Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 18, 1993, 93–103

FUNCTIONAL IDENTITIES FOR SPECIAL FUNCTIONS OF QUASICONFORMAL THEORY

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Abstract. Let $\Phi_{K,n}$ be the distortion function for K-quasiconformal maps of $\mathbf{B}^n$ into $\mathbf{B}^n$, $n \geq 2$. We describe the family $\mathscr{H}_n$ of all differentiable involutions h such that the functional identity $h \circ \Phi_{K,n} = \Phi_{1/K,n} \circ h$ holds for every $K > 0$. An infinite sequence of elementary involutions of $\mathscr{H}_2$ is explicitly constructed. A basic convergence theorem for $\Phi_{K,n}$, $K > 0$, $n = 2, 3, \ldots$, together with some other results for related functions are proved.

0. Introduction

The distortion function $\Phi_{K,n}$: $[0;1] \to [0;1]$, in the n-dimensional ($n \geq 2$) quasiconformal version of the Schwarz lemma (see [MRV] and [V]), is defined for $K > 0$, $n = 2, 3, 4, \ldots$, by $\Phi_{K,n}(0) = 0$, $\Phi_{K,n}(1) = 1$, and

$$\Phi_{K,n}(t) = M_n^{-1}\Big(\frac{1}{K_n} M_n(t)\Big) \tag{0.1}$$

for $0 < t < 1$, where $K_n = K^{1/(n-1)}$ and M_n is given by

$$\gamma_n(1/t) = \omega_{n-1}^{\circ} M_n^{1-n}(t). \tag{0.2}$$

Here, γ_n denotes the conformal capacity of the Grötzsch condenser in $\mathbf{R}^n$, $\omega_{n-1}^{\circ} = (2/\pi)^{n-1}\omega_{n-1}$, where ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere S^{n-1} in $\mathbf{R}^n$. Introducing any positive constant multiplier to the formula (0.2), defining M_n, we do not alter $\Phi_{K,n}$. Thus, for our convenience, we normalize ω_{n-1} as above. For $n = 2$ the function $M_2 = \mu$, and

$$\mu(t) = \frac{\mathscr{K}(\sqrt{1-t^2}\,)}{\mathscr{K}(t)}, \tag{0.3}$$

where

$$\mathscr{K}(t) = \int_0^{\pi/2} (1 - t^2 \sin^2 u)^{-1/2}\, du \tag{0.4}$$

1991 Mathematics Subject Classification: Primary 30C60; Secondary 33A25, 33A70.

is the complete elliptic integral of the first kind. From this several functional identities follow (see [AVV1] and [V]).

For the higher-dimensional case $n \geq 3$ neither such explicit expressions nor functional identities were known.

The main purpose of this paper is to show how one can obtain: one parameter family of identities satisfied by $\Phi_{K,n}$, $K > 0$, $n = 2, 3, \ldots$; equivalent identities for M_n, γ_n, τ_n and for $\mathscr{K}$, when $n = 2$; and a basic convergence theorem for $\Phi_{K,n}$, as an application. Our idea can be realized since we know that M_n is differentiable in $(0;1)$ for $n = 2, 3, \ldots$, (see [An2]).

The author wishes to thank M. Vuorinen for his assistance in preparing this material, Ken-ichi Sakan for useful remarks, the referee for helpful comments and suggestions, and to the Department of Mathematics of the University of Joensuu for hospitality while working on this paper.

1. Conjugate distortion functions

In the theory of quasiconformal mappings in $\mathbf{R}^n$, $n = 2, 3, \ldots$, we are particularly interested in two rings having extremal properties. The first is the Grötzsch ring $R_{G,n}(s)$, $s > 1$, whose complementary components are the closed unit ball $\overline{\mathbf{B}}^n$ and the ray $[se_1, \infty]$, where e_1 is the first unit vector of the rectangular coordinates axes in $\mathbf{R}^n$. The other is the Teichmüller ring $R_{T,n}(t)$, $t > 0$, whose complementary components are the segment $[-e_1, 0]$ and the ray $[te_1, \infty]$. The conformal capacities $\gamma_n(s)$ and $\tau_n(t)$ of $R_{G,n}(s)$ and $R_{T,n}(t)$, respectively, are decreasing functions related by the following functional identity

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1) \qquad \text{for } s > 1. \tag{1.1}$$

(see [G]).

