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ABELIAN NONDISCRETE CONVERGENCE GROUPS IN THE PLANE

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Abstract. Let g generate a nondiscrete convergence group in the 2-sphere S^2. We show that then there exists a homeomorphism f of S^2 onto itself such that $(f^{-1}\circ g\circ f)(z)$ equals $e^{2\pi i\alpha}z$ or $e^{2\pi i\alpha}/\bar{z}$ where α is an irrational number. In combination with known results this implies that every sense-preserving cyclic convergence group in S^2 is topologically conjugate to a group of Möbius transformations. Further, we prove that any abelian nondiscrete convergence group on S^2 is topologically conjugate to a Möbius group.

1. Introduction and results

1.1. The essential topological properties of uniformly quasiconformal groups are captured by the notion of a convergence group, defined by Gehring and Martin in [4, p. 335]. We say that a group G of homeomorphisms of the n-dimensional sphere S^n onto itself is a *convergence group* if it has the following *convergence property*: every sequence of elements of G contains a subsequence, say g_j, such that

(i) $g_j \to g$ and $g_j^{-1} \to g^{-1}$ uniformly on S^n, where g is a homeomorphism; or

(ii) there are $x_0, y_0 \in S^n$ (possibly $x_0 = y_0$) such that $g_j \to x_0$ and $g_j^{-1} \to y_0$ locally uniformly on $S^n \setminus \{y_0\}$ and $S^n \setminus \{x_0\}$, respectively.

Any group of K-quasiconformal, or, for $n = 1$, of K-quasisymmetric functions, for a fixed $K \ge 1$, is a convergence group. A group G is *discrete* if it does not contain a sequence of distinct elements converging to the identity mapping uniformly on S^n; otherwise, G is *nondiscrete*.

We say that the group G (or the function g) is *topologically conjugate* to a Möbius group (or to a Möbius transformation) if there is a homeomorphism f of S^n onto itself such that $f^{-1}\circ G\circ f$ (or $f^{-1}\circ g\circ f$) is a Möbius group (or transformation). Of course, a Möbius group is any group of Möbius transformations, and we allow the elements of G as well as Möbius transformations to be sense-reversing. We denote the identity mapping by Id and write $g^0 = \mathrm{Id}$, $g^j = g\circ g^{j-1}$ and $g^{-j} = (g^{-1})^j$ for $j \ge 1$.

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1.2. We consider under what circumstances a cyclic convergence group G on S^n generated by $g \neq \mathrm{Id}$ is topologically conjugate to a Möbius group, or, equivalently, the question of when such a function g is conjugate to a Möbius transformation. Suppose first that $n = 1$. If G is discrete then g is conjugate to an elliptic, parabolic, hyperbolic or orientation reversing Möbius transformation of S^1. If G is nondiscrete then g is also conjugate to a Möbius transformation h which we may take to be an irrational rotation $z \mapsto e^{2\pi i\alpha} z$ where α is an irrational number. For these results, see [5] and [21].

Suppose then that $n \geq 2$, and let G be discrete. Gehring and Martin ([4, p. 340]) classified the functions g that can occur as generators into three classes:

(i) g is called *elliptic* if g has finite order;

(ii) g is *parabolic* if g has a unique fixed point x_0 and then $g^j(x) \to x_0$ as $j \to \infty$ or $j \to -\infty$ locally uniformly on $S^n \setminus \{x_0\}$;

(iii) g is *loxodromic* if g has exactly two fixed points x_1 and x_2 and then, say, $g^j(x) \to x_1$ and $g^{-j}(x) \to x_2$ as $j \to \infty$, locally uniformly on $S^n \setminus \{x_2\}$ and $S^n \setminus \{x_1\}$, respectively.

When $n \geq 3$, we refer to [4, pp. 354–356] for discussion and references. Suppose that $n = 2$. Kerékjártó [14] proved that a sense-preserving loxodromic function is topologically conjugate to the Möbius transformation $h(z) = 2z$. The same proof applies to sense-reversing loxodromic functions and shows that they are conjugate to $h(z) = 2\bar{z}$. Furthermore, a result of Sperner [20] and Kerékjártó [13] shows that a sense-preserving parabolic mapping is topologically conjugate to $h(z) = z + 1$. In [6], it was proved that a sense-reversing parabolic function is conjugate to $h(z) = \bar{z} + 1$.

A theorem due in part to Brouwer, Kerékjártó, and Eilenberg [3] shows that an elliptic element g in S^2 is topologically conjugate to an orthogonal transformation, and thus, to $h(z) = cz$ or $h(z) = c/\bar{z}$ where c is a root of unity.

1.3. The question remains of what can be said in the situation that g generates a nondiscrete convergence group. This is settled by the following theorem, which is one of our main results.

Theorem 1. *Let g generate a nondiscrete convergence group in S^2. Then there is a homeomorphism f of S^2 onto itself such that*

$$(f^{-1} \circ g \circ f)(z) = e^{2\pi i\alpha} z \tag{1.1}$$

if g is sense-preserving, while

$$(f^{-1} \circ g \circ f)(z) = e^{2\pi i\alpha} / \bar{z} \tag{1.2}$$

if g is sense-reversing, where α is an irrational number.

