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ON THE MULTIPLICATIVE BEHAVIOR OF BLOCH FUNCTIONS

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Abstract. We prove that for each unbounded Bloch function f in the open unit disk D there exists a singular inner function S and a bounded outer function Q such that neither product $f\cdot S$ nor $f\cdot Q$ is a normal function.

A function f meromorphic in $D=\{z:|z|<1\}$ is called a normal function if

$$\sup_{z\in D}\left(1-|z|^2\right)\frac{|f'(z)|}{1+|f(z)|^2}<\infty.$$

A well known subfamily of the normal functions is the so-called Bloch space $\mathscr{B}$, which consists of those functions f analytic in D for which

$$\|f\|_{\mathscr{B}}=\sup_{z\in D}\left(1-|z|^2\right)|f'(z)|<\infty$$

(see, for example, [1]). It is well known that the Bloch functions form a linear space, but the collection of all normal functions do not, and, moreover, neither class is closed under multiplication (see [5], [6]). It has been shown that for each unbounded normal analytic function in D, there exists a Blaschke product B such that the product $f\cdot B$ is not a normal function (see [2], [6]), and that, in fact, the Blaschke product B can be replaced by a function g of the disk algebra, where g has zeros in D (see [3]).

In this paper, we show the following result.

Theorem. *If f is an unbounded Bloch function in D, then there exists both a singular inner function S and a bounded outer function Q such that neither the product $f\cdot S$ nor the product $f\cdot Q$ is a normal function.*

Since neither S nor Q has zeros in D, this result is not contained in any of the results mentioned above.

We recall that a *singular inner function* is a function of the form

$$S(z)=\exp\left\{-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right\}$$

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where μ is a finite positive Borel measure on ∂D which is singular with respect to Lebesgue measure. Also, a *bounded outer function* is a function of the form

$$Q(z) = e^{i\gamma} \exp\Big\{ - \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log \psi(e^{it})\, dt \Big\}$$

where γ is a real number, $\psi \in L^\infty(\partial D)$, and $\log \psi(e^{it}) \in L^1(\partial D)$ (see [4], [8]).

Proof of the Theorem. We first construct the appropriate singular inner function S.

We note that an unbounded Bloch function f must satisfy the growth condition

$$\big|f(z)\big| = O\Big(\log \frac{1}{1-|z|}\Big) \qquad \text{as } |z| \to 1,$$

(see [1, p. 13]), so it follows that there exists a sequence $\{z_n\}$ in D for which

$$\text{(1)} \qquad \big|f(z_n)\big| \to \infty \qquad \text{and} \qquad \big(1-|z_n|\big) \log \big|f(z_n)\big| \to 0 \text{ as } n \to \infty.$$

Without loss of generality, we may assume that $z_n \to 1$, that $0 < \arg z_{n+1} < \arg z_n < \frac{1}{2}\pi$, and that $\big|f(z_{n+1})\big| > \big|f(z_n)\big| > 1$ for each n.

Now consider the angular domain

$$A = \Big\{ z \in D : \operatorname{Im} z > 0, \big|\arg(1-z)\big| < \frac{3\pi}{8} \Big\}$$

and an arc $\Gamma \subset E = \{z \in D \setminus A : \operatorname{Im} z > 0\}$ where Γ has an endpoint at $z = 1$. If all but a finite number of the points z_n lie in A, then, by a result of J. McMillan (see [1, p. 33] or [7, p. 269]), $\big|f(z)\big|$ cannot be bounded on Γ, which means that we may assume that the sequence $\{z_n\}$ lies on the arc Γ, in addition to the other properties we have assumed above. Now an elementary calculation shows that $1-|z| < |e^{i\arg z} - 1|$ for each $z \in E$. From this, it is easy to see that we may assume that the sequence $\{z_n\}$ is such that it also has all of the following properties:

$$\text{(2)} \qquad 1-|z_n| < |e^{i\arg z_n} - z_k| \qquad \text{for each } k \neq n,$$

$$\text{(3)} \qquad \big(1-|z_n|\big) \log \big|f(z_n)\big| < \big(1-|z_{n-1}|\big) \frac{\log 2}{2^{n+1}}, \qquad n \geq 1,$$

and

$$\text{(4)} \qquad \big(1-|z_n|^2\big) \sum_{k=1}^{n-1} \frac{\log \big|f(z_k)\big|}{1-|z_k|} < \frac{\log 2}{2}, \qquad n \geq 2.$$

Now, for each positive integer n, let

$$t_n = \arg z_n,$$
$$c_n = \frac{1-|z_n|}{1+|z_n|} \log\bigl|f(z_n)\bigr|,$$

and

$$g(z) = \sum_{n=1}^{\infty} c_n \frac{\exp(it_n)+z}{\exp(it_n)-z}.$$

It follows from (3) that $\sum_{n=1}^{\infty} c_n < \log 2$, and so the sum defining the function $g(z)$ is convergent in D. Finally, we set

$$S(z) = \exp\bigl(-g(z)\bigr),$$

and we note that $S(z)$ is a singular inner function.