Let $\mathscr{H}$ denote the family of all differentiable automorphisms of $(0;1)$, and $\mathscr{H}_n$, $n = 2, 3, \ldots$, be the set of all involutions $h \in \mathscr{H}$ such that

$$h \circ \Phi_{K,n} = \Phi_{1/K,n} \circ h \tag{1.2}$$

holds for each $K > 0$.

Moreover, let $Q = (0;1) \times (0;1)$. Then we prove

Theorem 1. *Let $n = 2, 3, \ldots$, be fixed. A function $h \in \mathscr{H}_n$ if and only if there is $L > 0$ such that*

$$h(t) = M_n^{-1}\big(L_n / M_n(t)\big) \tag{1.3}$$

holds for $0 < t < 1$ and $L_n = L^{1/(n-1)}$. Moreover, if $(\xi, \eta) \in Q$ is an arbitrary point then there is $L_n^{\xi\eta}$ such that $h(\xi) = \eta$.

Proof. ($\Rightarrow$) Introducing $\tilde{M}_n = M_n \circ h \circ M_n^{-1}$, we see that (1.2) with (0.1) may be written down as

$$\tilde{M}_n\Big(\frac{t}{K_n}\Big) = K_n \tilde{M}_n(t) \tag{1.4}$$

for $0 < t < \infty$, $K_n = K^{1/(n-1)}$, $K > 0$ and $n = 2, 3, \ldots$. By the definition of $\mathscr{H}_n$, and the regularity of M_n (see [An2]), it follows that $\tilde{M}_n$ is differentiable in $0 < t < \infty$. Hence, the well-known Euler identity implies that all the solutions of (1.4) can be written as

$$\tilde{M}_n(t) = \frac{L_n}{t}$$

for $0 < t < \infty$, $L_n = L^{1/(n-1)}$, $L > 0$ and $n = 2, 3, \ldots$. Thus

$$h(t) = M_n^{-1}\big(L_n / M_n(t)\big). \tag{1.5}$$

Let $(\xi, \eta) \in Q$ be an arbitrary point. Setting $L_n^{\xi\eta} = M_n(\xi) M_n(\eta)$ we see that $M_n^{-1}\big(L_n^{\xi\eta}/M_n(\xi)\big) = \eta$, so the second part of our theorem follows.

In the case ($\Leftarrow$) it is evident that each function of the form (1.5) belongs to $\mathscr{H}_n$ for $n = 2, 3, \ldots$, which ends the proof.

Let

$$\mathscr{H}^\infty = \bigcup_{n=2}^{\infty} \mathscr{H}_n. \tag{1.6}$$

We may additionally assume that each function of $\mathscr{H}^\infty$ maps 0 and 1 onto 1 and 0, respectively. A function of $\mathscr{H}^\infty$ class is called a *conjugate distortion function*. To justify the name let us consider the family of functions defined as

$$\Psi_{K,\alpha,\beta}[\nu, t] = \nu^{-1}\big(K^\alpha \nu^\beta(t)\big), \tag{1.7}$$

for $\alpha, \beta \in \mathbf{R}$, $\beta \neq 0$ and $K > 0$, where ν is a differentiable homeomorphism mapping $(0;1)$ onto $(0;\infty)$. The family of functions defined by (1.7) forms a group of automorphisms of (0;1), under composition. With each automorphism $\Psi_{K,\alpha,\beta}[\nu, \cdot\,]$ we associate the automorphism

$$\Psi^*_{K,\alpha,\beta}[\nu, \cdot\,] = \Psi_{K,-\alpha,-\beta}[\nu, \cdot\,]. \tag{1.8}$$

The correspondence $\Psi_{K,\alpha,\beta}[\nu, \cdot\,] \to \Psi^*_{K,\alpha,\beta}[\nu, \cdot\,]$ we call a *conjugation*.

