional Teichmüller spaces. It is well known that a finite-dimensional Teichmüller space is a straight geodesic space in the sense of Buseman (S. Kravetz [4]), that is, for any pair of points in a finite-dimensional Teichmüller space there is a unique geodesic line through them. But we do not know whether or not the same is true for the infinite-dimensional case. We further explain the question as follows.

Let $T(S)$ be a Teichmüller space of a Riemann surface S. $T(S)$ is defined as a quotient space of the Beltrami differentials on S. If $\dim T(S) < \infty$, each point $[\mu] \in T(S)$ contains a unique extremal Beltrami differential, say μ_0, and the path $[t\mu_0]$ $(0 \le t \le 1)$ is the unique geodesic joining $[0]$ and $[\mu]$. But in the infinite-dimensional case the situation is different. When $\dim T(S) = \infty$, a point $[\mu]$ of $T(S)$ may contain more than one extremal differential. The first example of such a point in the universal Teichmüller space was given by K. Strebel, known as the Strebel chimney (K. Strebel [11] or see O. Lehto [6]). Suppose the point $[\mu]$ contains two extremal differentials μ_1 and μ_2. The question is how to determine whether $[t\mu_1]$ is the same as $[t\mu_2]$ $(0 \le t \le 1)$. This question was proposed by F.P. Gardiner [1].

In [7] the author answers this question by constructing two extremal differentials μ_1 and μ_2 such that they are in the same point of the universal Teichmüller

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space but the geodesic paths $[t\mu_1]$ and $[t\mu_2]$ $(0 \leq t \leq 1)$ are different. This example shows that the geodesics joining two points in the universal Teichmüller space are not unique and hence the universal Teichmüller space is not a straight geodesic space.

After this result, a natural question that follows concerns the general infinite-dimensional Teichmüller space except the universal Teichmüller space. The purpose of this paper is to investigate the geodesic problem for any infinite-dimensional Teichmüller space.

The main result is Theorem 3.1, which states that if a point $[\mu] \in T(S)$ contains two extremal differentials μ_1 and μ_2 such that $\mu_1 - \mu_2$ does not belong to the N-class of Ahlfors, then $[t\mu_1]$ and $[t\mu_2]$ $(0 \leq t \leq 1)$ are different geodesics joining $[0]$ and $[\mu]$.

As a consequence of this main result, we get a criterion for the non-uniqueness of the geodesics in any infinite-dimensional Teichmüller space: If μ is an extremal Beltrami differential on S and the set $\{p \in S \mid |\mu(p)| < \|\mu\|_\infty - \varepsilon\}$ $(\varepsilon > 0)$ has an interior point, there are infinitely many geodesic lines through $[0]$ and $[\mu]$ (see Theorem 3.2).

As as application of Theorem 3.2, we shall construct in Section 4 a pair of points in any infinite-dimensional Teichmüller space such that they have infinitely many geodesics.

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Remark. Having prepared this paper, the author received a preprint of a paper [12] by Harumi Tanigawa, who shows a similar result to Theorem 3.2 of this paper. Her result is as follows: If μ is an extremal Beltrami differential which vanishes on a Jordan domain U and does not vanish identically on the whole Riemann surface, there exists a family of geodesic discs through $[0]$ and $[\mu]$ with a complex analytic parameter. She constructs an extremal Beltrami differential which satisfies the conditions in her theorem. The method of her proof is completely different from ours.

From the proofs of Theorem 3.1 and 3.2 of this paper one may find that these theorems still hold if the "geodesics" in the statements of Theorem 3.1 and 3.2 is replaced by "geodesic discs".

