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HOLOMORPHIC CURVATURE OF INFINITE DIMENSIONAL SYMMETRIC COMPLEX BANACH MANIFOLDS OF COMPACT TYPE

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Abstract. We show that the tangent norm on an infinite dimensional symmetric complex Banach manifold of compact type has constant positive holomorphic curvature (in the sense of [2], 8.7).

Introduction

Symmetric Banach manifolds generalise to infinite dimensions the Hermitian symmetric spaces. The finite dimensional Hermitian symmetric spaces were classified by Cartan, in [1], using the theory of Lie groups and Lie algebras. The theory of J^*-triple systems appears to be the more appropriate tool in infinite dimensions and a classification of the symmetric Banach manifolds in terms of J^*-triple systems is given by Kaup in [5].

The J^*-triple systems, or J^*-triples, which are Banach spaces endowed with a certain triple product structure, generalise the concepts of C^*-algebra, J^*-algebra, and JB^*-algebra. The J^*-triples which are positive in a certain sense, corresponding to a certain class of linear operators on the space having positive spectrum, are called JB^*-triples. It is known that, for the above classification, the JB^*-triples characterise the bounded symmetric domains, or, in finite dimensions, the Hermitian symmetric spaces of non-compact type.

Changing the sign of the triple product on a J^*-triple, U say, produces another J^*-triple which is referred to as the dual triple of the J^*-triple U. In particular, the dual triple of a JB^*-triple is negative in a certain sense. Moreover, the symmetric Banach manifold associated, via Kaup's construction, to the dual triple of a JB^*-triple is called a symmetric manifold of compact type, since in finite dimensions these manifolds are exactly the Hermitian symmetric spaces of compact type.

In this way, to every JB^*-triple we can associate a bounded symmetric domain and a symmetric manifold of compact type (classified by the dual triple).

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This gives an exact analogy of the duality in finite dimensions between the Hermitian symmetric spaces of compact and non-compact type. While the bounded symmetric domains are relatively well understood and studied, little is known of the symmetric manifolds of compact type in infinite dimensions.

Examples include certain Grassmann manifolds (*cf.* [6]) and certain function spaces (*cf.* [7]).

As infinite dimensional analogues of the compact Hermitian symmetric spaces, these symmetric manifolds of compact type may be expected to display some properties reminiscent of compactness. In this paper we show this to be the case, by proving that the holomorphic curvature of an arbitrary symmetric manifold of compact type is constant and positive (with a value of $+4$).

The concept of holomorphic curvature that we use, is one that may be defined, for an arbitrary infinitesimal Finsler metric on a complex Banach manifold, in terms of a generalised Hessian. It is important to note, that this does not coincide with the concept of holomorphic curvature as understood in the Riemannian sense. For a survey in this direction, see [2].

Notation

Throughout, we let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ and for r, $0 < r < 1$, $r\mathbf{D} = \{rz : z \in \mathbf{D}\}$. We use $\mathrm{Dom}(f)$ to denote the domain of a function f. For complex Banach spaces E and F, let $L(E,F)$ denote the Banach space of all continuous linear maps: $E \to F$ and let $L(E) = L(E,E)$. We denote by $L^k(E)$ the space of all continuous homogeneous polynomials: $E \to E$ of degree k. We say $T \in L(E)$ is hermitian if $e^{i\lambda T}$ is an isometry for all $\lambda \in \mathbf{R}$.

Let U be a complex Banach space with continuous conjugate-linear mapping $*\colon U \to L^2(U)$ and write a^* for $*(a)$. For all a, b, z in U define

$$\{a, b^*, z\} := \frac{1}{2}\big(b^*(a+z) - b^*(a) - b^*(z)\big)$$

and define $a \,\square\, b^*$ in $L(U)$ by $a \,\square\, b^*(z) := \{a, b^*, z\}$, for all z in U.

Then $(U, *)$ is called a J^*-triple system, or J^*-triple, if and only if

(i) $\{\alpha, \beta^*, \{x, y^*, z\}\} = \{\{\alpha, \beta^*, x\}, y^*, z\} - \{x, \{\beta, \alpha^*, y\}^*, z\} + \{x, y^*, \{\alpha, \beta^*, z\}\}$ for all α, β, x, y and z in U.

(ii) $\alpha \,\square\, \alpha^* \in L(U)$ is hermitian for all α in U.

If, in addition, the following hold

(iii) $\alpha \,\square\, \alpha^* \geq 0$ for all $\alpha \in U$

(iv) $\|\alpha \,\square\, \alpha^*\| = \|\alpha\|^2$ for all α in U

we call U a JB^*-triple.

