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NULL SETS FOR DOUBLING AND DYADIC DOUBLING MEASURES

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Abstract. In this note, we study sets on the real line which are null with respect to all doubling measures on $\mathbf{R}$, or with respect to all dyadic doubling measures on $\mathbf{R}$. We give some sufficient conditions for the former, a test for the latter, and some examples.

Our work is motivated by a characterization of dyadic doubling measures by Fefferman, Kenig and Pipher [5], and by a result of Martio [8] on porous sets and sets of total $\mathscr{A}$-harmonic measure zero for certain class of nonlinear $\mathscr{A}$-operators.

A measure μ on $\mathbf{R}$ is said to have the *doubling property* with constant λ if, whenever I and J are two neighboring intervals of same length then $\mu(I) \le \lambda\mu(J)$; denote by $\mathscr{D}(\lambda)$ the collection of all doubling measures with constant λ, and $\mathscr{D} = \cup_{\lambda\ge 1}\mathscr{D}(\lambda)$. A measure μ on $\mathbf{R}$ has the *dyadic doubling property* with constant λ if $\mu(I) \le \lambda\mu(J)$ whenever I and J are two dyadic neighboring intervals of same length and $I\cup J$ is also a dyadic interval; denote by $\mathscr{D}_d(\lambda)$ and $\mathscr{D}_d$ the corresponding collections of dyadic doubling measures.

Given $\{a_n\}$, $0<\alpha_n<1$, a set $E\subseteq\mathbf{R}$ is called $\{\alpha_n\}$*-porous* if there exists a sequence of coverings $\mathscr{E}_n = \{E_{n,j}\}$ of E, by intervals with mutually disjoint interiors, so that each $E_{n,j}\setminus E$ contains an interval $J_{n,j}$ of length $\ge \alpha_n|E_{n,j}|$, $\cup_{\mathscr{E}_{n+1}}E_{n+1,k}$ is contained in $\cup_{\mathscr{E}_n}(E_{n,j}\setminus J_{n,j})$ and $\sup|E_{n,j}|\to 0$ as $n\to\infty$. The Cantor ternary set is $\{\frac13\}$-porous. The porous sets studied by Martio [8] are $\{\alpha\}$-porous for some $\alpha>0$.

Theorem 1. *If $0<\alpha_n<1$, $\sum_1^\infty \alpha_n^K=\infty$ for all $K\ge 1$, and E is $\{\alpha_n\}$-prorous, then E is null for all doubling measures on $\mathbf{R}$.*

Corollary. *There exist sets of Hausdorff dimension one which are null for all doubling measures.*

The condition given in Theorem 1 cannot be improved:

Theorem 2. *If $0<\alpha_n<\frac14$ is a decreasing sequence satisfying $\sum_n \alpha_n^K<\infty$ for some $K\ge 1$, then there exists a perfect set which is $\{\alpha_n\}$-porous, but carries a positive measure for some $\mu\in\mathscr{D}$.*

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Let E be a closed set in $[0,1]$ and $\lambda > 1$. In Theorem 3, we give a deterministic procedure of testing whether E is $\mathscr{D}_d(\lambda)$-null. In this process, an optimal measure $\mu_{E,\lambda}$ among $\mathscr{D}_d(\lambda)$ is selected for E. The precise statement is given in Section 2.

Denote by $\mathscr{N}$ the collection of null sets for doubling measures $\{E : \mu(E) = 0$ for all $\mu \in \mathscr{D}\}$, and $\mathscr{N}_d$ its dyadic counterpart $\{E : \mu(E) = 0$ for all $\mu \in \mathscr{D}_d\}$. Clearly $\mathscr{N}_d \subseteq \mathscr{N}$ and $\mathscr{N}$ is translation invariant. The assertion that $\mathscr{N}_d \neq \mathscr{N}$ is not suprising, however it requires a lot of work.

Theorem 4. *There exists a perfect set $S \subseteq [0,1]$ which is in $\mathscr{N} \setminus \mathscr{N}_d$. And corresponding to this S, there exists a set T of dimension one, so that $t + S \in \mathscr{N}_d$ for each $t \in T$.*

It would be interesting to know whether a pair of sets S, T can be chosen to satisfy length $(T) > 0$ in addition to the properties in Theorem 4.

Theorem 5. *Let t be any number whose binary expansion has infinitely many zeros and infinitely many ones. Then there exists a perfect set S_t so that $S_t \in \mathscr{N} \setminus \mathscr{N}_d$ but $t + S_t \in \mathscr{N}_d$.*

Finally, in Section 4, we shall comment on relations betwen sets in $\mathscr{N}$ and null sets of the harmonic measures with respect to the p-Laplacians in the upper half plane.

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1. Proofs of Theorems 1 and 2

We first state a useful lemma.

Lemma 1. *Let μ be a dyadic doubling measure on $[0,1]$ and I be any subinterval. Then there exists $K > 1$ depending on the dyadic doubling constant λ only, so that*

$$4|I|^{1/K}\mu\big([0,1]\big) \geq \mu(I) \geq \tfrac{1}{4}|I|^{K}\mu\big([0,1]\big).$$

Proof. Let I_1 and I_2 be two adjacent dyadic closed intervals in $[0,1]$ such that $I \subseteq I_1 \cup I_2$ and $2|I| \geq |I_1| + |I_2|$. Then

$$\begin{aligned}\mu(I) &\leq \mu(I_1) + \mu(I_2)\\ &\leq \bigg(\Big(\frac{\lambda}{1+\lambda}\Big)^{-\log_2|I_1|} + \Big(\frac{\lambda}{1+\lambda}\Big)^{-\log_2|I_2|}\bigg)\mu\big([0,1]\big)\\ &\leq 2\big(2|I|\big)^{\log_2((1+\lambda)/\lambda)}\mu\big([0,1]\big) \leq 4|I|^{\log_2((1+\lambda)/\lambda)}\mu\big([0,1]\big).\end{aligned}$$