Setting $\nu = M_n$, $\alpha = 1/(1-n)$ and $\beta = 1$, into (1.7), we obtain $\Phi_{K,n}$, whereas

$$\Phi^*_{K,n} := \Psi^*_{K,-1/(1-n),-1}[M_n, \cdot\,] \tag{1.9}$$

is a *conjugate distortion function*.

For each $n = 2, 3, \ldots$, let

$$\mathscr{F}_n = \bigcup_{K>0} \Phi_{K,n}. \tag{1.10}$$

Then we have

Theorem 2. *Let $n = 2, 3, \ldots$ be fixed. Then $\mathscr{F}_n^* = \mathscr{H}_n$ and $\mathscr{F}_n \cup \mathscr{H}_n$ is a group with composition.*

The basic properties of the *conjugate distortion functions* we state as

Theorem 3. *For each $K, L > 0$, $n = 2, 3, \ldots$, we have:*

(i) *for every fixed $L \in (0;\infty)$ $\Phi^*_{L,n}$ is a decreasing automorphism of $(0;1)$ and for each $t \in (0;1)$ $\Phi^*_{L,n}$ is a decreasing diffeomorphism of $(0;\infty)$ onto $(0;1)$;*
(ii) $\Phi^*_{L,n} \circ \Phi^*_{K,n} = \Phi_{K/L,n}$;
(iii) $\Phi^*_{L,n} \circ \Phi_{K,n} = \Phi^*_{LK,n}$;
(iv) $\Phi^*_{L,n} = \Phi_{1/L,n} \circ \Phi^*_{1,n}$;
(v) $\Phi^*_{L,n} \circ \Phi_{K,n} \circ \Phi^*_{L,n} = \Phi_{1/K,n}$;
(vi) $\Phi^*_{1,2}(t) = \sqrt{1-t^2}$, $\Phi^*_{2,2}(t) = (1-t)/(1+t)$, $\Phi^*_{4,2}(t) = (1-\sqrt{t}\,)/(1+\sqrt{t}\,)^2$, *etc.*, $0 \le t \le 1$.

Proof. The properties (i), …, (v) follow from the definition of $\Phi^*_{L,n}$ and the properties of $\Phi_{K,n}$. Since $M_2(\sqrt{1-t^2}\,) = \mu(\sqrt{1-t^2}\,) = \mathscr{K}(t)/\mathscr{K}(\sqrt{1-t^2}\,) = 1/\mu(t)$ then $\mu^{-1}\big(1/\mu(t)\big) = \sqrt{1-t^2}$. Making use of (iii) we obtain: $\Phi^*_{2,2}$; $\Phi^*_{4,2}$; $\Phi^*_{8,2}$; etc.

The functional identities (1.2) have a natural geometric interpretation. To this let us consider the mapping

$$\tilde{\Phi}_{K,n}(t,x) = \big(\Phi_{K,n}(t), \Phi_{1/K,n}(x)\big) \tag{1.11}$$

that maps $\overline{Q}$ onto itself for each $K > 0$ and $n = 2, 3, \ldots$. We have

$$\begin{aligned}\tilde{\Phi}_{K,n}\big(t, \Phi^*_{L,n}(t)\big) &= \big(\Phi_{K,n}(t), \Phi_{1/K,n} \circ \Phi^*_{L,n}(t)\big)\\ &= \big(\Phi_{K,n}(t), \Phi^*_{L,n}\big(\Phi_{K,n}(t)\big)\big)\end{aligned} \tag{1.12}$$

for each $0 \le t \le 1$, $K, L > 0$, and $n = 2, 3, \ldots$. Thus every curve

$$\Gamma_{L,n} = \big\{\,(t,x) : x = \Phi^*_{L,n}(t),\ 0 < t < 1\,\big\}, \qquad L > 0 \tag{1.13}$$

is the *invariant curve* under each mapping from $\mathscr{F}_n$, for $n = 2, 3, \ldots$. Moreover, by Theorem 1, we have

$$Q = \bigcup_{L>0} \Gamma_{L,n} \tag{1.14}$$

for $n = 2, 3, \ldots$.