Thus such a sense-preserving g is conjugate to an irrational rotation.

All of the above together yields the following consequence.

Corollary 1. *If g generates a convergence group in S^2, then g is topologically conjugate to a Möbius transformation.*

It follows that among the homeomorphisms of S^2, the functions topologically conjugate to Möbius transformations are exactly the ones that generate cyclic groups with the convergence property expressed in the definition of a convergence group, which is analogous to the characteristic property of normal families of analytic functions.

We note that in spite of this result for cyclic groups, not every discrete convergence group in S^2 containing only sense-preserving elements is topologically conjugate to a Möbius group, as has been pointed out in [4, Theorem 7.31, p. 356]. For various results on noncyclic discrete convergence groups on S^2, see [16].

Our second main result is the following extension of Theorem 1.

Theorem 2. *Let G be an abelian nondiscrete convergence group on S^2. Then there is a homeomorphism f of S^2 onto itself such that $f^{-1} \circ G \circ f$ is a Möbius group.*

Let G be a (not necessarily cyclic) nondiscrete convergence group on S^2. We say that $x \in S^2$ is in the *limit set* $L^*(G)$ of G if and only if there is $y \in S^2$ (possibly $y = x$) and a sequence $g_j \in G$ such that $g_j \to x$ locally uniformly in $S^2 \setminus \{y\}$ as $j \to \infty$. (Note that the more usual definition of $L^*(G)$ by means of properly discontinuous action would lead to $L^*(G)$ being always equal to S^2.) The proof of [4, Theorems 4.5 and 4.9] shows that either $L^*(G) = S^2$, or $L^*(G)$ is a perfect nowhere dense subset of S^2, or $L^*(G)$ consists of at most two points. In the last case we say that G is *elementary*. By Theorem 1, a nondiscrete cyclic convergence group has an empty limit set and is thus elementary.

In general one might ask what can be said about the structure of an elementary convergence group G. If $L^*(G)$ consists of exactly one point, then the example in [4, p. 356] shows that G need not be topologically conjugate to a Möbius group if G is discrete. A simple modification of that same example, obtained by replacing the Fuchsian group acting on the unit disk by the group of all Möbius transformations of the unit disk onto itself, gives a counterexample for nondiscrete groups.

However, it seems to us that when $L^*(G)$ is empty or consists of two points, positive conjugacy results might be obtained by means of introducing a certain metric on S^2 that is invariant under G. As the development of the theory of this metric involves a lot of detailed work, we will postpone it to another paper.

Throughout the paper, we denote the complex plane by $\mathbf{C}$ and identify S^2 with $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, whenever convenient. We denote by $H(S^2)$ the group of all homeomorphisms of S^2 onto itself. We write $B(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}$, $B(r) = B(0, r)$, $S(z_0, r) = \{z \in \mathbf{C} : |z - z_0| = r\}$ and $S(r) = S(0, r)$ whenever

$z_0 \in \mathbf{C}$ and $r > 0$. We also write S^1 for $S(0,1)$. We denote the real axis by $\mathbf{R}$ and write $\overline{\mathbf{R}}$ for the extended real axis $\mathbf{R} \cup \{\infty\}$. The set of integers is denoted by $\mathbf{Z}$.

We next note the following result.

Theorem 3. *Let G be a closed abelian subgroup of $H(S^2)$ with only sense-preserving elements and with the convergence property. Then G is isomorphic as a topological group to one of the following:*

(1) $\mathbf{R}$, $\mathbf{R}^2$, $\mathbf{R} \oplus \mathbf{Z}$, $\mathbf{R} \oplus \mathbf{Z}_m$, $\mathbf{C} \setminus \{0\}$, S^1, $S^1 \oplus \mathbf{Z}$, *or*
(2) $\mathbf{Z}$, $\mathbf{Z}^2$, $\mathbf{Z}_m$, $\mathbf{Z} \oplus \mathbf{Z}_m$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$,

the group operation being multiplication on $\mathbf{C} \setminus \{0\}$, S^1, and the S^1-part of $S^1 \oplus \mathbf{Z}$, and addition otherwise. The possibilities in (1) *occur if and only if G is nondiscrete.*

Here $\mathbf{Z}_m = \mathbf{Z}/(m\mathbf{Z})$, where $m \geq 2$. The group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, also denoted by D_2, is the Vierergruppe. For example, $\{z \mapsto \lambda z : \lambda > 0\}$ is isomorphic to the additive group $\mathbf{R}$. Not all closed nondiscrete abelian subgroups of $H(S^2)$ are convergence groups. One example of this is

$$G = \bigl\{(x,y) \mapsto (sx, ty) : s, t \in \mathbf{R} \setminus \{0\}\bigr\}.$$

To prove Theorem 3, we note that if G is as in Theorem 3, then G is topologically conjugate to a Möbius group. If G is nondiscrete, this follows from Theorem 2 above, and if G is discrete, the same follows by combining [4, Theorem 5.15, p. 343], and [16, Theorem 4.3, p. 397]. Considering what kind of closed abelian sense-preserving Möbius groups there can be, we arrive at the list given in (1) and (2). For discrete groups, all possibilities (up to conjugacy by a Möbius transformation, which does not affect the conclusion of Theorem 3) are listed in [2, pp. 84–90]. Note that all abelian Möbius groups are elementary ([2, pp. 69, 70, 83]). For nondiscrete groups, the possibilities can be gathered from the proof of Theorem 2. Alternatively, they may be obtained from the description of elementary Möbius groups with only sense-preserving elements as given in [2, p. 84], considering the possibilities that can occur when the group is, in addition, abelian and closed. This proves Theorem 3.