2. Preliminaries

Let S be a Riemann surface with a holomorphic universal covering map: $\pi\colon \Delta \to S$, where Δ is the unit disc. Then S can be expressed as a quotient space Δ/Γ, where Γ is a Fuchsian group acting on Δ. Denote by $\mathrm{Bel}(S)$ the space of the Beltrami differentials on S with the supremum norm smaller than 1. For each element $\mu \in \mathrm{Bel}(S)$ there exists a Riemann surface S^μ and a quasiconformal mapping $f^\mu: S \to S^\mu$ such that the complex dilatation of f^μ is μ. We denote by $\tilde{f}^\mu: \Delta \to \Delta$ the lift of f^μ with the points 1, i and -1 fixed. Then $\tilde{f}^\mu$ is uniquely determined by μ. An element $\mu_1 \in \mathrm{Bel}(S)$ is said to be equivalent to $\mu_2 \in \mathrm{Bel}(S)$ if and only if $\tilde{f}^{\mu_1}|_{\partial\Delta} = \tilde{f}^{\mu_2}|_{\partial\Delta}$. Then the Teichmüller space of S, denoted by $T(S)$, can be defined as the quotient space of $\mathrm{Bel}(S)$ under the equivalence relations. The Teichmüller metric between two points $[\mu]$ and $[\nu]$ is defined as follows:

$$d\big([\mu],[\nu]\big) = \tfrac{1}{2} \inf_f \log K[f],$$

where f runs through all quasiconforml mappings of S^μ onto S^ν in the homotopy class $\left[f^\nu \circ (f^\mu)^{-1}\right]$ and $K[f]$ is the maximal dilatation of f.

An element $\mu \in \mathrm{Bel}(S)$ is said to be extremal if and only if its norm is the smallest among the elements of $[\mu]$. It is known that the extremal Beltrami differential always exists for any point $[\mu] \in T(S)$. Moreover, if μ is extremal, then $t\mu$ is also extremal for t $(0 \le t \le 1)$ and the path $[t\mu]$ is a geodesic (in the Teichmüller metric) joining the points $[0]$ and $[\mu]$. If $\dim T(S) < +\infty$, the extremal differential of $[\mu]$ is unique for each point $[\mu] \in T(S)$ and must be of the following form: $k\overline{\varphi}/|\varphi|$, where φ is a holomorphic quadratic differential with finite norm and k is a constant non-negative and smaller than 1.

Let $Q(S)$ be the space of quadratic differentials on S with L_1-norms finite. The dimension of $Q(S)$ is finite if and only if $\dim T(S)$ is finite.

An element $\mu \in \mathrm{Bel}(S)$ is extremal if and only if

$$\sup_{\substack{\varphi \in Q(S)\\ \|\varphi\|=1}} \left\{ \left| \iint_S \mu\varphi \, dx\, dy \right| \right\} = \|\mu\|_\infty . \tag{2.1}$$

This result is due to R.S. Hamilton [3] and S. Krushkal [5] (for necessity), and E. Reich and K. Strebel [8] (for sufficiency). Condition (2.1) is called Hamilton–Krushkal condition.

If μ is extremal and $\varphi_n \in Q(S)$ such that $\|\varphi_n\| = 1$ and

$$\lim_{n\to\infty} \left| \iint_S \mu\varphi_n \, dx\, dy \right| = \|\mu\|_\infty , \tag{2.2}$$

the sequence $\{\varphi_n\}$ is called a Hamilton sequence.

Another known result we need is the main inequality of Reich and Strebel (see [9]). If $\mu \in \mathrm{Bel}(S)$ is equivalent to 0, i.e., $\tilde{f}^\mu|_{\partial\Delta} = \mathrm{id}$, we have

$$\|\varphi\| \le \iint_S |\varphi| \frac{\left|1 - \mu\varphi/|\varphi|\right|^2}{1-|\mu|^2} \, dx\, dy \tag{2.3}$$

for all $\varphi \in Q(S)$.

A Beltrami differential $\mu \in \mathrm{Bel}(S)$ is said to be an element of $N(S)$, known as the N-class of Beltrami differentials, if and only if

$$\iint_S \mu\varphi \, dx\, dy = 0, \qquad \text{for all } \varphi \in Q(S). \tag{2.4}$$

3. Main results

In this section we shall give some sufficient conditions of a point $[\mu] \in T(S)$ having more than 1 geodesics joining $[0]$ and $[\mu]$.