If

$$|I| \geq 1 - 16^{-(\log_2((1+\lambda)/\lambda))^{-1}} \equiv A,$$

then

$$\mu\bigl([0,1]\setminus I\bigr) \le 8\bigl(1-|I|\bigr)^{\log_2((1+\lambda)/\lambda)}\mu\bigl([0,1]\bigr) \le \tfrac{1}{2}\mu\bigl([0,1]\bigr).$$

Hence $\mu(I) \ge \frac{1}{2}\mu\bigl([0,1]\bigr)$. If $|I| < A$, let J be the largest dyadic interval contained in I. Therefore $|J| \ge |I|/4$ and

$$\begin{aligned}\mu(I) \ge \mu(J) &\ge \Bigl(\frac{1}{1+\lambda}\Bigr)^{-\log_2|J|}\mu\bigl([0,1]\bigr) \ge \bigl(|I|/4\bigr)^{\log_2(1+\lambda)}\mu\bigl([0,1]\bigr)\\ &\ge \mu\bigl([0,1]\bigr)|I|^{(\log_2(1+\lambda))(1-2(\log_2 A)^{-1})}.\end{aligned}$$

Proof of Theorem 1. Assume $E \subseteq [0,1]$, and let $\mathscr{E}_n = \{E_{n,j}\}$ be the coverings of E and $\{J_{n,j}\}$ be the subintervals of $E_{n,j}\setminus E$ in defining $\{\alpha_n\}$-porosity. Let $\mu \in \mathscr{D}$, it follows from Lemma 1 that $\mu(J_{n,j}) \ge \frac{1}{4}\alpha_n^K\mu(E_{n,j})$ for some $K \ge 1$ depending on μ only. Thus

$$\mu(E_{n,j}\setminus J_{n,j}) \le (1-\tfrac{1}{4}\alpha_n^K)\mu(E_{n,j}).$$

Summing over j, we obtain

$$\sum_k \mu(E_{n+1,k}) \le (1-\tfrac{1}{4}\alpha_n^K)\sum_j \mu(E_{n,j}).$$

Therefore

$$\mu(E) \le \prod_n (1-\tfrac{1}{4}\alpha_n^K)\mu\bigl([0,1]\bigr) = 0.$$

Proof of Theorem 2. Let N_n be a rapidly increasing sequence of odd integers with $N_1 = 1$, $N_n \ge \alpha_{n-1}^{-1}$ for $n \ge 2$. After replacing α_n by a number which is at most twice its size, we may assume that $\alpha_n = m_n N_{n+1}^{-1}$ for some odd integer m_n. The construction of E resembles that of the Cantor set. First we remove the open interval which constitutes the middle α_1 position of $[0,1]$, and subdivide the two remaining closed intervals into subintervals of equal length N_2^{-1}, call this collection of subintervals $\mathscr{S}_1$. This subdivision is possible due to the modification on α_n's. On each interval in $\mathscr{S}_1$, remove the open interval which constitutes its middle α_2 portion, and subdivide the remaining intervals into subintervals of equal length $(N_2N_3)^{-1}$, call this new collection of subintervals $\mathscr{S}_2$. Continue the process indefinitely and let

$$E = \bigcap_n\Bigl(\bigcup_{I\in\mathscr{S}_n} I\Bigr).$$

Clearly E is $\{\alpha_n\}$-porous.

It remains to choose N_n so that $\mu(E) > 0$ for some $\mu \in \mathscr{D}$. Our idea comes from Ahlfors and Beurling [2; Theorem 3]. First, we construct a function h which plays the role of $1+\lambda$cosine in [2].

Lemma 2. *Given $0<\alpha<\frac14$, $K>2$, there exists a function h continuous on $\mathbf{R}$, of period 1, monotonic in $[0,\frac12]$ and in $[\frac12,1]$ respectively, which satisfies $\int_0^1 h(x)\,dx=1$,*

$$h(x)=\begin{cases}\alpha^{K-1} & \text{on } [\frac12(1-\alpha),\frac12(1+\alpha)],\\ 1+\sqrt{\alpha} & \text{on } [0,\frac12-\sqrt{\alpha}]\cup[\frac12+\sqrt{\alpha},1],\end{cases}$$

and $h(x)\,dx$ is in $\mathscr{D}(B^K)$ for some absolute constant $B>2$.

As an example, we may choose

$$h(x)=\alpha^{K-1}+\alpha^{-K}\Bigl(x-\frac{1+\alpha}{2}\Bigr)^{2K-1}$$

on

$$\Bigl[\frac{1+\alpha}{2},\frac{1+\alpha}{2}+\frac{\alpha^{(2K-1)/(4(K-1))}}{4}\Bigr],$$

piecewise linear on

$$\Bigl[\frac{1+\alpha}{2}+\frac{\alpha^{(2K-1)/(4(K-1))}}{4},\frac12+\sqrt{\alpha}\Bigr]$$

with derivatives between $1/4\sqrt{\alpha}$ and $4/\sqrt{\alpha}$, and $h(x)=h(1-x)$ for x in $[\frac12-\sqrt{\alpha},\frac12(1-\alpha)]$, so that the continuity, monotonicity and $\int_0^1 h(x)\,dx=1$ are satisfied. For this h, $h\,dx\in\mathscr{D}(B^k)$ for some absolute constant $B>2$.