In the proof of Theorem 2, we shall make use of a result of Kerékjártó ([12, p. 115]) that states the following. If h_t is a one-parameter family of sense-preserving homeomorphisms of S^2 onto itself, all of them, other than the identity, fixing the same point and no other point, then there is a homeomorphism f of $\overline{\mathbf{C}}$ such that $(f^{-1} \circ h_t \circ f)(z) = z + t$ for all real t. Later on in the proof of Theorem 2, we shall need a similar result for a one-parameter family of loxodromic mappings. As it seems to us that this result has not been obtained in previous literature, we shall prove it here. We state it separately for possible later reference.

Theorem 4. *Let h_t be a one-parameter family of loxodromic mappings of S^2. Thus each h_t (for $t \in \mathbf{R}$) is a sense-preserving homeomorphism of S^2 onto itself, each h_t for $t \neq 0$ can be topologically conjugated to a loxodromic or hyperbolic Möbius transformation, and $h_{t+u} = h_t \circ h_u$ for all real t and u. Further, the mapping of $\mathbf{R}$ into $H(S^2)$ taking t onto h_t is a continuous function of t. Then there is a homeomorphism f of S^2 onto itself such that $(f \circ h_t \circ f^{-1})(z) = e^t z$ for all $z \in \overline{\mathbf{C}} = S^2$ and all real t.*

If each h_t commutes with every rotation $z \mapsto cz$, where $|c| = 1$, then we may choose f so that f also commutes with every such rotation.

The proof of Theorem 4 will be given in Subsection 6.13.

Finally, we want to observe the following two interesting corollaries of our work here, suggested by the referee.

Corollary 2. *There is a neighbourhood $\mathscr{U}$ of the identity mapping in $H(S^2)$ such that no nontrivial convergence group of S^2 is contained in $\mathscr{U}$.*

Thus every convergence group has the "no small subgroups" property. It is clear that in order to establish the result it is enough to consider only cyclic groups. A nondiscrete cyclic group is necessarily a conjugate of an irrational rotation by Corollary 1. The sphere is then foliated by invariant topological circles, one of which must have large diameter. It is not too difficult to see that the result follows by consideration of the nearly transitive action of the cyclic group on this circle. If the cyclic group is discrete and either parabolic or loxodromic the result is immediate. There remains only the case that the cyclic group has finite order. Again however the existence of an invariant foliation by topological circles, since the map is topologically conjugate to a rotation, leads directly to a proof. Indeed we see that the neighbourhood $\mathscr{U}$ can be taken to be of the form

$$\mathscr{U} = \bigl\{ f \in H(S^2) : \sup\bigl\{ q\bigl(x, f(x)\bigr) : x \in S^2 \bigr\} < \varepsilon \bigr\}$$

for some suitable $\varepsilon > 0$ where $q(x, y)$ denotes the chordal distance between x and y.

The following result is a consequence of a group having no small subgroups (see [10, Theorem 6, p. 95]).

Corollary 3. *Any closed nondiscrete convergence group of S^2 contains a nontrivial one-parameter subgroup.*

We would like to thank the referee for his detailed comments. In particular, he suggested the argument given in Subsection 6.7, which is perhaps simpler than our original argument for this case, he pointed out a difficulty in our original argument for Case II in Section 6, which prompted us to provide the present proof, and he suggested the previous two corollaries.

We would also like to thank Pekka Tukia for useful suggestions.

2. Outline of the proof of Theorem 1

2.1. As the proof of Theorem 1 involves several steps that sometimes require lengthy proofs, we formulate two lemmas that essentially imply Theorem 1. Then we prove these lemmas in the sections that follow. We assume throughout that g generates a nondiscrete convergence group G on S^2.

Lemma 1. *Suppose that g is topologically conjugate to the function $z \mapsto e^{2\pi i\alpha}z$ for some irrational α whenever g is sense-preserving. Then g is topologically conjugate to $z \mapsto e^{2\pi i\alpha}/\bar{z}$ for some irrational α whenever g is sense-reversing.*

In view of Lemma 1, we may assume from now on that g is sense-preserving. One of the most difficult auxiliary results we need to prove is the following.

Lemma 2. *If g is as in Theorem 1 and is sense-preserving, then there is no sequence $m_j \to \infty$ such that $g^{m_j} \to x_0$ and $g^{-m_j} \to y_0$ uniformly on compact subsets of $S^2 \setminus \{y_0\}$ and $S^2 \setminus \{x_0\}$, respectively.*

Thus any sequence of iterates of g contains a subsequence, say g^{k_j}, such that $g^{k_j} \to h$ and $g^{-k_j} \to h^{-1}$ uniformly on S^2 where h is a homeomorphism.