Lemma 3.1. *Let $\mu \in \mathrm{Bel}(\Delta)$ and $F_t\colon \Delta \to \Delta$ be a quasiconformal mapping of the unit disc Δ onto itself with the points 1, i and -1 fixed and with the complex dilatation $t\mu$ $(0 \le t \le 1)$. Then we have a sequence $\{t_n\}$, $0 < t_n < 1$, such that*

$$\partial_z F_{t_n}(z) \to 1, \qquad \text{as } t_n \to 0, \tag{3.1}$$

almost everywhere in Δ.

Proof. It is known that the Beltrami equation

$$\partial_{\bar z} w = t\mu \partial_z w \qquad (0 < t < 1)$$

has a solution of the following form:

$$w_t(z) = z - \frac{1}{\pi} \iint_\Delta \frac{\omega_t(\zeta)}{\zeta - z}\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta),$$

where ω_t is a solution of the equation

$$\omega - t\mu S(\omega) = t\mu, \tag{3.2}$$

with S a singular integration operator:

$$S(\omega) = -\frac{1}{\pi} \iint_\Delta \frac{\omega(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta.$$

Let Λ_p be the norm of the operator $S\colon L_p \to L_p$. Then $\Lambda_2 = 1$ and Λ_p is a continuous function of $p > 1$ (see [13]). Suppose

$$\bigl|\mu(z)\bigr| \le k < 1, \qquad \text{a.e. } z \in \Delta,$$

where k is a constant. We choose $p_0 > 2$ such that $\Lambda_{p_0} k < 1$. Then (3.2) has a solution $\omega_t \in L_{p_0}$ satisfying

$$\|\omega_t\|_{L_{p_0}} \leq tk/(1 - \Lambda_{p_0} k).$$

Hence we get $\|\omega_t\|_{L_{p_0}} \to 0$ (as $t \to 0$) and

$$w_t(z) \to z \qquad \text{(uniformly for } z \in \Delta) \tag{3.3}$$

as $t \to 0$. On the other hand, we see

$$\partial_z w_t = 1 + S(\omega_t).$$

It follows that

$$\|\partial_z w_t - 1\|_{L_{p_0}} \leq \Lambda_{p_0} \|\omega_t\|_{L_{p_0}} \to 0 \qquad (\text{as } t \to 1),$$

which implies that there exists a sequence $\{t_n\}$ such that

$$\partial_z w_{t_n} \to 1 \qquad \text{(a.e.) as } t_n \to 0. \tag{3.4}$$

By the assumption of the lemma, F_t can be expressed as $\varphi_t(w_t)$, where φ_t is a conformal mapping of $w_t(\Delta)$ onto Δ with the following conditions:

$$\varphi_t\big(w_t(1)\big) = 1, \qquad \varphi_t\big(w_t(i)\big), \qquad \varphi_t\big(w_t(-1)\big) = -1.$$

By (3.3) and the Caratheodory theorem, it follows from the above conditions that φ is locally uniformly convergent to z and hence φ_t' is locally convergent to 1 as $t \to 0$. The lemma is proved by (3.4). QED.

Theorem 3.1. *Let μ_1 and μ_2 be two extremal Beltrami differentials in $[\mu] \in T(S)$. If $\mu_1 - \mu_2$ does not belong to the N-class of Beltrami differentials on S, then $[t\mu_1]$ and $[t\mu_2]$ $(0 \leq t \leq 1)$ are two different geodesics joining $[0]$ and $[\mu]$.*

Proof. Suppose $[t\mu_1] = [t\mu_2]$ for every $t \in [0,1]$. The theorem will be proved by obtaining a contradiction with the assumption that $\mu_1 - \mu_2 \notin N(S)$.

Let $S = \Delta/\Gamma$, where Γ is a Fuchsian group. For each $t \in [0,1]$ there are a Riemann surface S_t and a quasiconformal mapping $f_{t,j} = f^{t\mu_j}\colon S \to S_t$ $(j = 1,2)$ such that $f_{t,1}$ is homotopic to $f_{t,2}$ modulo the boundary. Suppose $F_{t,j}\colon \Delta \to \Delta$ is the lift of $f_{t,j}$ with the points 1, i and -1 fixed $(j = 1,2)$. Then we have

$$F_{t,1}\,|_{\partial\Delta} = F_{t,2}\,|_{\partial\Delta}\,.$$

Let $S_t = \Delta/\Gamma_t$, where Γ_t is a Fuchsian group. It is easy to see that we can assume that

$$\Gamma_t = \{F_{t,j} \circ \gamma \circ F_{t,j}^{-1} \mid \gamma \in \Gamma\},$$

where $j = 1$ or 2.