In the hypothesis $\sum\alpha_n^K<\infty$, we may assume $K>2$. Corresponding to each pair (α_n,K), we fix a function h_n which satisfies properties in Lemma 2 with $\alpha=\alpha_n$. Denote by

$$A_n=\bigcup_{k=-\infty}^{\infty}\bigl([k,k+\tfrac12-\tfrac12\alpha_n]\bigr)\cup\bigl([k+\tfrac12+\tfrac12\alpha_n,k+1]\bigr),$$

$$F_k=\bigcup_{I\in\mathscr{S}_k}I,\qquad M_n=\prod_1^n N_k,\quad\text{and}\quad f_n(x)=\prod_1^n h_k(M_kx).$$

We shall choose N_n inductively so that $N_{n+1}\gg N_n$ and that $f_n(x)$ is "nearly constant" on each interval of length M_{n+1}^{-1}.

Recall that $N_1=1$ and assume that odd integers $N_2, N_3,\dots, N_n$ have been chosen so that

$$\int_{F_k}f_k(x)\,dx\ge\prod_{j=1}^k(1-2\alpha_j^K)\qquad(1\le k\le n),\tag{1.1}$$

and whenever $|x-x'|\le M_k^{-1}$, $(2\le k\le n)$,

$$\frac{k-1}{k} < h_{k-1}(M_{k-1}x)/h_{k-1}(M_{k-1}x') < \frac{k+1}{k}, \tag{1.2}$$

and

$$\frac{k-1}{k} < f_{k-1}(x)/f_{k-1}(x') < \frac{k+1}{k}. \tag{1.3}$$

Note that $\int_0^1 h_k(x)\,dx = 1$, $\int_{[0,1]\cap A_k} h_k(x)\,dx = 1-\alpha_k^K$ and that $f_k(x)$ is uniformly continuous on $\mathbf{R}$ for each $k\ge 1$. Let $F\subseteq[0,1]$ be any measurable set. Then $\chi_F f_n$ is the pointwise a.e. and L^1 limit of an increasing sequence of simple functions, each of which has the form $\sum a_j\chi_{I_j}$, where $\{a_j\}$ are constants and $\{I_j\}$ are finitely many mutually disjoint open intervals with rational end points. Therefore,

$$\int_0^1 \chi_F(x)f_n(x)\chi_{A_{n+1}}(Mx)h_{n+1}(Mx)\,dx \to (1-\alpha_{n+1}^K)\int_F f_n(x)\,dx$$

as $M\to\infty$. Thus a large odd integer N_{n+1} can be found so that $N_{n+1}>\alpha_n^{-1}$, and (1.1), (1.2) and (1.3) hold with $k=n+1$. Here we have used the fact that $F_{n+1}=\{x\in F_n : M_{n+1}x\in A_{n+1}\}$.

Let μ be a weak limit point of $f_n(x)\,dx$. Then $\mu(E)>0$ in view of (1.1) and $\sum\alpha_n^K<\infty$. To verify that μ is a doubling measure we consider two neighboring intervals I and I' satisfying

$$M_{n+1}^{-1}\le |I| = |I'| \le M_n^{-1}.$$

In view of (1.3)

$$\Big(\frac{n-1}{n}\Big)^2 \le \frac{f_{n-1}(x)}{f_{n-1}(x')} \le \Big(\frac{n+1}{n}\Big)^2$$

whenever $x\in I$ and $x'\in I'$. We note that $f_m(x)/f_n(x)$ has period M_{n+1}^{-1} if $m\ge n+1$, and that

$$\frac{n}{n+1} < \frac{h_n(M_nx)}{h_n(M_nx')} < \frac{n+2}{n+1}$$

whenever $|x-x'|\le M_{n+1}^{-1}$. Writing

$$f_m(x) = f_{n-1}(x)h_n(M_nx)\frac{f_m(x)}{f_n(x)}, \qquad \text{for } m\ge n+1,$$

we deduce from the fact $h_n(M_nx)\,dx\in\mathscr{D}(B^K)$ that

$$(CB^K)^{-1} \le \int_I f_m(x)\,dx\Big/\int_{I'} f_m(x)\,dx \le CB^K$$

for every $m\ge n+1$. Therefore $\mu\in\mathscr{D}$.

2. A test for $\mathscr{D}_d(\lambda)$-null sets

Given a closed set E in $[0,1]$, we shall develop a procedure to test whether E is in $\mathscr{D}_d(\lambda)$ for some $\lambda > 1$.

Let $I_{n,j}$, $1 \le j \le 2^n$, be the dyadic closed intervals in $[0,1]$ of length 2^{-n}, $I^r_{n,j}$ be the closed interval which forms the right half of $I_{n,j}$ and let $I^l_{n,j} = I_{n,j} \setminus I^r_{n,j}$. Let

$$h_{n,j}(x) = \begin{cases} 1 & \text{on } I^r_{n,j} \\ -1 & \text{on } I^l_{n,j} \\ 0 & \text{on } \mathbf{R} \setminus I_{n,j}. \end{cases}$$

For a fixed integer $n \ge 2$, define

$$f^{(n)}_n \equiv \prod_{j=1}^{2^n} \bigl(1 + \delta(n,j)\tau h_{n,j}\bigr),$$

and $d\nu^{(n)}_n \equiv f^{(n)}_n dx$, where $\tau = (\lambda-1)/(\lambda+1)$ and

$$\delta(n,j) = \begin{cases} 1, & E \cap I^r_{n,j} \ne \emptyset, \\ -1, & E \cap I^r_{n,j} = \emptyset. \end{cases}$$

Denote by $E_n = \cup\bigl\{I_{n+1,j} : I_{n+1,j} \cap E \ne \emptyset, 1 \le j \le 2^{n+1}\bigr\}$. Let

$$f^{(n)}_{n-1} \equiv \prod_{j=1}^{2^{n-1}} \bigl(1 + \delta(n-1,j)\tau h_{n-1,j}\bigr),$$

and $d\nu^{(n)}_{n-1} \equiv f^{(n)}_{n-1} d\nu^{(n)}_n$, where

$$\delta(n-1,j) = \begin{cases} 1, & \nu^{(n)}_n(I^r_{n-1} \cap E_n) \ge \nu^{(n)}_n(I^l_{n-1} \cap E_n), \\ -1, & \text{otherwise.} \end{cases}$$