2.2. Recall that $H(S^2)$ is the group of all homeomorphisms of S^2 onto itself. For $x, y \in S^2$ we write $q(x,y)$ for the chordal distance of x and y induced by the Euclidean metric in $\mathbf{R}^3$. For $f_1, f_2 \in H(S^2)$ we set

$$d(f_1, f_2) = \sup\{q\bigl(f_1(x), f_2(x)\bigr) : x \in S^2\}.$$

The metric d induces the topology of uniform convergence on S^2, the usual topology of $H(S^2)$.

Following Kerékjártó, we say that $h \in H(S^2)$ is regular at $x \in S^2$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that if $q(x,y) < \delta$ then $q\bigl(h^n(x), h^n(y)\bigr) < \varepsilon$ for all integers n. Kerékjártó proved in [14, p. 250], that if h is a nonperiodic sense-preserving homeomorphism of S^2 that is regular at each point of S^2, then h is topologically conjugate to a rotation $z \mapsto e^{2\pi i\alpha}z$, where α is irrational. To complete the proof of Theorem 1, it thus suffices to show that g is regular on S^2.

If g is as in Theorem 1 and is sense-preserving, pick $x \in S^2$ and $\varepsilon > 0$. If g is not regular at x, then there is a sequence of points y_k with $q(x, y_k) \to 0$ and of integers m_k with $|m_k| \to \infty$ such that

$$q\bigl(g^{m_k}(x), g^{m_k}(y_k)\bigr) \geq \varepsilon$$

for all k. By Lemma 2, we may pass to a subsequence without changing notation and assume that $g^{m_k} \to h$ uniformly on S^2 where h is a homeomorphism. But then $g^{m_k}(x) \to h(x)$ and $g^{m_k}(y_k) \to h(x)$, which gives a contradiction. Thus g is regular at x. This proves Theorem 1, subject to Lemmas 1 and 2.

Remark. It appears that well-known results of Montgomery and Zippin on compact connected transformation groups in $\mathbf{R}^3$ (and S^2), such as [17, Theorem 6.7.1, p. 260], do not directly apply here, for even if Lemma 2 implies that the closure of G in $H(S^2)$ is compact, there seems to be no obvious way to prove that it is also connected. In retrospect, of course, it is seen that this closure is indeed connected.

3. Proof of Lemma 1

Let the assumptions of Lemma 1 be satisfied, and let g be sense-reversing. We may perform a preliminary conjugation and assume that $g^2(z) = e^{2\pi i\alpha}z$ where α is irrational. Then g commutes with $e^{2\pi i\alpha}z$ so that

$$g(cz) = cg(z) \quad \text{for all } z \in \mathbf{C} \tag{3.1}$$

whenever $c = e^{2\pi in\alpha}$ for some integer n, and thus, by continuity, whenever $|c| = 1$. Thus g maps each circle $S(r)$ onto a circle $S\big(u(r)\big)$ where u is a homeomorphism of $(0,\infty)$ onto itself. Also $u \circ u = \mathrm{Id}$ since $g^2(z) = e^{2\pi i\alpha}z$.

If u is increasing, then (3.1) implies that g is sense-preserving, which is a contradiction. Thus u is strictly decreasing and has a unique fixed point t on $(0,\infty)$. We let v be any increasing homeomorphism of $[t,\infty)$ onto $[1,\infty)$ so that $v(t) = 1$. For $0 < r < t$, we set $v(r) = 1/v\big(u(r)\big)$. Thus v is an increasing homeomorphism of $(0,\infty)$ onto itself, and $(v \circ u \circ v^{-1})(r) = 1/r$ for all $r > 0$.

By (3.1), for all $r > 0$ there is $\psi(r)$ with $|\psi(r)| = 1$ such that

$$g(re^{i\theta}) = u(r)\psi(r)e^{i\theta}$$

for all r and θ. Clearly ψ is a continuous function of r on $(0,\infty)$. We have $\psi(r)\psi\big(u(r)\big) = e^{2\pi i\alpha}$ for all $r > 0$ since $g^2(z) = e^{2\pi i\alpha}z$. In particular, $\psi(t)^2 = e^{2\pi i\alpha}$. We define $f(0) = 0$, $f(\infty) = \infty$, $f(re^{i\theta}) = v(r)e^{i\theta}$ for $0 < r \le t$, and

$$f(re^{i\theta}) = v(r)\psi(r)\overline{\psi(t)}\,e^{i\theta} \quad \text{for } t < r < \infty,$$

so that f is a homeomorphism of S^2 onto itself. One can verify that

$$(f \circ g \circ f^{-1})(z) = \psi(t)/\bar{z}.$$

This proves Lemma 1.