Let φ be an arbitrary fixed element of $Q(S)$ and $\tilde{\varphi}(z)\,dz^2$ be the lift of φ. Then $\tilde{\varphi}$ satisfies

$$\tilde{\varphi}\big(\gamma(z)\big)\big[\gamma'(z)\big]^2 = \tilde{\varphi}(z), \qquad z \in \Delta.$$

There is a holomorphic function $\psi(z)$ in Δ with the condition

$$\iint_{\Delta} \big|\psi(z)\big|\,dx\,dy < +\infty,$$

such that the Poincaré series of ψ (see [2, Chapter 4, Theorem 3]).

$$\Theta\psi(z) = \sum_{\gamma\in\Gamma} \psi\big(\gamma(z)\big)\big[\gamma'(z)\big]^2$$

is equal to $\tilde{\varphi}$. Using this function ψ, we define

$$\tilde{\varphi}_t(z) = \sum_{\gamma_t\in\Gamma_t} \psi\big(\gamma_t(z)\big)\big[\gamma_t'(z)\big]^2.$$

Then we get

$$\tilde{\varphi}_t\big(\gamma_t(z)\big)\big[\gamma_t'(z)\big]^2 = \tilde{\varphi}_t(z)$$

for $z \in \Delta$ and $\gamma_t \in \Gamma_t$. This means that $\tilde{\varphi}_t(z)\,dz^2$ is a lift of a holomorphic quadratic differential φ_t on S_t with finite norm.

Let g_t be the composition mapping of $f_{t,1}$ and $f_{t,2}^{-1}$, i.e., $g_t = f_{t,1} \circ f_{t,2}^{-1}\colon S_t \to S_t$. Denote by σ_t the complex dilatation of g_t. Then it follows from the assumption that $[t\mu_1] = [t\mu_2]$, that σ_t is equivalent to 0. Hence by the main inequality we have

$$\|\varphi_t\| \le \iint_{S_t} |\varphi_t| \frac{\big|1 - \sigma_t\varphi_t/|\varphi_t|\big|^2}{1 - |\sigma_t|^2}\,d\xi\,d\eta,$$

which can be rewritten in the following form:

$$\operatorname{Re} \iint_{S_t} \frac{\sigma_t\varphi_t}{1 - |\sigma_t|^2}\,d\xi\,d\eta \le \iint_{S_t} \frac{|\sigma_t|^2|\varphi_t|}{1 - |\sigma_t|^2}\,d\xi\,d\eta. \tag{3.5}$$

Let Ω be a fundamental domain of S. Then $\Omega_t = F_{t,2}(\Omega)$ is a fundamental domain of S_t. Let $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\sigma}_t$ be the lifts of μ_1, μ_2 and σ_t, respectively. Then we have

$$\tilde{\sigma}_t(\zeta) = \frac{t(\tilde{\mu}_1 - \tilde{\mu}_2)}{1 - t^2\tilde{\mu}_1\overline{\tilde{\mu}_2}} \cdot \frac{\partial_z F_{t,2}}{\overline{\partial_z F_{t,2}}} \circ F_{t,2}^{-1}(\zeta).$$