After defining $f^{(n)}_k$ and $d\nu^{(n)}_k$, we let

$$f^{(n)}_{k-1} \equiv \prod_{j=1}^{2^{k-1}} \bigl(1 + \delta(k-1,j)\tau h_{k-1,j}\bigr),$$

and $d\nu^{(n)}_{k-1} \equiv f^{(n)}_{k-1} d\nu^{(n)}_k$, where

$$\delta(k-1,j) = \begin{cases} 1, & \nu^{(n)}_k(I^r_{k-1,j} \cap E_n) \ge \nu^{(n)}_k(I^l_{k-1,j} \cap E_n), \\ -1, & \text{otherwise.} \end{cases}$$

Continue this process until we arrive at $\nu_1^{(n)}$. Define $\mu_n \equiv \nu_1^{(n)}$.

Notice that

$$\nu_k^{(n)}(I_{k,j}) = 2^{-k} \qquad (1 \le k \le n) \tag{2.1}$$

with the understanding that $I_{0,j} \equiv I_{0,1} \equiv [0,1]$; and that from $d\nu_k^{(n)}$ to $d\nu_{k-1}^{(n)}$, total measure in each $I_{k-1,i}$ is kept unchanged, but the total measures of $I_{k-1,i}^r$ and $I_{k-1,i}^l$ are redistributed in the most advantageous way.

Repeat for each $n \ge 2$, to obtain a sequence of measures $\{\mu_n\}$. Let $\mu_{E,\lambda}$ be a weak limit point of $\{\mu_n\}$, extended to $\mathbf{R}$ with period 1.

Theorem 3. *Among all the measures in $\mathscr{D}_d(\lambda)$ which have mass one on $[0,1]$, $\mu_{E,\lambda}$ has the maximum measure on E. In particular, E is $\mathscr{D}_d(\lambda)$-null if and only if $\mu_{E,\lambda}(E) = 0$.*

Proof. It is clear that $\mu_{E,\lambda}\big([0,1]\big) = 1$ and $\mu_{E,\lambda} \in \mathscr{D}_d(\lambda)$.

Let ω be any measure in $\mathscr{D}_d(\lambda)$ with $\omega\big([0,1]\big) = 1$. We claim, in fact, that

$$\omega(E_n) \le \mu_n(E_n). \tag{2.2}$$

Let m be the largest integer in $[1,n]$, if it exists, such that there exists at least one interval $I_{m,j}$ on which

$$\frac{\omega(I_{m,j}^r)}{\omega(I_{m,j}^l)} \ne \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)}. \tag{2.3}$$

(If such m does not exist, then $\omega(E_n) = \mu_n(E_n)$.) We shall redistribute the measure ω on these $I_{m,j}$'s and keep ω unchanged elsewhere. Denote by $\mathscr{I}_m = \{I_{m,j} : \text{(2.3) holds on } I_{m,j}\}$; and let $\omega_m = \omega$ on $[0,1] \setminus \cup_{\mathscr{I}_m} I_{m,j}$, and

$$d\omega_m = \begin{cases} \dfrac{\omega(I_{m,j})\mu_n(I_{m,j}^r)}{\omega(I_{m,j}^r)\mu_n(I_{m,j})}\, d\omega & \text{on } I_{m,j}^r, \\[2ex] \dfrac{\omega(I_{m,j})\mu_n(I_{m,j}^l)}{\omega(I_{m,j}^l)\mu_n(I_{m,j})}\, d\omega & \text{on } I_{m,j}^l, \end{cases} \tag{2.4}$$

for each $I_{m,j} \in \mathscr{I}_m$. Clearly, if $I_{m,j} \in \mathscr{I}_m$, then

$$\frac{\omega_m(I_{m,j}^r)}{\omega_m(I_{m,j}^l)} = \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)},$$

and ω_m is dyadic on $I_{m,j}$ with constant $\le \lambda$. Actually, ω_m is in $\mathscr{D}_d(\lambda)$ by the following lemma.

Lemma 3. *Let I be a dyadic subinterval of $[0,1]$, and μ and ν be dyadic doubling measures on $[0,1]$ and I respectively, satisfying $\mu(I)=\nu(I)$. Then the new measure ω defined by $\omega\equiv\nu$ on I, $\equiv\mu$ on $[0,1]\setminus I$ is dyadic doubling on $[0,1]$ with constant bounded by the maximum of those of μ and ν.*

First, we shall verify that $\omega(E_n)\le\omega_m(E_n)$. To show this, it is enough to prove

$$\omega(E_n\cap I_{m,j})\le\omega_m(E_n\cap I_{m,j}) \tag{2.5}$$

for each $I_{m,j}\in\mathscr{I}_m$.

Fix $I_{m,j}\in\mathscr{I}_m$, clearly (2.5) holds when $m=n$. Thus we assume $m<n$ and note from the definition of m that

$$\frac{\omega(I^r_{k,i})}{\omega(I_{k,i})}=\frac{\mu_n(I^r_{k,i})}{\mu_n(I_{k,i})},\qquad\text{and}\qquad\frac{\omega(I^l_{k,i})}{\omega(I_{k,i})}=\frac{\mu_n(I^l_{k,i})}{\mu_n(I_{k,i})}$$

for $m+1\le k\le n$. Therefore

$$\frac{\omega(I^r_{m,j}\cap E_n)}{\omega(I^r_{m,j})}=\frac{\mu_n(I^r_{m,j}\cap E_n)}{\mu_n(I^r_{m,j})}$$

and

$$\frac{\omega(I^l_{m,j}\cap E_n)}{\omega(I^l_{m,j})}=\frac{\mu_n(I^l_{m,j}\cap E_n)}{\mu_n(I^l_{m,j})}.$$