4. Results from two-dimensional topology

4.1. We shall often need to refer to *convergence of continua*. Recall that a subset C of S^2 is called a *continuum* if C is closed and connected and contains at least two points. If C and C_i for $i \geq 1$ are continua in S^2, we say that C_i converges to C as $i \to \infty$, and write $C_i \to C$, if any neighbourhood of any point of C intersects all but finitely many C_i, and if any point outside C has a neighbourhood that intersects only finitely many C_i. A basic result that can be found in [22, Theorem 7.1, p. 8 and (9.12), p. 12], states the following.

Lemma 3. *Any sequence C_i of continua, contained in a fixed compact subset of the plane, contains a subsequence converging to a nonempty closed connected set C. If each C_i has diameter at least ε, for some fixed positive ε, then C is a continuum of diameter at least ε.*

4.2. The next lemma is a purely topological result which is easy to believe and which should be known, but for which we have not been able to find a direct reference. Recall that a set Γ is *locally connected* at $z_0 \in \Gamma$ if any neighbourhood U_1 of z_0 contains a neighbourhood U_2 of z_0 such that $\Gamma \cap U_2$ is contained in a single component of $\Gamma \cap U_1$ ([22, p. 15]).

When we talk about straight lines on S^2, we are identifying S^2 with $\overline{\mathbf{C}}$.

Lemma 4. *Let D be a simply connected domain in S^2, set $\Gamma = \partial D$ and suppose that Γ is not locally connected at $z_0 \in \Gamma$. Then Γ contains a continuum of points ζ, contained in a preassigned neighbourhood of z_0, with the following properties. There are parallel lines L_1 and L_2, which we can choose to intersect a preassigned neighbourhood of ζ, containing ζ in the open strip domain S determined by them, and there are distinct components C_i of $S \cap \Gamma$ such that the disjoint continua $\overline{C}_i$ tend to a continuum $\mathscr{C}$ containing ζ and such that if $i < j < k$, then C_j separates C_i from C_k and from $\mathscr{C}$ in S. The continuum $\mathscr{C}$ is contained in the component of $\overline{S} \cap \Gamma$ containing ζ. Each set $\overline{C}_i$ as well as $\mathscr{C}$ intersects both L_1 and L_2. If the coordinate axes are chosen so that L_1 and L_2 are parallel to the y-axis, then for $i \geq 1$ and $j = 1, 2$, there are points in $\overline{C}_i \cap L_j$ with y-coordinate y_{ij} and points in $\mathscr{C} \cap L_j$ with y-coordinate y_j such that y_{ij} strictly increases to y_j as $i \to \infty$, for $j = 1, 2$, or such that y_{ij} strictly decreases to y_j as $i \to \infty$, for $j = 1, 2$.*

Furthermore, any neighbourhood W_0 of ζ contains a neighbourhood W of ζ such that among the components of $W \cap D$, there are distinct components V_i for $i \geq 1$ containing points w_i that tend to ζ as $i \to \infty$.

If D_0 is a domain with $D_0 \subset D$ and $\partial D \subset \partial D_0$, then there are distinct components V_i of $W \cap D$ with $w_i \in V_i \cap D_0$ such that $w_i \to \zeta$ as $i \to \infty$.

We remark that $\mathscr{C}$ need not be the closure of any component C of $\Gamma \cap S$ but may be larger than any such closure. For instance, $\Gamma \cap S$ might be a horizontal segment while $\mathscr{C}$ could contain an additional vertical segment of L_1 and/or L_2.

To get an example, suppose that

$$\Gamma = \{iy : -1 \le y \le 1\} \cup \{x + iy : y = \sin(1/x),\ 0 < x \le 1\}$$

in Lemma 4. Then Γ is not locally connected at the origin. For the lines L_1 and L_2, we may take the horizontal lines $y = \pm\frac{1}{2}$, so that S is the strip $\{x + iy : |y| < \frac{1}{2}\}$. Clearly $\overline{S} \cap \Gamma$ has infinitely many components, and they cluster to $\mathscr{C} = \{iy : |y| \le \frac{1}{2}\}$, which is the component of $\overline{S} \cap \Gamma$ containing the origin. In this case $\mathscr{C}$ happens to be the closure of a component of $S \cap \Gamma$. The continuum of points ζ referred to in Lemma 4 can be taken to be the closed line segment from $-iy_0$ to iy_0, where $0 < y_0 < 1$.

4.3. Let Γ be a continuum. Recall that a point $z_0 \in \Gamma$ is called a *cut point* if $\Gamma \setminus \{z_0\}$ is not connected. We shall need to show that various continua are Jordan curves. For this purpose, we quote the converse of Jordan's curve theorem ([22, Statement (2.4), p. 34]).

Lemma 5. *Let Γ be a continuum in S^2. Let D be one of the components of $S^2 \setminus \Gamma$, and suppose that $\Gamma = \partial D$. If, in addition, Γ is locally connected at each of its points and if Γ has no cut points, then Γ is a Jordan curve.*

We will use Lemma 4 to prove that a given continuum Γ is locally connected. It is also necessary to have a criterion that guarantees that Γ has no cut points.