From inequality (3.5) it follows that

$$\operatorname{Re}\iint_{\Omega_t} \tilde{\sigma}_t\tilde{\varphi}_t\,d\xi\,d\eta \le \iint_{\Omega_t} \frac{|\tilde{\sigma}_t|^2|\tilde{\varphi}_t|}{1-|\tilde{\sigma}_t|^2}\,d\xi\,d\eta + O(t^2)$$

and hence

$$\limsup_{t\to 0}\operatorname{Re}\iint_{\Omega_t} \frac{\tilde{\mu}_1-\tilde{\mu}_2}{1-t^2\tilde{\mu}_1\overline{\tilde{\mu}_2}}\cdot\frac{\partial_z F_{t,2}}{\overline{\partial_z F_{t,2}}}\circ F_{t,2}^{-1}(\zeta)\tilde{\varphi}_t(\zeta)\,d\xi\,d\eta \le 0$$

or

$$\limsup_{t\to 0}\operatorname{Re}\iint_{\Omega_t} \frac{\tilde{\mu}_1-\tilde{\mu}_2}{1-t^2\tilde{\mu}_1\overline{\tilde{\mu}_2}}\cdot\frac{\partial_z F_{t,2}}{\overline{\partial_z F_{t,2}}}\tilde{\varphi}_t(F_{t,2})\,dx\,dy \le 0.$$

It is easy to see that $\tilde{\varphi}_t(z)\to\tilde{\varphi}(z)$. By Lemma 3.1 there is a sequence $\{t_n\}$ $(0<t_n<1)$ such that $\partial_z F_{t_n,2}\to z$ as $t_n\to 0$. Then we have

$$\operatorname{Re}\iint_{\Omega}(\tilde{\mu}_1-\tilde{\mu}_2)\tilde{\varphi}\,dx\,dy \le 0$$

or

$$\operatorname{Re}\iint_{S}(\mu_1-\mu_2)\varphi\,dx\,dy \le 0.$$

Since φ is an arbitrary element of $Q(S)$, this inequality implies

$$\iint_{S}(\mu_1-\mu_2)\varphi\,dx\,dy = 0, \qquad \text{for all } \varphi\in Q(S),$$

which means $\mu_1-\mu_2\in N(S)$. This is a contradiction with the assumption of the theorem. QED.

Theorem 3.1 gives a sufficient condition of the non-uniqueness of geodesics. But the condition that $\mu_1-\mu_2\notin N(S)$ is not very explicit for practical use. Below there follows a more explicit condition which can be regarded as a criterion of the non-uniqueness of geodesics.

Theorem 3.2. *Let μ be an extremal Beltrami differential on a Riemann surface S. If the set*

$$A_\mu = \big\{p\in S \;\big|\; |\mu|(p) < \|\mu\|_\infty - \varepsilon\big\} \qquad (\text{for some } \varepsilon>0)$$

has an interior point, there are infinitely many geodesics joining the points $[0]$ and $[\mu]$ in $T(S)$.

Proof. Suppose $p_0 \in S$ is an interior point of A_μ and $U \subset A_\mu$ is a neighbourhood of p_0. Without any loss of generality we assume that μ is C^2 smooth in U (otherwise one may make a small perturbation of $\mu|_U$ such that the resulting differential is C^2 smooth and its absolute values are smaller than $\|\mu\|_\infty - \varepsilon'$ for another $\varepsilon' > 0$).

Let $\varphi \in Q(S) \setminus \{0\}$ and $\varphi|_{p_0} \neq 0$. We choose U so small that $\varphi|_p \neq 0$ for every p in U. Suppose z is a natural parameter of φ such that $z(p_0) = 0$ and $\varphi|_U = dz^2$. Let $f^\mu\colon S \to S^\mu$ be a quasiconformal mapping of S onto another Riemann surface S^μ with the Beltrami differential μ. Let ζ be a local parameter of a neighbourhood V of $q_0 = f^\mu(p_0)$ with $\zeta(q_0) = 0$. Suppose $f^\mu(V) \subset U$ and the local expression of $f^\mu|_U$ is $\zeta = f(z)$, the complex dilatation of which is $\mu(z)$. Without any loss of generality we assume

$$\partial_\zeta\big[\partial_z f\big(f^{-1}(\zeta)\big)\big] \not\equiv 0 \qquad \text{in } V. \tag{3.6}$$