We first show that $\frac{1}{2} < \bigl|f(z_n)S(z_n)\bigr| < 1$. Note that

$$\begin{aligned}
\operatorname{Re}\bigl(g(z_n)\bigr) &= \sum_{j=1}^{\infty} c_n \frac{\bigl(1-|z_n|^2\bigr)}{\bigl|\exp(it_j)-z_n\bigr|^2} \\
&= c_n \frac{\bigl(1-|z_n|^2\bigr)}{\bigl|\exp(it_n)-z_n\bigr|^2} + \sum_{j\neq n} c_j \frac{\bigl(1-|z_n|^2\bigr)}{\bigl|\exp(it_j)-z_n\bigr|^2} \\
&< c_n \frac{\bigl(1-|z_n|^2\bigr)}{\bigl|\exp(it_n)-z_n\bigr|^2} + \sum_{j=1}^{n-1} \frac{c_j\bigl(1-|z_n|^2\bigr)}{\bigl(1-|z_j|\bigr)^2} + \sum_{j=n+1}^{\infty} \frac{c_j}{1-|z_n|}.
\end{aligned}$$

From (4), we have

$$\sum_{j=1}^{n-1} \frac{c_j\bigl(1-|z_n|^2\bigr)}{\bigl(1-|z_j|\bigr)^2} < \frac{\log 2}{2}$$

and from (3),

$$\begin{aligned}
\sum_{j=n+1}^{\infty} \frac{c_j}{1-|z_n|} &= \sum_{j=n+1}^{\infty} \frac{\bigl(1-|z_j|\bigr)\log\bigl|f(z_j)\bigr|}{\bigl(1+|z_j|\bigr)\bigl(1-|z_n|\bigr)} \\
&< \sum_{j=n+1}^{\infty} \frac{1-|z_{j-1}|}{1-|z_n|}\frac{\log 2}{2^j} < \frac{\log 2}{2}.
\end{aligned}$$

Finally,

$$c_n \frac{\bigl(1-|z_n|^2\bigr)}{\bigl|\exp(it_n)-z_n\bigr|^2} = \frac{\bigl(1-|z_n|\bigr)^2 \log\bigl|f(z_n)\bigr|}{\bigl|\exp(it_n)-z_n\bigr|^2} = \log\bigl|f(z_n)\bigr|,$$

so we may combine these inequalities to obtain that

$$\log\left|f(z_n)\right| < \operatorname{Re}\big(g(z_n)\big) < \log\left|f(z_n)\right| + \log 2.$$

It follows that

$$\frac{1}{2\left|f(z_n)\right|} < \left|S(z_n)\right| < \frac{1}{\left|f(z_n)\right|},$$

which is the desired conclusion. We note also that $\left|S(z_n)\right| < 1$.

Also, if $h(z) = f(z)S(z)$ then $\frac{1}{2} < \left|h(z_n)\right| < 1$, and so

$$\begin{aligned}(1-|z_n|^2)\frac{\left|h'(z_n)\right|}{1+\left|h(z_n)\right|^2} &\geq \tfrac{1}{2}\big(1-|z_n|^2\big)\big(\left|S'(z_n)f(z_n)\right| - \left|S(z_n)f'(z_n)\right|\big)\\ &> \tfrac{1}{4}\big(1-|z_n|^2\big)\left|g'(z_n)\right| - \tfrac{1}{2}\,\|f\|_{\mathscr{B}}\end{aligned}$$

and we will be finished if we can show that

$$\big(1-|z_n|^2\big)\left|g'(z_n)\right| \to \infty.$$

But

$$\begin{aligned}\big(1-|z_n|^2\big)\left|g'(z_n)\right| &= \big(1-|z_n|^2\big)\left|\sum_{j=1}^{\infty} 2c_j \frac{\exp(it_j)}{\big(\exp(it_j)-z_n\big)^2}\right|\\ &> 2c_n\frac{1+|z_n|}{1-|z_n|} - 2\big(1-|z_n|^2\big)\sum_{j=1}^{n-1}\frac{\big(1-|z_j|\big)\log\left|f(z_j)\right|}{\left|\exp(it_j)-z_n\right|^2}\\ &\quad - 2\big(1-|z_n|^2\big)\sum_{j=n+1}^{\infty}\frac{\big(1-|z_j|\big)\log\left|f(z_j)\right|}{\left|\exp(it_j)-z_n\right|^2}\\ &> 2\log\left|f(z_n)\right| - 2\log 2 \to \infty,\end{aligned}$$

again making use of (2), (3), and (4). This completes the proof of the existence of the appropriate singular inner function.