Moreover, from the construction of μ_n,

$$\frac{\mu_n(I_{n+1,l})}{\mu_n(I_{k,i})}=\frac{\nu_k^{(n)}(I_{n+1,l})}{\nu_k^{(n)}(I_{k,i})}, \tag{2.6}$$

if $1\le k\le n$ and $I_{n+1,l}\subseteq I_{k,i}$. Thus by (2.1) and the above identities,

$$\frac{\omega(I^r_{m,j})\cap E_n}{\omega(I^r_{m,j})}=\nu^{(n)}_{m+1}(I^r_{m,j}\cap E_n)2^{m+1} \tag{2.7}$$

and

$$\frac{\omega(I^l_{m,j})\cap E_n}{\omega(I^l_{m,j})}=\nu^{(n)}_{m+1}(I^l_{m,j}\cap E_n)2^{m+1}. \tag{2.8}$$

Writing I in place of $I_{m,j}$ for the rest of this paragraph, we obtain

$$\begin{aligned}\omega(E_n\cap I)&=\Big[\frac{\omega(E_n\cap I^r)}{\omega(I^r)}\frac{\omega(I^r)}{\omega(I)}+\frac{\omega(E_n\cap I^l)}{\omega(I^l)}\frac{\omega(I^l)}{\omega(I)}\Big]\omega(I)\\&=2^{m+1}\omega(I)\Big[\nu^{(n)}_{m+1}(E_n\cap I^r)\frac{\omega(I^r)}{\omega(I)}+\nu^{(n)}_{m+1}(E_n\cap I^l)\frac{\omega(I^l)}{\omega(I)}\Big]\\&\le 2^{m+1}\omega(I)\big(\nu^{(n)}_{m+1}(E_n\cap I^r)A+\nu^{(n)}_{m+1}(E_n\cap I^l)(1-A)\big)\end{aligned}$$

where $A = \frac{1}{2}(1+\tau)$ if $\nu_{m+1}^{(n)}(E_n \cap I^r) \geq \nu_{m+1}^{(n)}(E_n \cap I^l)$, and $A = \frac{1}{2}(1-\tau)$ otherwise. From the definition of $\nu_m^{(n)}$, (2.6), (2.7), and (2.8), it follows that

$$\begin{aligned} \omega(E_n \cap I) &\leq 2^{m+1}\omega(I)\Big[\nu_{m+1}^{(n)}(E_n \cap I^r)\frac{\nu_m^{(n)}(I^r)}{\nu_m^{(n)}(I)} + \nu_{m+1}^{(n)}(E_n \cap I^l)\frac{\nu_m^{(n)}(I^l)}{\nu_m^{(n)}(I)}\Big] \\ &= \omega(I)\Big[\frac{\omega(E_n \cap I^r)}{\omega(I^r)}\frac{\mu_n(I^r)}{\mu_n(I)} + \frac{\omega(E_n \cap I^l)}{\omega(I^l)}\frac{\mu_n(I^l)}{\mu_n(I)}\Big] = \omega_m(E_n \cap I). \end{aligned}$$

This proves (2.5) and hence $\omega(E_n) \leq \omega_m(E_n)$.

We proceed to make modifications of ω_m on each dyadic interval $I_{m-1,i}$ of size 2^{-m+1} on which (2.3) holds with m, j, ω replaced by $m-1$, i and ω_m respectively, according to the rule (2.4) adapted for $m-1$, i and ω_m; call this new measure ω_{m-1}. Continue to modify ω_{m-1} on dyadic intervals of size 2^{-m+2} if necessary to obtain $\omega_{m-2}, \ldots$. Finally we arrive at a measure ω_1, and obtain

$$\omega(E_n) \leq \omega_m(E_n) \leq \omega_{m-1}(E_n) \leq \cdots \leq \omega_1(E_n)$$

and

$$\frac{\omega_1(I_{m,j}^r)}{\omega_1(I_{m,j}^l)} = \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)},$$

for all $1 \leq m \leq n$, $1 \leq j \leq 2^m$. Therefore $\omega_1(E_n) = \mu_n(E_n)$ and (2.2) is proved.

We note that $E = \cap_m E_m$. Therefore for any $\varepsilon > 0$ and sufficiently large m and n with $m > m(\varepsilon)$ and $n > n(\varepsilon, m)$, we have

$$\begin{aligned} \mu_{E,\lambda}(E) &\geq \mu_{E,\lambda}(E_m) - \varepsilon \geq \mu_n(E_m) - 2\varepsilon \geq \mu_n(E_n) - 2\varepsilon \\ &\geq \omega(E_n) - 2\varepsilon \geq \omega(E) - 2\varepsilon. \end{aligned}$$

This shows that $\omega(E) \leq \mu_{E,\lambda}(E)$.

3. Proofs of Theorems 4 and 5

Given a, ε, $\delta \in (0,1)$, $\varepsilon a < \delta < \varepsilon$, we choose a sequence of integers $\{n_k\}$ satisfying $n_k \geq 4$ and

$$n_{n+1} > n_k + [\varepsilon \log_2 k]. \tag{3.1}$$

For $k \geq 2 + [2^{1/\delta}]$ and $0 \leq j \leq 2^{n_k} - 1$, denote by

$$L_{k,j} = \Big[\frac{j}{2^{n_k}}, \frac{j+1}{2^{n_k}}\Big],$$

$$I_{k,j} = \Big[\frac{j}{2^{n_k}}, \frac{j}{2^{n_k}} + \frac{1}{2^{n_k} k^{\delta}}\Big],$$

and

$$J_{k,j} = \Big[\frac{j+1}{2^{n_k}} - \frac{1}{2^{n_k}\dot{k}^{\varepsilon}}, \frac{j+1}{2^{n_k}}\Big],$$

where $\dot{k}^{\delta} = 2^{[\delta \log_2 k]}$, $\dot{k}^{\varepsilon} = 2^{[\varepsilon \log_2 k]}$ and $[\ ]$ is the greatest integer function. Note that intervals L's, I's and J's are dyadic,

$$(3.2) \qquad |J_{k,j}|/|I_{k,j'}| = O(k^{\delta-\varepsilon}) = o(1) \qquad \text{as } k \to \infty,$$

and

$$(3.3) \qquad |J_{k,j}|/|L_{k+1,j'}| = 2^{n_{k+1}-n_k-[\varepsilon \log_2 k]} > 2.$$