(Otherwise one can change the parameter ζ or take a small perturbation of $f|_U$). We look at a disc $\Delta_r = \big\{\zeta \mid |\zeta| < r\big\}$, where $r > 0$ is sufficiently small to allow $\Delta_r \subset V$. Define a mapping of Δ_r onto itself:

$$h_\alpha(\zeta) = \zeta + \alpha\eta(\zeta),$$

where α is a complex parameter and $\eta \in C^\infty$ has a compact supporting set in Δ. Then we have

$$h_\alpha(\zeta) \equiv \zeta, \qquad \text{for all } \zeta \in \partial\Delta_r. \tag{3.7}$$

Let $\nu_\alpha = \partial_{\bar\zeta} h_\alpha / \partial_\zeta h_\alpha$. It is easy to see that $\|\nu_\alpha\|_\infty$ is less than 1 when $|\alpha|$ is sufficiently small. By the argument principle for quasiregular functions and condition (3.7), h_α is an $1-1$ mapping and hence a quasiconformal mapping of Δ_r onto itself.

Let us look at the composed mapping $g_\alpha = h_\alpha \circ f$. It is easy to see that the complex dilatation of g_α is

$$\sigma_\alpha = \frac{\mu(z) + \nu_\alpha(\zeta)\theta(z)}{1 + \overline{\mu(z)}\nu_\alpha(\zeta)\theta(z)}, \tag{3.8}$$

where $\zeta = f(z)$ and $\theta(z) = \partial_z f / \overline{\partial_z f}$.

Let α_1 and α_2 be two sufficiently small parameters and $\alpha_1 \neq \alpha_2$. Then we have

$$\sigma_{\alpha_1}(z) - \sigma_{\alpha_2}(z) = \frac{\big[\nu_{\alpha_1}(\zeta) - \nu_{\alpha_2}(\zeta)\big]\theta(z)\big[1 - |\mu(z)|^2\big]}{\big[1 + \bar\mu(z)\nu_{\alpha_1}(\zeta)\theta(z)\big]\big[1 + \bar\mu(z)\nu_{\alpha_2}(\zeta)\theta(z)\big]}.$$

Noting the fact that $\nu_\alpha(\zeta) = O(|\alpha|)$, we get

$$\sigma_{\alpha_1}(z) - \sigma_{\alpha_2}(z) = \big[\nu_{\alpha_1}(\zeta) - \nu_{\alpha_2}(\zeta)\big]\theta(z)\big[1 - |\mu(z)|^2\big] \cdot \big[1 + O\big(|\alpha_1| + |\alpha_2|\big)\big], \qquad \zeta = f(z).$$

It follows from the definition of ν_α that

$$\nu_{\alpha_1}(\zeta) - \nu_{\alpha_2}(\zeta) = (\alpha_1 - \alpha_2)\partial_{\bar\zeta}\eta\big[1 + O\big(|\alpha_1| + |\alpha_2|\big)\big],$$

and hence

$$\sigma_{\alpha_1}(z) - \sigma_{\alpha_2}(z) = (\alpha_1 - \alpha_2)\partial_{\bar\zeta}\eta(\zeta)\theta(z)\big[1 - |\mu(z)|^2\big] \cdot \big[1 + O\big(|\alpha_1| + |\alpha_2|\big)\big]. \tag{3.9}$$

We are now going to show that

$$\iint_{D_r} \big[\sigma_{\alpha_1}(z) - \sigma_{\alpha_2}(z)\big]\, dx\, dy \neq 0 \tag{3.10}$$

for sufficiently small α_1 and α_2 ($\alpha_1 \neq \alpha_2$), where $D_r = f^{-1}(\Delta_r)$. By (3.9) it is easy to see that, to show (3.10), it is sufficient to prove

$$\iint_{D_r} \partial_{\bar\zeta}\eta\big(f(z)\big)\theta(z)\big[1 - |\mu(z)|^2\big]\, dx\, dy \neq 0$$

or

$$\iint_{\Delta_r} \partial_{\bar\zeta}\eta(\zeta)/\overline{\big[\partial_z f\big(f^{-1}(\zeta)\big)\big]^2}\, d\xi\, d\eta \neq 0. \tag{3.11}$$