To show the existence of an appropriate outer function, we again begin by selecting a sequence $\{z_n\}$ in D such that, without loss of generality, $z_n \to 1$, $\left|f(z_n)\right| \to \infty$, $0 < |z_n| < |z_{n+1}|$, and $0 < \arg z_{n+1} < \arg z_n < \frac{1}{2}\pi$ for each positive integer n. As in the proof of the first part, we can assume that the sequence $\{z_n\}$ also satisfies the following conditions:

(5) $$\big(1-|z_n|\big)\big(\log\left|f(z_n)\right|\big)^2 < \frac{1}{16} \qquad \text{for each } n,$$

(6) $$1-|z_j| < \left|\exp(it_j)-z_n\right| \qquad \text{for } j \neq n \text{ and } t = \arg(z_j),$$

$$(7) \qquad 2t_{n+1} < t_n \qquad \text{for each } n,$$

$$(8) \qquad \bigl(1-|z_n|^2\bigr)\sum_{j=1}^{n-1}\frac{\log\bigl|f(z_j)\bigr|}{1-|z_j|} < \frac{1}{16}\log 2, \qquad \text{for } n \ge 2,$$

and

$$(9) \qquad \bigl(1-|z_n|\bigr)\log\bigl|f(z_n)\bigr| < \bigl(1-|z_{n-1}|\bigr)\frac{\log 2}{2^{n+1}}, \qquad \text{for } n \ge 2.$$

Now set $c_n = 4\bigl(1-|z_n|\bigr)\log\bigl|f(z_n)\bigr|$ for each n, let

$$\chi_n(t) = \begin{cases} 1, & \text{for } t \in [t_n, t_n + c_n^2], \\ 0, & \text{otherwise} \end{cases}$$

let $\psi(e^{it}) = \exp\bigl(\sum_{j=1}^{\infty}\chi_n(t)/c_n\bigr)$, and define

$$\varphi(z) = \int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\psi(e^{it})\,dt$$

and $Q(z) = \exp\bigl\{-\varphi(z)\bigr\}$.

First, we show that the intervals $[t_j, t_j + c_j^2]$ are mutually disjoint. If we fix $j > 1$, the definition of c_j and (5) yield

$$t_j + c_j^2 = t_j + 16\bigl(1-|z_j|\bigr)^2\bigl(\log\bigl|f(z_j)\bigr|\bigr)^2 < t_j + 1 - |z_j|,$$

and from (6), we get $t_j + 1 - |z_j| < t_j + \bigl|\exp(it_j) - z_n\bigr|$ for each $n \ne j$. But $z_n \to 1$, and $0 < t_j = \arg(z_j) < \frac{1}{2}\pi$ so we conclude that

$$t_j + 1 - |z_j| < t_j + \bigl|\exp(it_j) - 1\bigr| < 2t_j.$$

Now, (7) says that $2t_j < t_{j-1}$, so by combining the inequalities in this paragraph we conclude that $t_j + c_j^2 < t_{j-1}$, which means that the intervals $[t_j, t_j + c_j^2]$ are mutually disjoint.

From the disjointness of the intervals and (9), we have that

$$\int_0^{2\pi}\log\psi(e^{it})\,dt = \sum_{j=1}^{\infty}\Bigl(\frac{1}{c_j}\Bigr)c_j^2 = \sum_{j=1}^{\infty}c_j < \frac{\log 2}{2}.$$

Noting that $\operatorname{Re}\bigl(\varphi(z)\bigr) > 0$, we conclude that $Q(z)$ is a bounded outer function.