As an immediate consequence of Theorem 1 we have

Theorem 4. *Let $n = 2, 3, \ldots$ be fixed. The following identities:*

(i) $M_n(t)M_n\big(\Phi^*_{L,n}(t)\big) = L_n$;

(ii) $\gamma_n(1/t)\gamma_n\big(1/\Phi^*_{L,n}(t)\big) = (\omega^\circ_{n-1})^2/L$;

(iii) $\tau\big((1/t)^2 - 1\big)\tau_n\big(\big(1/\Phi^*_{L,n}(t)\big)^2 - 1\big) = \pi^{2(1-n)}\omega^2_{n-1}/L$

holds for $0 < t < 1$ and each $L > 0$. Moreover, these three identities and the identity (1.2) *are equivalent.*

Proof. By (1.9), (0.1) and (1.2) we obtain (i). The identities (ii) and (iii) follow from (0.2) and (1.1).

In the case of $n = 2$ we additionally have

Corollary 1. *The identity*

$$\mu = L\frac{\mathscr{K} \circ \Phi^*_{L,2}}{\mathscr{K} \circ \Phi_{L,2}} \tag{1.15}$$

holds for each $L > 0$, where $\mathscr{K}$ is the elliptic integral.

This observation is an immediate consequence of (0.3) and the identity (i) of Theorem 4. It is worth-while to note that (1.15) generalize the identities of Landen (cf. [AVV2, p. 6]).

Corollary 2. *By Theorem* 3 *we have:* $\Phi^*_{4,2}(t) = \big((1-\sqrt{t}\,)/(1+\sqrt{t}\,)\big)^2$, $\Phi^*_{8,2}(t) = \big((\sqrt{1+t} - \sqrt[4]{t}\,)/(\sqrt{1+t} + \sqrt[4]{t}\,)\big)^2$, $\ldots$ *and, generally,* $\Phi^*_{2^n,2} = \Phi^*_{1,2} \circ \Phi^n_{2,2}$, $n = 1, 2, \ldots$, *where* $\Phi_{2,2}(t) = 2\sqrt{t}/(1+t)$, *(see* [V, *p.* 68]*).*

Corollary 3. *By Theorem* 3 *and Corollary* 2 *we have:*

(i) *setting $n = 2$, then $L = 1$ or $L = 2$ into* (1.2) *we obtain the well-known identities* (3.4), (3.5), $\ldots$, (3.9), *presented in* [AVV1];

(ii) *setting $n = 2$, then $L = 4$ with $t = r^2$ into* (1.2) *we obtain* (3.10) *of* [AVV1];

(iii) *the identity* (1.7) *of* [AVV1] *follows from* (v) *of Theorem* 3;

(iv) *setting $n = 2$, then $L = 1$ or $L = 2$ into* (i) *of Theorem* 4 *then, in view of* (0.2), *we obtain the identities* (5.57) *of* [V].

Corollary 4. *For every $K > 0$ and $0 \le t \le 1$, we have*

(i) $\Phi^*_{K,2}(t) = \sqrt{1 - \Phi^2_{K,2}(t)}$;

(ii) $\Phi^*_{2K,2}(t) = \big(1 - \Phi_{K,2}(t)\big)/\big(1 + \Phi_{K,2}(t)\big)$;

(iii) $\Phi^*_{4K,2}(t) = \big(1 - \sqrt{\Phi_{K,2}(t)}\,\big)^2/\big(1 + \sqrt{\Phi_{K,2}(t)}\,\big)^2$;

(iv) $\Phi^*_{8K,2}(t) = \big(\sqrt{1+\Phi_{K,2}(t)} - \sqrt[4]{4\Phi_{K,2}(t)}\,\big)^2/\big(\sqrt{1+\Phi_{K,2}(t)} + \sqrt[4]{4\Phi_{K2}(t)}\,\big)$; *etc.*

Corollary 5. *There exists no $L > 0$ such that $\Phi^*_{L,2}(t) = 1 - t$.*

We recall that an explicit estimate

$$t^{1/K_n} \le \Phi_{K,n}(t) \le \lambda_n^{1-1/K_n} t^{1/K_n} \tag{1.16}$$

holds for $K \ge 1$, $n = 2, 3, \ldots$, and $0 \le t \le 1$, where $K_n = K^{1/(n-1)}$. The constant λ_n is known only when $n = 2$, in this case $\lambda_2 = 4$ [LV, p. 62]. Generally, $2e^{0.76(n-1)} \le \lambda_n \le 2e^{n-1}$, for $n \ge 3$ (see [An1], [AF] and [V, p. 89]).