Lemma 6. *Let Γ be a continuum in S^2, and let D and E be distinct components of $S^2 \setminus \Gamma$. If $\Gamma = \partial D = \partial E$ then Γ has no cut points.*

Note that $S^2 \setminus \Gamma$ might have more than two components, and yet Γ might be the boundary of each of them (cf. Lakes of Wada [8, p. 143]).

Proof of Lemma 6. Let z_0 be a cut point of Γ, and let x and y be points in distinct components of $\Gamma \setminus \{z_0\}$. We apply [23, Theorem IV.5.6, p. 112], taking the closed set F in that theorem to be $\{z_0\}$. We conclude that there is a continuum $\gamma \subset D \cup \{z_0\}$ that separates x and y. Since $x, y \in \partial E$, there are points of E in two distinct components of $S^2 \setminus \gamma$. Thus E, being connected, intersects γ, which is impossible since $\gamma \subset D \cup \{z_0\}$. Hence Γ has no cut points, and Lemma 6 is proved.

4.4. *Proof of Lemma* 4. Let the assumptions of Lemma 4 be satisfied. Then Γ is not locally connected, and in particular, Γ is a continuum (rather than a point or the empty set). We shall assume that $z_0 \ne \infty$, leaving the simple modifications required in the case $z_0 = \infty$ to the reader.

By [22, Theorem 10.2, p. 13], there is a disk neighbourhood U of z_0 contained in a preassigned neighbourhood of z_0, and an infinite sequence C_i' of distinct components of $\Gamma \cap \overline{U}$ converging to a nondegenerate limit continuum C' which contains z_0 and is disjoint from each C_i'. We have $C' \subset \Gamma$, and indeed C' is contained in the component of $\Gamma \cap \overline{U}$ containing z_0. Note that there are points $z_i' \in C_i'$ with $z_i' \to z_0$ as $i \to \infty$.

In the rest of the proof, we shall make various assumptions about the existence of limits that can be justified by passing to subsequences. For the sake of convenience, we shall assume that this is done automatically but without changing notation. Also we may say that certain statements that are true for all large i, are true for all i.

Since Γ is connected, we have $C_i' \cap \partial U \neq \emptyset$ for all i. We find $\zeta_i' \in C_i' \cap \partial U$ with $\zeta_i' \to \zeta' \in C' \cap \partial U$ as $i \to \infty$. Let L_1 and L_2 be parallel lines orthogonal to the line segment from z_0 to ζ' such that z_0 and ζ' are outside $\overline{S}$ and indeed lie in distinct components of $\mathbf{C} \setminus \overline{S}$, where S is the open strip domain bounded by L_1 and L_2. Each C_i' intersects both L_1 and L_2. Let C_i be a component of $C_i' \cap S$ whose closure intersects both L_1 and L_2. Note that $\overline{C}_i \to \mathscr{C}$ where $\mathscr{C}$ is contained in a component of $C' \cap \overline{S}$, and that the continuum $\mathscr{C}$ intersects both L_1 and L_2. Without loss of generality, we assume that L_1 and L_2 are vertical. For $j = 1, 2$, let y_{ij} be the y-coordinate of a point in $\overline{C}_i \cap L_j$. Then $y_{ij} \to y_j$ as $i \to \infty$, where y_j is the y-coordinate of a point in $\mathscr{C} \cap L_j$. For any distinct i and k, we have

$$(y_{i1} - y_1)(y_{i2} - y_2) > 0 \qquad \text{and} \qquad (y_{i1} - y_{k1})(y_{i2} - y_{k2}) > 0$$

since C_i', C_k' and C' are disjoint. Thus we may assume, for example, that y_{i1} and y_{i2} are strictly decreasing sequences.

For a point ζ that is to have the properties specified in Lemma 4, we could take any point in $\mathscr{C} \cap S \subset S \cap \Gamma$, and so it is clear that there is a continuum of such points. Given ζ and a neighbourhood W_0 of ζ, which we may assume to be a disk, we can replace L_1 and L_2 by some other vertical lines to ensure that they intersect W_0, and we still obtain continua $\overline{C}_i$ with all of the above properties. We assume that this has been done without changing notation. It is now obvious that if $i < j < k$, then C_j separates C_i from C_k and from $\mathscr{C}$ in S.

There are points $\zeta_i \in C_i$ with $\zeta_i \to \zeta$ as $i \to \infty$. Let $U_i = B(\zeta_i, \varrho_i)$ be a neighbourhood of ζ_i disjoint from $\Gamma \setminus C_i$, where $\varrho_i \to 0$. Since $\zeta_i \in \partial D_0$, there is a point $\alpha_i \in U_i \cap D_0$.

Now $W = W_0 \cap S$ is a neighbourhood of ζ contained in W_0, and $\alpha_i, \zeta_i \in W$ for all i.

We claim that if $k > n$, then α_{2n} and α_{2k} lie in distinct components of $W \cap D$. If not, we can join α_{2n} and α_{2k} by a polygonal arc in $S \cap D$. But this arc must intersect C_{2n+1}, which is a contradiction. Thus $W \cap D$ has distinct components V_i for $i \geq 1$ with $w_i = \alpha_{2i} \in V_i \cap D_0$ and $\alpha_{2i} \to \zeta$. This proves Lemma 4.