The construction of a set $S \in \mathscr{N} \setminus \mathscr{N}_d$ is similar to the Cantor set; collections of nested intervals from $\{J_{k,j}\}$ are used. The measure μ in $\mathscr{D}_d$ to be produced with $\mu(S) > 0$ will satisfy

$$\frac{\mu(J_{k,j})}{\mu(L_{k,j})} = \Big(\frac{|J_{k,j}|}{|L_{k,j}|}\Big)^a$$

on infinitely many $J_{k,j}$'s.

Let $\{K_i\}$ be an increasing sequence of integers with $K_0 \equiv 2 + [2^{1/\delta}]$ and some other properties to be specified later. Let $S_0 = [0,1]$, $\mathscr{C}^I_{1+K_0}$ be the collection of all $I_{1+K_0,j} \subseteq S_0$ and $\mathscr{C}^J_{1+K_0}$ be the collection of all $J_{1+K_0,j} \subseteq S_0$. After $\mathscr{C}^I_k$ and $\mathscr{C}^J_k$ have been defined for some k, $1 + K_0 \le k \le K_1 - 1$, we let

$$\mathscr{C}^I_{k+1} = \mathscr{C}^I_k \cup \big\{I_{k+1,j} \subseteq S_0 : I_{k+1,j} \text{ is not contained in any interval in } \mathscr{C}^I_k \cup \mathscr{C}^J_k\big\},$$

$$\mathscr{C}^J_{k+1} = \mathscr{C}^J_k \cup \big\{J_{k+1,j} \subseteq S_0 : J_{k+1,j} \text{ is not contained in any interval in } \mathscr{C}^I_k \cup \mathscr{C}^J_k\big\};$$

and let

$$S^I_1 = \text{union of all intervals in } \mathscr{C}^I_{K_1},$$

$$S_1 = \text{union of all intervals in } \mathscr{C}^J_{K_1}.$$

Next let $\mathscr{C}^I_{1+K_1}$ be the collection of all $I_{1+K_1,j} \subseteq S_1$ and $\mathscr{C}^J_{1+K_1}$ be the collection of all $J_{1+K_1,j} \subseteq S_1$. And define for each k, $1 + K_1 \le k \le K_2 - 1$,

$$\mathscr{C}^I_{K+1} = \mathscr{C}^I_k \cup \big\{I_{k+1,j} \subseteq S_1 : I_{k+1,j} \text{ is not contained in any interval in } \mathscr{C}^I_k \cup \mathscr{C}^J_k\big\},$$

$$\mathscr{C}^J_{K+1} = \mathscr{C}^J_k \cup \big\{J_{k+1,j} \subseteq S_1 : J_{k+1,j} \text{ is not contained in any interval in } \mathscr{C}^I_k \cup \mathscr{C}^J_k\big\},$$

$$S^I_2 = \text{union of all intervals in } \mathscr{C}^I_{K_2},$$

and

$$S_2 = \text{union of all intervals in } \mathscr{C}^J_{K_2}.$$

Clearly $S^I_2 \subseteq S_1$ and $S_2 \subseteq S_1$.

Continue this procedure to obtain $\mathscr{C}^I_{K_3}$, $\mathscr{C}^J_{K_3}$, S^I_3 and S_3, ..., and so on, and let

$$S = \bigcap_1^\infty S_m.$$

To construct $\mu \in \mathscr{D}_d$ with $\mu(S) > 0$, we shall use scale invariant versions of Lemma 3 and the following lemma repeatedly.

Lemma 4. *Given a, α, $\beta \in (0,1)$ with $\alpha^a + \beta < 1/16$ and $c_1, c_2 \in (\frac{1}{2}, 2)$, there exists a measure $\mu \in \mathscr{D}_d(10^{1/a})$, which satisfies $\mu\big([0,1]\big) = 1$, $\mu\big([0,\alpha]\big) = c_1\alpha^a$, and $\mu\big([1-\beta,1]\big) = c_2\beta$.*

As an example, we may choose

$$\mu\big([0,t]\big) = \begin{cases} c_1 t^a, & 0 \le t \le t_0 \equiv \big(\frac{1}{8}\big)^{1/a}, \\ \frac{1}{8}c_1 + \big(1 - \frac{1}{8}(c_1+c_2)\big)(t-t_0)/(\frac{7}{8} - t_0), & t_0 \le t \le \frac{7}{8}, \\ c_2 t + 1 - c_2, & \frac{7}{8} \le t \le 1. \end{cases}$$

Then extend μ periodically to $\mathbf{R}$ with period 1.