Now we prove

Theorem 5. *For each* $L \ge 1$, $n = 2, 3, \ldots$, *the following inequalities*

$$\begin{cases} \lambda_n^{1-L_n} \big(\Phi^*_{1,n}(t)\big)^{L_n} \le \Phi^*_{L,n}(t) \le \big(\Phi^*_{1,n}(t)\big)^{L_n}, \\ \quad \big(\Phi^*_{1,n}(t)\big)^{1/L_n} \le \Phi^*_{1/L,n}(t) \le \lambda_n^{1-1/L_n} \big(\Phi^*_{1,n}(t)\big)^{1/L_n} \end{cases} \tag{1.17}$$

holds for $0 \le t \le 1$, *with* $L_n = L^{1/(n-1)}$.

Proof. Using (iv) of Theorem 3, with $L = 1$, then by the inequality (1.16) we obtain the second row of (1.17). Since $\Phi_{1/K,n} \circ \Phi^*_{1,n} = \Phi^*_{K,n}$, then the first row of (1.17) follows, if we apply inequality (1.16) to $\Phi^{-1}_{K,n} = \Phi_{1/K,n}$.

In the particular case $n = 2$ we have

Corollary 6. *By* (vi) *of Theorem* 3 *then by Theorem* 5, *we have*

$$\begin{cases} 4^{1-L}(1-t^2)^{L/2} \le \Phi^*_{L,2}(t) \le (1-t^2)^{L/2}, \\ \quad (1-t^2)^{1/2L} \le \Phi^*_{1/L,2}(t) \le 4^{1-1/L}(1-t^2)^{1/2L}, \end{cases} \tag{1.17$'$}$$

for each $0 \le t \le 1$ *and* $L \ge 1$.

2. Applications

From (v) of Theorem 3 and (1.16), applied to $\Phi_{1/K,n} = \Phi^{-1}_{K,n}$, we see that

$$\Phi^*_{L,n}\Big(\big(\Phi^*_{L,n}(t)\big)^{K_n}\Big) \le \Phi_{K,n}(t) \le \Phi^*_{L,n}\Big(\lambda_n^{1-K_n}\big(\Phi^*_{L,n}(t)\big)^{K_n}\Big) \tag{2.1}$$

holds for every $0 \le t \le 1$, $n = 2, 3, \ldots$, $K \ge 1$ and $L > 0$, where $K_n = K^{1/(n-1)}$.

Let

$$b_n[K,L](t) = \Phi^*_{L,n}\Big(\big(\Phi^*_{L,n}(t)\big)^{K_n}\Big) \tag{2.2}$$

and

$$B_n[K,L](t) = \Phi^*_{L,n}\Big(\lambda_n^{1-K_n}\big(\Phi^*_{L,n}(t)\big)^{K_n}\Big) \tag{2.3}$$

for $K \geq 1$, $L > 0$, $0 \leq t \leq 1$, $n = 2, 3, \ldots$, and $K_n = K^{1/(n-1)}$. From the properties of $\Phi^*_{L,n}$ and the inequality (1.16), it follows that

$$\begin{cases} b_n[K,L](t) \leq \Phi_{K,n}(t) \leq B_n[K,L](t), \\ \Phi^*_{1,n}(B_n[K,L](t)) \leq \Phi^*_{K,n}(t) \leq \Phi^*_{1,n}(b_n[K,L](t)) \end{cases} \tag{2.4}$$

holds for all $L > 0$, $0 \leq t \leq 1$, $n = 2, 3, \ldots$, and $K \geq 1$.