We denote by U_z the closed J^*-subtriple of U generated by z in U.

A morphism of J^*-triple systems $\lambda\colon (U,*) \to (V,*)$ is a continuous linear map $\lambda\colon U \to V$ such that

$$\lambda\bigl(\{u,v^*,w\}\bigr) = \bigl\{\lambda(u),\lambda(v)^*,\lambda(w)\bigr\}$$

for all u,v,w in U.

A manifold M modelled locally on open subsets of complex Banach spaces with biholomorphic coordinate transformations is called a complex Banach manifold. Let TM denote the tangent bundle of M.

A mapping $\alpha\colon TM \to \mathbf{R}$ is called a norm on TM if the restriction of α to every tangent space T_x, $x \in M$, is a norm on T_x with the following property: there is a neighbourhood U of x in M which can be realised as a domain in a complex Banach space E such that

$$c\|a\| \le \alpha(u,a) \le C\|a\|$$

for all $(u,a) \in TU \cong U \times E$ and suitable constants $0 < c \le C$. We then refer to (M,α), or simply M, as a normed manifold. If $(\tilde{M},\tilde{\alpha})$ is another complex normed manifold, we say that a holomorphic mapping $\varphi\colon M \to \tilde{M}$ is an isometry if for all

$$(z,v) \in TM, \qquad \tilde{\alpha}\bigl(\varphi(z),\varphi'(z)v\bigr) = \alpha(z,v).$$

Let $\mathrm{Aut}(M)$ denote the group of all biholomorphic isometries of M.

A connected complex normed manifold M is called symmetric if for every $a \in M$ there exists an involution $s_a \in \mathrm{Aut}(M)$ having a as an isolated fixed point.

A morphism of the symmetric manifolds M and $\tilde{M}$ is a holomorphic mapping $h\colon M \to \tilde{M}$ such that $h \circ s_x = s_{h(x)} \circ h$ for all x in M. The characterisation of symmetric manifolds in terms of J^*-triple systems is given in [5].

Let u be a real-valued upper semicontinuous function with $\mathrm{Dom}(u) \subset \mathbf{C}$. For $p \in \mathrm{Dom}(u)$, we define the generalised Hessian of u at p, as

$$\Delta u(p) = 4 \liminf_{\substack{r\to 0\\ r\neq 0}} \frac{1}{r^2}\left\{\frac{1}{2\pi}\int_0^{2\pi} u(p+re^{i\theta})\,d\theta - u(p)\right\} \in \mathbf{R}\cup\{\infty\}.$$

Let $(p,v) \in TM$. We say $f \in \mathscr{H}(p,v)$ if for some r, $0 < r < 1$, f is a holomorphic mapping: $r\mathbf{D} \to M$ with $f(0) = p$, $f'(0) = v$. For $f \in \mathscr{H}(p,v)$ we define $\lambda_f\colon r\mathbf{D} \to \mathbf{R}$ by $\lambda_f(z) = \alpha^2\bigl(f(z),f'(z)\bigr)$.

For $(p,v) \in TM$ with $v \neq 0$, we may then define the holomorphic curvature of α at p in the direction of v, *cf.* [1], denoted $\kappa_\alpha(p,v)$, as

$$\kappa_\alpha(p,v) = \sup\left\{\frac{\Delta \log\lambda_f(0)}{-2\alpha^2(p,v)} : f \in \mathscr{H}(p,v)\right\}.$$

Clearly, for $\lambda \neq 0 \in \mathbf{C}$, $\kappa_\alpha(p,v) = \kappa_\alpha(p,\lambda v)$, for all $(p,v) \in TM$ with $v \neq 0$.

If α_1 and α_2 are tangent norms on the complex manifolds D_1 and D_2 respectively and $\varphi\colon D_1 \to D_2$ is a biholomorphic mapping which is an isometry for α_1 and α_2 then

$$\kappa_{\alpha_1}(p,v) = \kappa_{\alpha_2}\big(\varphi(p), \varphi'(p)v\big) \qquad \text{for } (p,v) \in TD_1.$$

Results

Let U be an arbitrary JB^*-triple and let M be the associated symmetric manifold of compact type (*i.e.* M is the symmetric manifold associated via Kaup's construction to the dual triple of U). Let m_0 denote a base point of M and let $\alpha\colon TM \to \mathbf{R}$ be the tangent norm of M. Since $\mathrm{Aut}(M)$ acts transitively on M, it suffices to consider $\kappa_\alpha(m_0, v)$ where $v \in T_{m_0}M = U$, $\|v\| = 1$. Let $Q(z)$ denote the conjugate linear mapping on U given by

$$Q(z)a = \{z, a^*, z\},$$

for all a in U.