All measures μ_k defined below are periodic with period 1. Choose $\mu_{1+K_0} \in \mathscr{D}_d(10^{1/a})$ so that

$$\begin{aligned} \mu_{1+K_0}(L_{1+K_0,j}) &= |L_{1+K_0,j}|, \\ \mu_{1+K_0}(I_{1+K_0,j}) &= |L_{1+K_0,j}|(1+K_0)^{-\delta}, \\ \mu_{1+K_0}(J_{1+K_0,j}) &= |L_{1+K_0,j}|(1+K_0)^{-\varepsilon a} \end{aligned}$$

for each $0 \le j \le 2^{1+K_0} - 1$. After μ_k is selected for some k, $1 + K_0 \le k \le K_1$, we choose $\mu_{k+1} \in \mathscr{D}_d(10^{1/a})$, so that $\mu_{k+1} = \mu_k$ on each interval in $\mathscr{C}_k^I \cup \mathscr{C}_k^J$, and μ_{k+1} is a redistribution of μ_k on each $L_{k+1,j}$ which is not contained in any interval in $\mathscr{C}_k^I \cup \mathscr{C}_k^J$:

$$\mu_{k+1}(L_{k+1,j}) = \mu_k(L_{k+1,j}), \tag{3.4}$$

$$\mu_{k+1}(I_{k+1,j}) = (1+k)^{-\delta}\mu_{k+1}(L_{k+1,j}), \tag{3.5}$$

$$\mu_{k+1}(J_{k+1,j}) = (1+k)^{-\varepsilon a}\mu_{k+1}(L_{k+1,j}). \tag{3.6}$$

The measure μ_{K_1} so chosen has the properties that

$$\mu_{K_1}(S_1^I \cup S_1) = 1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a} - k^{-\delta})$$

and

$$\begin{aligned} \mu_{K_1}(S_1) &\ge \mu_{K_1}(S_1^I \cup S_1) \inf_{1+K_0 \le k \le K_1} \frac{k^{-\varepsilon a}}{k^{-\varepsilon a} + k^{-\delta}} \\ &\ge \Big(1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a})\Big)(1 - K_0^{\varepsilon a - \delta}), \end{aligned}$$

because $\varepsilon a < \delta$.

Next choose $\mu_{1+K_1} \in \mathscr{D}_d(10^{1/a})$ so that $\mu_{1+K_1} = \mu_{K_1}$ on $S_0 \setminus S_1$, and on each $L_{1+K_1,j} \subseteq S_1$ it is a redistribution of μ_{K_1} satisfying (3.4), (3.5) and (3.6) with $k = K_1$. After μ_k is constructed for some k, $1 + K_1 \le k < K_2$, build μ_{k+1} from μ_k following the same steps as in the case $1 + K_0 \le k \le K_1$. The dyadic doubling measure μ_{K_2} so obtained belongs to $\mathscr{D}_d(10^{1/a})$, moreover

$$\mu_{K_2}(S_2^I \cap S_2) = \Bigl(1 - \prod_{1+K_1}^{K_2} (1 - k^{-\varepsilon a} - k^{-\delta})\Bigr)\mu_{K_1}(S_1),$$

and

$$\begin{aligned}\mu_{K_2}(S_2) &\ge \mu_{K_2}(S_2^I \cup S_2) \inf_{1+K_1 \le k \le K_2} \frac{k^{-\varepsilon a}}{k^{-\varepsilon a} + k^{-\delta}} \\ &\ge \mu_{K_2}(S_2^I \cup S_2)(1 - K_1^{\varepsilon a - \delta}) \\ &\ge \Bigl(1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a})\Bigr)\Bigl(1 - \prod_{1+K_1}^{K_2} (1 - k^{-\varepsilon a})\Bigr)(1 - K_0^{\varepsilon a - \delta})(1 - K_1^{\varepsilon a - \delta}).\end{aligned}$$

Whenever μ_{K_m} is constructed, keep $\mu_{1+K_m} = \mu_{K_m}$ on $S_0 \setminus S_m$, redistribute the mass on each $L_{1+K_m,j} \subseteq S_m$ according to (3.4), (3.5) and (3.6) with $k = K_m$, and keep the dyadic doubling constant bounded by $10^{1/a}$. Continue this indefinitely. Thus, we obtain a sequence of measures $\mu_{K_m} \in \mathscr{D}_d(10^{1/a})$, with $\mu_{K_m}\bigl([0,1]\bigr) = 1$ and

$$\mu_{K_m}(S_m) \ge \prod_{i=0}^{m-1} \bigl((1 - K_i^{\varepsilon a} - \delta)(1 - A_i)\bigr)$$

where $A_i = \prod_{1+K_i}^{K_{1+i}} (1 - k^{-\varepsilon a})$. Let μ be a weak limit point of $\{\mu_{K_m}\}$. Clearly $\mu \in \mathscr{D}_d(10^{1/a})$.

Since $\varepsilon a < 1$, it is possible to choose $\{K_i\}$ so that

$$\sum_{i=1}^{\infty} K_i^{\varepsilon a - \delta} + \sum_{i=1}^{\infty} A_i < +\infty. \tag{3.7}$$

With respect to this choice of $\{K_i\}$, we have $\mu(S) > 0$, hence $\mu \notin \mathscr{N}_d$.

It remains to show that $S \in \mathscr{N}$. Let $\nu \in \mathscr{D}$. Recall that $J_{k,j}$ and $I_{k,j+1}$ have the common boundary point $(j+1)/(2^{n_k})$; by the doubling property

$$\nu(J_{k,j} \cup I_{k,j+1}) \ge A^{(\varepsilon - \delta)\log_2 k - 5} \nu(J_{k,j})$$

for some $A > 1$ depending only on the doubling constant of ν. For $m \ge 2$, intervals in $\mathscr{C}^J_{K_m} \cup \{I_{k,j+1} : J_{k,j} \in \mathscr{C}^J_{K_m}\}$ ($\ne \mathscr{C}^J_{K_m} \cup \mathscr{C}^I_{K_m}$) may meet in their interiors; however, because of (3.3), every point in $[0,1]$ is covered by at most three such intervals. Therefore

$$3\nu\big([0,1]\big) \geq \sum_{J_{k,j}\in\mathscr{C}^J_{K_m}} \nu(J_{k,j}\cup I_{k,j+1}) \geq A^{(\varepsilon-\delta)\log_2 K_{m-1}-5}\nu(S_m)$$
$$\geq A^{(\varepsilon-\delta)\log_2 K_{m-1}-5}\nu(S).$$

Hence $\nu(S)=0$. Therefore $S\in\mathscr{N}\setminus\mathscr{N}_d$.