Setting $n = 2$ then $L = 2$ into the first row of (2.4), we immediately obtain (vii) of [Z]. Setting $n = 2$ then $L = 4$, we see that

$$\Big(\frac{(1+\sqrt{t}\,)^K - (1-\sqrt{t}\,)^K}{(1+\sqrt{t}\,)^K + (1-\sqrt{t}\,)^K}\Big)^2 \leq \Phi_{K,2}(t) \leq \Big(\frac{(1+\sqrt{t}\,)^K - 2^{1-K}(1-\sqrt{t}\,)^K}{(1+\sqrt{t}\,)^K + 2^{1-K}(1-\sqrt{t}\,)^K}\Big)^2 \tag{2.5}$$

holds for $0 \leq t \leq 1$ and $K \geq 1$. The right-hand inequality is [AVV3, Theorem 5.7].

By Corollary 2 then (2.4), we can improve (2.5) when taking $L = 2^i$, $i \geq 3$. It can be easily checked by computer, which is also useful to illustrate (2.2), (2.3) and (2.4).

To explain the nature of (2.4) and the idea of the *conjugate distortion functions* we prove first

Lemma. *For each* $K > 0$ *and* $n = 2, 3, \ldots$,

$$\lim_{L\to\infty} \Phi^*_{L,n} \circ \varphi \circ \Phi^*_{L,n}(t) = \Phi_{K,n}(t) \tag{2.6}$$

for $0 \leq t \leq 1$, *where* φ: $[0;1] \to [0;1]$ *is any function such that*

$$\lim_{t\to 0^+} \frac{\log \varphi(t)}{\log t} = K_n. \tag{2.7}$$

Proof. Let $K > 0$, $n = 2, 3, \ldots$ be arbitrary. It follows from (2.7) and [P, Theorem 3.1] that for $0 \leq t \leq 1$,

$$\lim_{L\to\infty} \Phi_{L,n} \circ \varphi \circ \Phi_{L,n}(t) = \Phi_{1/K,n}(t).$$

Hence, and by Theorem 3, we get

$$\begin{aligned} \Phi^*_{L,n} \circ \varphi \circ \Phi^*_{L,n}(t) &= \Phi^*_{1,n} \circ (\Phi_{L,n} \circ \varphi \circ \Phi_{L,n}) \circ \Phi^*_{1,n}(t) \\ &\to \Phi^*_{1,n} \circ \Phi_{1/K,n} \circ \Phi^*_{1,n}(t) = \Phi_{K,n}(t) \qquad \text{as } L \to \infty \end{aligned}$$

for $0 \leq t \leq 1$, which ends the proof.

Now we can prove

Theorem 6. *For each* $K \geq 1$, $n = 2, 3, \ldots$ *and* $0 \leq t \leq 1$,

$$(2.8) \quad \begin{cases} \lim_{L\to\infty} b_n[K,L](t) = \lim_{L\to\infty} B_n[K,L](t) = \Phi_{K,n}(t), \\ \lim_{L\to\infty} \Phi^*_{1,n}(b_n[K,L](t)) = \lim_{L\to\infty} \Phi^*_{1,n}(B_n[K,L](t)) = \Phi^*_{K,n}(t). \end{cases}$$

Moreover, b_n *is an increasing function of* L *whereas* B_n *is a decreasing function of* $L \in (0,\infty)$.

Proof. Setting $\varphi_1(t) = t^{K_n}$ and $\varphi_2(t) = \lambda_n^{1-K_n} t^{K_n}$, we have

$$\lim_{t\to 0^+} \frac{\log \varphi_1(t)}{\log t} = \lim_{t\to 0^+} \frac{\log \varphi_2(t)}{\log t} = K_n, \qquad K_n = K^{1/(n-1)}.$$

This, in view of the Lemma and (iii) of Theorem 3 then by (2.2) and (2.3), gives (2.8). The second statement is derived from parallel properties of the distortion function $\Phi_{K,n}$.

By Theorem 3 and Theorem 6 it follows that the sequences $b_2[K,2^i]$ and $B_2[K,2^i]$, $i = 1, 2, \ldots$, of elementary functions converge to $\Phi_{K,2}$ uniformly. This gives a new, pure numerical, method to estimate $\Phi_{K,2}$ and any functional of it.

It seems worth-while to note that the basic approximation Theorem 6 was previously a conjecture, cf. [Z]. By a significant result obtained by D. Partyka [P, Theorem 3.1], relevant to this matter, it was possible to prove (2.8).