5. Proof of Lemma 2

5.1. To prove Lemma 2, suppose that $g^{m_j} \to x_0$ and $g^{-m_j} \to y_0$ locally uniformly on $S^2 \setminus \{y_0\}$ and $S^2 \setminus \{x_0\}$, respectively. Since g is sense-preserving, it follows from [15, Statement 12.1, p. 157], that g has at least one fixed point z_0 in S^2. One can verify that $z_0 = x_0$ or $z_0 = y_0$, and by replacing g by g^{-1}, if necessary, we may assume that $g(x_0) = x_0$. Recall that there is a sequence $n_j \to \infty$ such that $g^{n_j} \to Id$ and $g^{-n_j} \to \mathrm{Id}$ uniformly on S^2.

5.2. As the rest of the proof of Lemma 2 is rather complicated, it is convenient to first outline the proof, stating intermediate results as sublemmas. After that we proceed to prove the sublemmas.

Sublemma 1. *If U is a nonempty proper open subset of S^2 and if p is a positive integer, then we cannot have $\overline{g^p(U)} \subset U$.*

Sublemma 2. *We have $x_0 = y_0$, and x_0 is the unique fixed point of g. Similarly, x_0 is the only fixed point of g^q for any nonzero integer q. If g^{k_j} is another sequence with $k_j \to \infty$ such that $g^{k_j} \to x_1$ and $g^{-k_j} \to y_1$ locally uniformly on $S^2 \setminus \{y_1\}$ and $S^2 \setminus \{x_1\}$, respectively, then $x_1 = y_1 = x_0$.*

5.3. Our eventual aim is to find a Jordan curve Γ containing x_0 such that $g^k(\Gamma) = \Gamma$ for some positive integer k. Suppose that such a curve Γ has been found, and let h be a homeomorphism of $S^2 = \overline{\mathbf{C}}$ onto itself taking Γ onto $\overline{\mathbf{R}}$ with $h(x_0) = \infty$. Then $\tilde{g} = h \circ g^k \circ h^{-1}$ generates a nondiscrete convergence group, maps $\mathbf{R}$ onto itself, and fixes ∞. Since x_0 is the unique fixed point of g^k, it follows that $\tilde{g}$ has no fixed points on $\mathbf{R}$. Thus either $\tilde{g}(x) > x$ for all $x \in \mathbf{R}$, or $\tilde{g}(x) < x$ for all $x \in \mathbf{R}$. In either case,

$$\tilde{g}^{\,n}(x) \to \infty \tag{5.1}$$

as $n \to \infty$, for each real x.

Since $g^{n_j} \to \mathrm{Id}$, we have $g^{kn_j} = (g^{n_j})^k \to \mathrm{Id}$, and so $\tilde{g}^{\,n_j} \to \mathrm{Id}$ as $j \to \infty$. This contradicts (5.1), and it follows that the assumption that $g^{m_j} \to x_0$ must have been false. This then proves Lemma 2.

The curve Γ will arise as the common boundary of certain domains. To define the relevant open sets, we first need some more terminology and preliminary results.

5.4. The *orbit* Gx of $x \in S^2$ is defined as the set $Gx = \{g^n(x) : n \in \mathbf{Z}\}$. The sets $G^+x = \{g^n(x) : n \geq 0\}$ and $G^-x = \{g^n(x) : n \leq 0\}$ are called *half orbits*. We may assume that $x_0 = \infty$ and also consider g as a homeomorphism of $\mathbf{C}$ onto itself. Then by a result of Homma and Kinoshita ([9, Theorem 5, p. 370]), there is a point $x_1 \in S^2 \setminus \{x_0\}$ (and, indeed, an everywhere dense set of such points x_1) for which G^+x_1 is *not* dense in S^2.

Sublemma 3. *No point $x \in S^2$ has a dense orbit under G.*

The following technical result is necessary.

Sublemma 4. *If $U \subset S^2 \setminus \{x_0\}$ is a nonempty open set with $g(U) = U$, then $x_0 \in \partial U$. If U has only finitely many components $U_1, \dots, U_n$, then $x_0 \in \partial U_i$ for all i.*

5.5. Choose $x_1 \in S^2 \setminus \{x_0\}$ and suppose that $x_2 \neq x_0$ and $x_2 \notin \overline{Gx_1}$. By conjugating g by a Möbius transformation, if necessary, we may assume that x_2 and x_0 are antipodal points so that $q(x_0, x_2) = 2$. For any $y \in S^2$ we write

$$\varrho(y) = \sup\bigl\{q\bigl(x_0, g^n(y)\bigr) : n \in \mathbf{Z}\bigr\} \geq q(x_0, y).$$

The choice of x_1 and x_2 implies that $\varrho(x_1) < 2$. Choose r so that $\varrho(x_1) < r < 2$. We define $V = \bigl\{y \in S^2 : \varrho(y) > r\bigr\}$ and $E = \bigl\{y \in S^2 : q(x_0, y) > r\bigr\}$.