Let

$$T=\Big\{t=\sum_{n=1}^{\infty}t_n2^{-n},\ \text{where } t_n=0 \text{ or } 1,\ \text{but } t_{n_k+[\delta\log_2 k]+1}=1 \text{ and } t_{n_k+[\delta\log_2 k]+2}=0 \text{ for each integer } k>K_0\Big\}.$$

In view of (3.1), it has Hausdorff dimension 1.

Fix $t\in T$ and $\nu\in\mathscr{D}_d$ and let $J_{k,j}$ be any interval in $\mathscr{C}^J_{K_m}$. We note that

$$\frac{p+\frac12}{2^{n_k}\dot{k}^\delta}<t+\frac{j+1}{2^{n_k}}<\frac{p+\frac34}{2^{n_k}\dot{k}^\delta}$$

for some integer p, because

$$q+\tfrac12<t2^{n_k}\dot{k}^\delta<q+\tfrac34$$

for some integer q.

Therefore $t+J_{k,j}$ is contained in the middle half of some dyadic interval

$$M_{k,j}=\Big[\frac{p}{2^{n_k}\dot{k}^\delta},\frac{p+1}{2^{n_k}\dot{k}^\delta}\Big].$$

Recall that the interval $I_{k,j+1}$ shares an end point $(j+1)/2^{n_k}$ with $J_{k,j}$ and has length $1/2^{n_k}\dot{k}^\delta$. Therefore

$$\big|(t+I_{k,j+1})\cap M_{j,k}\big|>\tfrac14\frac{1}{2^{n_k}\dot{k}^\delta}.$$

The dyadic doubling property of ν, (3.2) and Lemma 1,

$$\nu\big(t+(J_{k,j}\cup I_{k,j+1})\cap M_{k,j}\big)\geq c(k,\nu)\nu(t+J_{k,j})$$

with $c(k,\nu)\to\infty$ as $k\to\infty$. Summing over all $J_{k,j}$ in $\mathscr{C}^J_{K_m}$ and reasoning as before, we obtain

$$3\nu\big([0,1]\big)\geq c(K_{m-1},\nu)\nu(t+S_m)\geq c(K_{m-1},\nu)\nu(t+S).$$

Letting $m\to\infty$, we have $\nu(t+S)=0$. This completes the proof of Theorem 4.

It would be interesting to characterize those t's so that $t+S$ is in $\mathscr{N}_d$. However this seems difficult.

To prove Theorem 5, we note that in the binary expansion of t, the event that a digit 1 is followed immediately by a digit 0 occurs infinitely often. Choose ε, δ and a as in Theorem 4, and $\{n_k\}$ depending on t, so that (3.1),

$$t_{n_k+[\delta \log_2 k]+1} = 1 \qquad \text{and} \qquad t_{n_k+[\delta \log_2 k]+2} = 0$$

hold for each $k > k_0$. Let $S_t \equiv S$ in Theorem 4 associated with this sequence $\{n_k\}$. Then $S_t \in \mathscr{N} \setminus \mathscr{N}_d$. The proof of $t + S_t \in \mathscr{N}_d$ is similar to that in Theorem 4.

4. Null sets for p-harmonic measures

Consider the p-Laplace equation ($1 < p < \infty$)

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0$$

in the half plane $\Omega \equiv \{x \in \mathbf{R}^2 : x_2 > 0\}$. For the definition and properties of p-harmonic measure (the harmonic measure for p-Laplacian) see [6; Chapter 10].

Let E be a compact set on $\partial\Omega$ which has positive p-harmonic measure for some p. Then there exists a nonconstant solution u ($0 \le u \le 1$) of the p-Laplacian in Ω, with continuous boundary value 0 on $\partial\Omega \setminus E$.

Following [1], we may apply a linearization technique in [7] or an approximation technique in [4], and Theorem 4.5 in [3], to write

$$u(x) = \int_{\partial\Omega} K(x,y) f(y)\, d\omega(y),$$

where K is a limit of kernel functions and ω is a weak limit of harmonic measures at a fixed point, corresponding to a sequence of uniformly elliptic operators of nondivergence form in Ω, with ellipticity constants depending only on p. Moreover ω has the doubling property and u has nontangential limit on $\partial\Omega$ ω-a.e.

Because u has zero boundary value on $\partial\Omega \setminus E$, $f(y)\, d\omega(y)$ is supported in E. This implies that $\omega(E) > 0$. Therefore, we have

Theorem 6. *Compact sets in $\mathscr{N}$ are null sets for any p-harmonic measure with respect to the half plane $\{x \in \mathbf{R}^2 : x_2 > 0\}$.*

Remark. Martio has defined a version of porosity and proved that a porous set on $\{x_2 = 0\}$ has zero $\mathscr{A}$-harmonic measure with respect to all those nonlinear operators $\mathscr{A}$ on $\{x_2 > 0\}$ considered in [8]; p-Laplacians are examples of such operators. We do not know whether Theorem 6 can be extended to all such $\mathscr{A}$-operators. However a compact set E on $\{x_2 = 0\}$ is $\mathscr{A}$-harmonic measure null for all such $\mathscr{A}$ if it satisfies a stronger $\{a_n\}$-porous condition for some $\{a_n\}$ ($\sum \alpha_n^K = \infty$ for all $K > 1$), namely, in defining $\{a_n\}$-porosity, $E \cap E_{n,j}$ is required to lie in the middle $1 - 2\alpha_n$ portion of $E_{n,j}$ for each n and j. Proof follows by combining the original proof of Martio and that of Theorem 1.

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