For every $0 \leq t < 1$, $n = 2, 3, \ldots$, and $K, L > 0$, set

$$(2.9) \qquad \lambda_n[K,L](t) = \frac{\Phi_{K,n}(t)}{\Phi^*_{L,n}(t)}.$$

It satisfies the following functional identities

$$(2.10) \qquad \lambda_n[K,L]\big(\Phi_{M,n}(t)\big) = \lambda_n[KM, LM](t)$$

and

$$(2.11) \qquad \lambda_n[K,L]\big(\Phi^*_{M,n}(t)\big) = 1/\lambda_n[M/L, M/K](t).$$

By Theorem 3, Corollary 6 and Corollary 4, the following inequalities

$$(2.12) \qquad \frac{b_n[K,M](t)}{\Phi^*_{1,n}\big(b_n[L,M](t)\big)} \leq \lambda_n[K,L](t) \leq \frac{B_n[K,M](t)}{\Phi^*_{1,n}\big(B_n[L,M](t)\big)}$$

hold for every $0 \leq t < 1$, $K, L \geq 1$ and each $M > 0$.

In connection with study of quasisymmetric functions of the real line [LV] and the unit circle [K] the distortion function $\lambda(K)$ introduced by Lehto, Virtanen and

Väisälä (see [LV, (6.4), p. 81]), has found applications. A generalization of this, introduced by Agard [Ag], namely $\lambda(K,t)$, has been studied by Vamanamurthy and Vuorinen [VV].

We have

$$\big(\lambda_2[K,K](t)\big)^2 = \lambda(K,t)$$

and

$$\big(\lambda_2[K,K](1/\sqrt{2}\,)\big)^2 = \lambda(K).$$

Setting in (2.12) $M=4$ and $L=K$, we get

$$\begin{aligned}\frac{\big(1/(1-t)\big)^K\big[(1+\sqrt{t}\,)^K-(1-\sqrt{t}\,)^K\big]^4}{8\big[(1+\sqrt{t}\,)^{2K}+(1-\sqrt{t}\,)^{2K}\big]} &\le \lambda(K,t)\\ &\le \frac{\big(2/(1-t)\big)^K\big[(1+\sqrt{t}\,)^K-2^{1-K}(1-\sqrt{t}\,)^K\big]^4}{16\big[(1+\sqrt{t}\,)^{2K}+4^{1-K}(1-\sqrt{t}\,)^{2K}\big]}.\end{aligned}\tag{2.13}$$

By (2.12) and Theorem 6 we see that $\lambda_2[K,L](t)$ can be approximated by elementary functions.

Other functionals of $\Phi_{K,n}$ and $\Phi^*_{L,n}$, with applications, will be considered in an additional paper.

A sharp estimation for $\max_{0\le t\le 1}\big[\Phi_{K,2}(t)-t\big]$, $K\ge 1$, has been obtained by the author [Z, Theorem 2]. It says that for each $K\ge 1$,

$$\max_{0\le t\le 1}\big[\Phi_{K,2}(t)-t\big] \le \begin{cases} 1-\dfrac{1+4^{1-K}}{2K}, & 1\le K\le K_0,\\[2mm] \dfrac{1-4^{1-K}}{1+4^{1-K}}, & K>K_0,\end{cases}\tag{2.14}$$

where K_0 satisfies the equation $(1+4^{1-K})^2 = K4^{2-K}$, $2.481 < K_0 < 2.482$.

Taking advantage of (2.4) we improve (2.14) obtaining

Theorem 7. *For each* $K\ge 1$,

$$\max_{0\le t\le 1}\big[\Phi_{K,2}(t)-t\big] \le B_2[K,4](t_0)-t_0\tag{2.15}$$

where t_0 *is such that* $B_2'[K,4](t_0)=1$.