There exists a function $\eta \in C_0^\infty(\Delta_r)$ such that (3.11) holds; otherwise we have

$$\iint_{\Delta_r} \partial_{\bar\zeta}\eta(\zeta)/\overline{\big[\partial_z f\big(f^{-1}(\zeta)\big)\big]^2}\, d\xi\, d\eta = 0,$$

for all $\eta \in C_0^\infty(\Delta_r)$, which implies

$$\partial_{\bar\zeta}\Big(1/\overline{\big[\partial_z f\big(f^{-1}(\zeta)\big)\big]^2}\Big) \equiv 0 \qquad (\text{in } \Delta_r)$$

or

$$\partial_\zeta\big[\partial_z f\big(f^{-1}(\zeta)\big)\big] \equiv 0 \qquad (\text{in } \Delta_r)$$

being a contradiction with (3.6). From now on we assume η to be chosen such that (3.11) holds and hence (3.10) holds.

We identify the points in U with their parameters. Then D_r are identified with the domain of S which D_r represent. Now we define a Beltrami differential on S:

$$\sigma_\alpha^* = \begin{cases} \mu, & \text{in } S \setminus D_r; \\ \sigma_\alpha, & \text{in } D_r. \end{cases}$$

Obviously, $\sigma_\alpha^* \in [\mu]$ and σ_α^* is extremal when $|\alpha|$ is sufficiently small. By the definition we have

$$\iint\limits_S (\sigma_{\alpha_1}^* - \sigma_{\alpha_2}^*)\varphi \, dx\, dy = \iint\limits_{D_r} (\sigma_{\alpha_1} - \sigma_{\alpha_2})\varphi \, dx\, dy,$$

where φ is the holomorphic quadratic differential on S which is given at the beginning of the proof. Recalling the fact that the local expression of $\varphi|_U$ is dz^2, we get

$$\iint\limits_S (\sigma_{\alpha_1}^* - \sigma_{\alpha_2}^*)\varphi \, dx\, dy = \iint\limits_{D_r} (\sigma_{\alpha_1} - \sigma_{\alpha_2}) \, dx\, dy.$$

It follows from (3.10) that

$$\iint\limits_S (\sigma_{\alpha_1}^* - \sigma_{\alpha_2}^*)\varphi \, dx\, dy \neq 0,$$

which implies that $(\sigma_{\alpha_1}^* - \sigma_{\alpha_2}^*)$ does not belong to the N-class. By Theorem 3.1 we see that $[t\sigma_{\alpha_1}^*]$ and $[t\sigma_{\alpha_2}^*]$ $(0 \le t \le 1)$ are different geodesics joining $[0]$ and $[\mu]$, provided $\alpha_1 \neq \alpha_2$ and both are sufficiently small. QED.

Remark. It is easy to prove that the mapping $t \mapsto \left[t\mu/\|\mu\|_\infty\right]$ (μ is extremal) is a holomorphic isometry of $\Delta = \left\{t : |t| < 1\right\}$ into $T(S)$ with the Poincaré metric and Teichmüller metric. So one can replace the geodesic lines in Theorem 3.2 with the geodesic discs [12].

In her paper [12], H. Tanigawa investigates the boundary behaviour of a holomorphic mapping of Δ into $T(S)$ and gives a sufficient condition for holomorphic mappings into $T(S)$ to be rigid.

4. Existence

Now we want to construct an extremal Beltrami differential which satisfies the conditions in Theorem 3.2. The main point is to construct an extremal Beltrami differential which is not of the Teichmüller form or the Teichmüller form while the associated quadratic differential has infinite norm.

In [12], H. Tanigawa gives a construction of such an extremal Beltrami differential. Here we shall give another construction.

Let $T(S)$ be an arbitrary infinite dimensional Teichmüller space of a Riemann surface S. Suppose that $S = \Delta/\Gamma$, where Γ is a Fuchsian group acting on Δ. Then Γ is of the second kind or of the first kind with infinite generators.