The following linear operators on U, called Bergman operators, play an important role:

$$B(z,w^*) = I_U - 2z \,\square\, w^* + Q(z)Q(w) \qquad \text{for } z, w \in U.$$

For all z in U,

$$D_z := B(z, -z^*)^{1/2} \qquad \text{and} \qquad D_z^{-1}$$

are well defined.

By [4, 4.5], there exists a canonical chart on M, identifying a neighbourhood W of $0 \in U$, with a neighbourhood of $m_0 \in M$, with respect to which the tangent norm on M may be expressed as follows:

$$\alpha(z,v) = \|D_z^{-1}(v)\| \qquad \text{for all } (z,v) \in TW \cong W \times U.$$

We may chose $0 < \varepsilon < 1$ such that the ball $B(0, 2\varepsilon) := \{x \in U : \|x\| < 2\varepsilon\} \subseteq W$.

Fix $v \in U$, $\|v\| = 1$, arbitrary. The mapping $\varphi\colon \varepsilon\mathbf{D} \to M$, given by $\varphi(t) = tv$, is in $\mathscr{H}(0,v)$. Showing now that

$$\frac{\Delta \log \lambda_\varphi(0)}{-2\|v\|^2} \geq 4$$

implies $\kappa_\alpha(0,v) \geq 4$.

By [5, Corollary 4.8], the JB^*-subtriple of U generated by v, U_v, is isometrically J^*-isomorphic to $C_0(S)$, for some locally compact space S.

For any $z, w \in C_0(S)$, then

$$D_z^{-1}(w) = \left(\frac{1}{1+|z|^2}\right)w$$

and hence

$$\left\|D_{re^{i\theta}v}^{-1}(v)\right\| = \sup_{s\in S}\left(\frac{|v(s)|}{1+r^2|v(s)|^2}\right) = \frac{\|v\|}{1+r^2\|v\|^2}.$$

Then

$$\Delta \log \lambda_\varphi(0) = -8 \liminf_{\substack{r\to 0\\ r\neq 0}} \frac{1}{r^2}\big(\log\big(1+r^2\|v\|^2\big)\big).$$

By elementary calculus, the limit involved here is exactly

$$g'(0) = \|v\|^2 \qquad \text{for } \; g(t) = \log\big(1+t\|v\|^2\big).$$

Therefore

$$\Delta \log \lambda_\varphi(0) = -8\|v\|^2.$$

In particular

$$\kappa_\alpha(0,v) \geq \frac{\Delta \log \lambda_\varphi(0)}{-2\|v\|^2} = 4.$$

In the other direction, we examine

$$\frac{\Delta \log \lambda_\psi(0)}{-2\|v\|^2}$$

for arbitrary $\psi \in \mathscr{H}(0,v)$.

Lemma 1.1. *Let U be a JB^*-triple. Then*

$$\big\|B(z,-z^*)\big\| = \big(1+\|z\|^2\big)^2$$

for all $z \in U$.

Proof. By [3, Corollary 3], we have that for all x, y, z in a JB^*-triple U, $\big\|\{x,y^*,z\}\big\| \leq \|x\|\|y\|\|z\|$. Therefore $\big\|B(z,-z^*)\big\| \leq \big(1+\|z\|^2\big)^2$, for all $z \in U$. On the other hand, since U_z is isometrically J^*-isomorphic to $C_0(S)$, for some locally compact space S,

$$\big\|B(z,-z^*)(z)\big\| = \big(1+\|z\|^2\big)^2\|z\|$$

and the result follows. ▫

Fix ψ arbitrary in $\mathscr{H}(0,v)$. Then

$$\alpha\big(\psi(re^{i\theta}),\psi'(re^{i\theta})\big)=\Big\|D^{-1}_{\psi(re^{i\theta})}\psi'(re^{i\theta})\Big\|\geq\frac{\big\|\psi'(re^{i\theta})\big\|}{\big\|D_{\psi(re^{i\theta})}\big\|}.$$

From this

$$\Delta\log\lambda_\psi(0)\geq 4\liminf_{\substack{r\to 0\\ r\neq 0}}\frac{1}{r^2}\bigg\{\frac{1}{2\pi}\int_0^{2\pi}\big(2\log\big\|\psi'(re^{i\theta})\big\| - 2\log\|D_{\psi(re^{i\theta})}\|\big)\,d\theta-2\log\|v\|\bigg\}.$$