Proof. At first we show that $B_2[K,4]$ is concave. To this end let us differentiate $B_2[K,4](t)$ with respect to t, $0<t<1$, we obtain

$$\begin{aligned}B_2'[K,4](t) = K2^{3-K}&\frac{(1-t)^{K-1}}{\big[(1+\sqrt{t}\,)^K+2^{1-K}(1-\sqrt{t}\,)^K\big]^2}\times\\ &\times\frac{1}{\sqrt{t}}\,\frac{1-2^{1-K}\big((1-\sqrt{t})/(1+\sqrt{t})\big)^K}{1+2^{1-K}\big((1-\sqrt{t})/(1+\sqrt{t})\big)^K}.\end{aligned}$$

Introducing $x = (1-\sqrt{t}\,)/(1+\sqrt{t}\,)$, and considering

$$f(x) = \frac{1+x}{1-x}\,\frac{1-2^{1-K}x^K}{1+2^{1-K}x^K}, \qquad 0 < x < 1$$

we can see that

$$\big[\ln f(x)\big]' = \frac{2}{1-x^2} - K2^{1-K}x^{K-1}\frac{2}{1-4^{1-K}x^{2K}} \geq 0$$

for $0 < x < 1$ and $K \geq 1$.

We shall prove that

$$(2.16) \qquad 1 - 4^{1-K}x^{2K} \geq K2^{1-K}x^{K-1}(1-x^2) \qquad \text{for } 0 \leq x \leq 1 \quad \text{and } K \geq 1.$$

Note, that for $K = 1$ the inequality attains the equality sign. Because

$$\partial_1 = \frac{\partial}{\partial K}(1-4^{1-K}x^{2K}) = -8\Big(\frac{x}{2}\Big)^{2K}\ln\frac{x}{2} > 0 \qquad \text{for } 0 \leq x \leq 1, \quad K \geq 1$$

and

$$\partial_2 = \frac{\partial}{\partial K}\big(K2^{1-K}x^{K-1}(1-x^2)\big) = \Big(\frac{x}{2}\Big)^{K-1}(1-x^2)\,\Big(1 + K\ln\frac{x}{2}\Big) \leq 0$$

holds for $0 < x < 2/e$ and $K \geq 1$, then (2.16) remains true for $0 \leq x \leq 2/e$ and $K \geq 1$.

Let $2/e \leq x \leq 1$ and $1 \leq K \leq 3/2$. Hence

$$\begin{aligned} 8\Big(\frac{x}{2}\Big)^{2K} &\geq \frac{8}{e^{K+1}}\Big(\frac{x}{2}\Big)^{K-1} \geq \frac{8}{e^{5/2}}\Big(\frac{x}{2}\Big)^{K-1} \\ &\geq \Big(\frac{x}{2}\Big)^{K-1}\Big(1-\frac{4}{e^2}\Big) \geq \Big(\frac{x}{2}\Big)^{K-1}(1-x^2) \end{aligned}$$

and

$$(2.17) \qquad -\ln\frac{x}{2} > 1 + K\ln\frac{x}{2} \qquad \Leftrightarrow \qquad (K+1)\ln\frac{x}{2} < -1.$$

Thus $\partial_1 - \partial_2 \geq 0$ for $2/e \leq x \leq 1$ and $1 \leq K \leq 3/2$. By this we see that (2.16) holds for $2/e \leq x \leq 1$ and $1 \leq K \leq 3/2$.

Suppose now that $K > 3/2$ and $0 \leq x \leq 1$. Then

$$(2.18) \qquad 1 + K\ln\frac{x}{2} \leq 1 + K\ln\frac{1}{2} \leq 1 + \ln\frac{1}{2^{3/2}} < 0$$

and thus $\partial_2 < 0$ for $K \geq 3/2$. In [Z, p. 7] it is proved that the first ratio of $B_2'[K,4](t)$ is decreasing. This fact, with our considerations on f, shows that $B_2[K,4]$ is concave, and our proof is complete.

The theory of the conjugate distortion functions presented in this paper afford us to state the following:

(i) $b_n[K,L]$ and $B_n[K,L]$ are concave as functions of variable $t \in [0;1]$ for every $K > 1$, $L > 0$ and $n = 2, 3, \ldots$;

(ii) $b_n[K,L]$ is increasing whereas $B_n[K,L]$ is decreasing as functions L, $L > 0$ for every $K > 1$ and $n = 2, 3, \ldots$.

The convexity and concavity of the conjugate distortion functions seems to be an interesting topic for investigation on special functions.

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Received 11 February 1992