First, we assume that Γ is a Fuchsian group of the second kind (finitely or infinitely generated). Let $z_0 \in \partial\Delta$ be a point of $\partial\Delta \setminus \wedge(\Gamma)$, where $\wedge(\Gamma)$ is the limit set of Γ. Then z_0 is a point on a free side of a fundamental polygon G of Γ. Take a function φ which is holomorphic on $\mathbf{C} \setminus \{z_0\}$ and has a pole of the second order at the point z_0. We look at its Poincaré series

$$\tilde{\psi} = \Theta\varphi = \sum_{\gamma\in\Gamma} \varphi\bigl(\gamma(z)\bigr)\gamma'(z)^2.$$

It is easy to see that z_0 is a pole of $\tilde{\psi}$ of the second order and

$$\iint_G |\tilde{\psi}|\, dx\, dy = \infty.$$

Obviously, $\tilde{\psi}$ induces a holomorphic quadratic differential on S, which is denoted by ψ. Then we have

$$\iint_S |\psi|\, dx\, dy = \infty.$$

Define a Beltrami differential on S:

$$\mu = k\bar{\psi}/|\psi|,$$

where $k \in (0,1)$ is a constant. Since $\tilde{\psi}$ has a pole of the second order on the boundary of Δ, the Beltrami differential $\tilde{\mu} = k\overline{\tilde{\psi}}/|\tilde{\psi}|$ with respect to Γ is extremal ([10]). Hence μ is an extremal Beltrami differential on S. By the Hamilton theorem there is a sequence of quadratic differentials, $\{\psi_n\}$, such that $\|\psi_n\| = 1$ and

$$\lim_{n\to\infty} \left| \iint_S \mu\psi_n\, dx\, dy \right| = k.$$

It is easy to see that there is a subsequence of $\{\psi_n\}$ which is uniformly convergent to a holomorphic quadratic differential ψ_0 in any compact set of S. Without any loss of generality we may assume that the subsequence is the sequence $\{\psi_n\}$ itself. Then we have

$$\|\psi_n - \psi_0\| \not\to 0 \qquad (n \to \infty).$$

Otherwise, $\|\psi_n - \psi_0\| \to 0$ $(n \to \infty)$ implies $\|\psi_0\| = 1$ and

$$\left| \iint_S \mu\psi_0 \, dx \, dy \right| = k,$$

which means $\mu = e^{i\theta} k\bar{\psi}_0/|\psi_0|$. This is a contradiction with the definition of μ.

Without loss of generality we assume that $\|\psi_n - \psi_0\| \neq 0$ for every integer n. Define $\eta_n = (\psi_n - \psi_0)/\,\|\psi_n - \psi_0\|$. Then η_n has norm 1 and $\{\eta_n\}$ is a degenerate Hamilton sequence of μ, namely,

$$\lim_{n\to\infty} \left| \iint_S \mu\eta_n \, dx \, dy \right| = k$$

and η_n locally tends to zero.

Now we get the desired Beltrami differential as follows. We define

$$\mu^* = \begin{cases} \mu, & \text{in } S \setminus U; \\ k', & \text{in } U, \end{cases}$$

where $k' \in (0,1)$ is a constant smaller than k and U is a small compact neighbourhood of a point. Obviously, μ^* is also an extremal Beltrami differential and $\{\eta_n\}$ is also a Hamilton sequence of μ^*. μ^* satisfies the conditions in Theorem 3.2.

Now we look at the case where Γ is infinitely generated and of the first kind, namely, the fundamental polygon of Γ has infinitely many of the vertexes and has no free side. In this case, we take the point z_0 as a limit point of the vertexes of a fundamental polygon of Γ. We define the quadratic differentials $\tilde{\psi}$ and ψ as above. By a result in [10], we see that the Beltrami differential $k\bar{\psi}/|\psi|$ is extremal and $\|\psi\| = \infty$. All of the above arguments hold.

We have proved

Theorem 4.1. *In any infinite dimensional Teichmüller space $T(S)$, there is a pair of points such that there exist infinitely many of geodesic lines through them.*

5. Problems.

The following problems seem naturally to present themselves:

(i) Does the uniqueness of the extremal differentials in a point $[\mu]$ always imply the uniqueness of the geodesics joining $[0]$ and $[\mu]$?

(ii) Does the uniqueness of the geodesics joining $[0]$ and $[\mu]$ always imply the uniqueness of the extremal differentials in $[\mu]$?

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