As the mapping $z\mapsto 2\log\big\|\psi'(z)\big\|$ is subharmonic

$$\frac{1}{2\pi}\int_0^{2\pi}2\log\big\|\psi'(re^{i\theta})\big\|\,d\theta\geq 2\log\|v\|$$

and

$$\Delta\log\lambda_\psi(0)\geq -8\liminf_{\substack{r\to 0\\ r\neq 0}}\frac{1}{r^2}\bigg\{\frac{1}{2\pi}\int_0^{2\pi}\log\big(\|D_{\psi(re^{i\theta})}\|\big)\,d\theta\bigg\}.$$

By the Lebesgue dominated convergence theorem

$$\liminf_{\substack{r\to 0\\ r\neq 0}}\frac{1}{2\pi}\int_0^{2\pi}\frac{\log\big(\|D_{\psi(re^{i\theta})}\|\big)}{r^2}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\bigg(\liminf_{\substack{r\to 0\\ r\neq 0}}\frac{\log\big(\|D_{\psi(re^{i\theta})}\|\big)}{r^2}\bigg)\,d\theta.$$

Let $\sum_{n=0}^{\infty}a_n(t-1)^n$ denote the Taylor expansion of the function $f(t)=\sqrt{t}$ for $|t-1|<1$.

Since $D_z=f\big(B(z,-z^*)\big)$ in the sense of the functional calculus, then

$$D_z=\sum_{n=0}^{\infty}a_n\big(B(z,-z^*)-I_U\big)^n$$

whenever

$$\sigma\big(B(z,-z^*)\big)\subset\big\{t\in\mathbf{R}:|t-1|<1\big\}.$$

In particular, for $\|z\|<1/2$, we have

$$\|D_z\|\leq\sum_{n=0}^{\infty}|a_n|\big((1+\|z\|^2)^2-1\big)^n.$$

As

$$a_n=\frac{(-1)^{n+1}1.3.5\cdots(2n-3)}{2^n n!}$$

for $n \geq 2$, $a_0 = 1$, and $a_1 = 1/2$, the power series

$$\sum_{n=0}^{\infty} |a_n|(t-1)^n$$

represents the function $g(t) = 2 - \sqrt{2-t}$ for $|t-1| < 1$, and hence

$$\|D_z\| \leq g\big((1+\|z\|^2)^2\big)$$

for $\|z\| < 1/2$.

The mapping $k(t) = \log(2 - \sqrt{2-t}\,)$ is represented about $t = 1$ by the expansion

$$k(t) = \frac{(t-1)}{2} + \frac{(t-1)^3 k'''\big(\xi(t)\big)}{3!}$$

where $\xi(t)$ lies between t and 1.

From above,

$$\log\big(\|D_z\|\big) \leq k\big((1+\|z\|^2)^2\big), \qquad \text{for } \|z\| < 1/2.$$

Moreover, since $\psi(0) = 0$ and $\psi'(0) = v$, it follows that

$$\liminf_{\substack{r\to 0\\ r\neq 0}} \frac{\big\|\psi(re^{i\theta})\big\|}{r} = \|v\|$$

and therefore

$$\liminf_{\substack{r\to 0\\ r\neq 0}} \frac{\log\big(\|D_{\psi(re^{i\theta})}\|\big)}{r^2} \leq \|v\|^2.$$

On the other hand, Lemma 1.1 implies that $\|D_z\| \geq 1 + \|z\|^2$, for all $z \in U$, so

$$\liminf_{\substack{r\to 0\\ r\neq 0}} \frac{\log\big(\|D_{\psi(re^{i\theta})}\|\big)}{r^2} \geq \liminf_{\substack{r\to 0\\ r\neq 0}} \frac{\log\Big(1 + \big\|\psi(re^{i\theta})\big\|^2\Big)}{r^2} = \|v\|^2.$$

Therefore

$$\liminf_{\substack{r\to 0\\ r\neq 0}} \frac{\log\big(\|D_{\psi(re^{i\theta})}\|\big)}{r^2} = \|v\|^2.$$

It follows then that

$$\frac{\Delta \log \lambda_\psi(0)}{-2\|v\|^2} \leq 4.$$

Since ψ was arbitrary

$$\kappa_\alpha(0,v) \leq 4$$

and we have proved the desired result.

Theorem 1.2. *Let U be a JB^*-triple and let M be the associated symmetric complex manifold of compact type with tangent norm α on TM.*

The holomorphic curvature of α at any point of M is identically equal to 4.

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