Further, for $t \in [t_n, t_n + c_n^2]$, we have

$$\begin{aligned}|e^{it} - z_n|^2 &= \bigl(1 - |z_n|\bigr)^2 + 4|z_n| \sin^2\bigl((t - t_n)/2\bigr)\\ &< \bigl(1 - |z_n|\bigr)^2 + (t - t_n)^2 \le \bigl(1 - |z_n|\bigr)^2 + c_n^4.\end{aligned}$$

Now, making use of (5), we obtain

$$\begin{aligned}\int_{t_n}^{t_n + c_n^2} \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} \frac{1}{c_n}\, dt &> \frac{c_n\bigl(1 - |z_n|\bigr)}{\bigl(1 - |z_n|\bigr)^2 + c_n^4}\\ &= \frac{4 \log\bigl|f(z_n)\bigr|}{1 + \bigl(4 \log\bigl|f(z_n)\bigr|\bigr)^4 \bigl(1 - |z_n|\bigr)^2} > \log\bigl|f(z_n)\bigr|\end{aligned} \tag{10}$$

for each n. Also

$$\sum_{j \ne n} \int_{t_j}^{t_j + c_j^2} \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} \frac{1}{c_j}\, dt \le 2\bigl(1 - |z_n|\bigr) \sum_{j=1}^{n-1} \frac{c_j}{\bigl(1 - |z_j|\bigr)^2} + \sum_{j=n+1}^{\infty} \frac{2c_j}{1 - |z_n|} < \log 2$$

as consequences of (6), (8), and (9). It follows that

$$\log\bigl|f(z_n)\bigr| < \operatorname{Re} \varphi(z_n) < \log\bigl|f(z_n)\bigr| + \log 2,$$

which means that

$$\tfrac{1}{2} < \bigl|f(z_n) Q(z_n)\bigr| < 1.$$

Further,

$$\varphi'(z_n) = \sum_{j=1}^{\infty} \int_{t_j}^{t_j + c_j^2} \frac{2e^{it}}{(e^{it} - z_n)^2} \frac{1}{c_j}\, dt,$$

and, for large n, the argument of the integrand of the term corresponding to $j = n$ is always approximately $-t_n$, so that the integral is at least $1/2$ of the what its value would be if the integrand were the constant $\exp(-it_n)\bigl(1 - |z_n|\bigr)^{-2}$. Now, applying (6), we obtain

$$\bigl|\varphi'(z_n)\bigr| \ge \frac{c_n}{\bigl(1 - |z_n|\bigr)^2} - \sum_{j=1}^{n-1} \frac{2c_j}{\bigl(1 - |z_j|\bigr)^2} - \sum_{j=n+1}^{\infty} \frac{2c_j}{\bigl(1 - |z_n|\bigr)^2}.$$

Applying (8), we get

$$\begin{aligned}\bigl(1 - |z_n|^2\bigr) \sum_{j=1}^{n-1} \frac{2c_j}{\bigl(1 - |z_j|\bigr)^2} &= \bigl(1 - |z_n|^2\bigr) \sum_{j=1}^{n-1} \frac{8\bigl(1 - |z_j|\bigr) \log\bigl|f(z_j)\bigr|}{\bigl(1 - |z_j|\bigr)^2}\\ &= \bigl(1 - |z_n|^2\bigr) \sum_{j=1}^{n-1} \frac{8 \log\bigl|f(z_j)\bigr|}{1 - |z_j|} < \frac{\log 2}{2}.\end{aligned}$$

Also, using (9), we get

$$\left(1-|z_n|^2\right)\sum_{j=n+1}^{\infty}\frac{2c_j}{\left(1-|z_j|\right)^2}=\left(1-|z_n|^2\right)\sum_{j=n+1}^{\infty}\frac{8\left(1-|z_j|\right)\log\left|f(z_j)\right)}{\left(1-|z_j|\right)^2}$$

$$<16\sum_{j=n+1}^{\infty}\frac{1-|z_{j-1}|}{1-|z_n|}\frac{\log 2}{2^{j+1}}<\log 2\quad\text{for } n\ge 3.$$

Finally,

$$\frac{c_n}{\left(1-|z_n|\right)^2}=\frac{4\log\left|f(z_n)\right|}{1-|z_n|}$$

and so

$$\left(1-|z_n|^2\right)\left|\varphi'(z_n)\right|\ge\frac{4\log\left|f(z_n)\right|}{1-|z_n|}-\frac{3}{2}\log 2\to\infty$$

as $n\to\infty$. Now, if $k(z)=f(z)Q(z)$, a computation similar to the one performed in the first part of the proof shows that

$$\left(1-|z_n|\right)\frac{\left|k'(z_n)\right|}{1+\left|k(z_n)\right|^2}\to\infty,$$

and this completes the proof.

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