Sublemma 5. *The set V is a nonempty open set, and in fact*

$$V = g(V) = \bigcup_{n=-\infty}^{\infty} g^n(E). \tag{5.2}$$

Furthermore, V has only finitely many components, and they are of the form

$$V_i = g^{i-1}(V_1) \quad \text{for } 1 \leq i \leq N$$

for some $N \geq 1$, where V_1 is the component of V containing E and $g^N(V_1) = V_1$.

We have $x_0 \in \partial V_i$ for all i. Furthermore, we have $\varrho(x) = r$ for all $x \in \partial V \setminus \{x_0\}$.

Sublemma 6. *The set $W = S^2 \setminus \overline{V}$ is a nonempty open set. If W_1 is a component of W then $g^M(W_1) = W_1$ for some $M \geq 1$, and $x_0 \in \partial W_1$. Furthermore, any such component W_1 is simply connected. We have $\partial W_1 \subset \partial W \subset \partial V$.*

Choose such a component W_1 of W, and write $\Gamma_1 = \partial W_1$. Then Γ_1 is a continuum containing x_0, and $g^M(\Gamma_1) = \Gamma_1$. Next let $\mathscr{V}$ be the component of $S^2 \setminus \overline{W}_1$ containing V_1, and let $\mathscr{W}$ be the component of $S^2 \setminus \overline{\mathscr{V}}$ containing W_1.

Sublemma 7. *The domains $\mathscr{V}$ and $\mathscr{W}$ are simply connected and have the same boundary Γ. We have $x_0 \in \Gamma$ and $g^{MN}(\Gamma) = \Gamma$. The continuum Γ has no cut points. We have $\Gamma \subset \Gamma_1 \subset \partial V$ so that $\varrho(x) = r$ for all $x \in \Gamma \setminus \{x_0\}$.*

The sole purpose of taking the complement so many times to obtain the sets W, $\mathscr{V}$ and $\mathscr{W}$ from V has been to ensure that we get two domains with the *same* boundary.

With the aid of Lemma 5, we now show that Γ is a Jordan curve. Since $g^{MN}(\Gamma) = \Gamma$, the proof of Lemma 2 can then be completed as explained in Subsection 5.3 above.

Sublemma 8. *The continuum* Γ *is locally connected at each of its points. Thus* Γ *is a Jordan curve.*

This completes the proof of Lemma 2, subject to Sublemmas 1–8.

5.6. *Proof of Sublemma* 1. Suppose that U is a nonempty proper open subset of S^2 and that p is a positive integer such that $F = \overline{g^p(U)} \subset U$. Then F is a fixed compact proper subset of U, and for $m \geq 2$, we have $g^{pm}(U) \subset F \subset U$. But $g^{pn_j} \to \mathrm{Id}$ uniformly on S^2 while $g^{pn_j}(U) \subset F$, which is a contradiction. This proves Sublemma 1.

5.7. *Proof of Sublemma* 2. Recall that $g(x_0) = x_0$. If $x_0 \neq y_0$, let U be a spherical disk centred at x_0 such that $y_0 \notin \overline{U}$. Since $g^{m_j} \to x_0$, there is j_0 such that if $j \geq j_0$ and $p = m_j$ then $\Gamma = g^p(\partial U) \subset U$. Let the components of $S^2 \setminus \Gamma$ be D_1 and D_2 where $\overline{D}_1 \subset U$. By Sublemma 1, we cannot have $g^p(U) = D_1$. Thus $g^p(U) = D_2$ and so $x_0 = g^p(x_0) \in D_2$.

Since $y_0 \in S^2 \setminus \overline{U}$, we have, with $p = m_j$,

$$g^{-p}(S^2 \setminus U) \subset g^{-p}(S^2 \setminus D_1) = g^{-p}(\overline{D}_2) = \overline{U}. \tag{5.3}$$

Since $S^2 \setminus U$ is a compact subset of $S^2 \setminus \{x_0\}$, we have $g^{-m_j} \to y_0$ uniformly on $S^2 \setminus U$. This contradicts (5.3), and it follows that $y_0 = x_0$.

Thus if $z_0 \neq x_0$, then $g^{m_j}(z_0) \to x_0$. So we cannot have $g(z_0) = z_0$, and it follows that x_0 is the unique fixed point of g.

Since g^k is a homeomorphism for any $k \geq 1$ and $g^k(x_0) = x_0$, we have $g^k(z_0) \neq x_0$ if $z_0 \neq x_0$. Fix $q \geq 2$ and write $m_j = qk_j + r_j$ where $0 \leq r_j \leq q-1$. If $z_0 \neq x_0$ and $g^q(z_0) = z_0$, then $g^{m_j}(z_0) = g^{r_j}(z_0)$ so that $g^{m_j}(z_0)$ belongs to a finite set not containing x_0. Thus $g^{m_j}(z_0)$ does not tend to x_0. This contradiction shows that x_0 is the unique fixed point of g^q for any $q \geq 1$ and hence for any nonzero q.

If $g^{k_j} \to x_1$ and $g^{-k_j} \to y_1$ then at least one of x_1 and y_1 must be equal to x_0. If, for example, $x_1 = x_0$, we argue as above to deduce that $y_1 = x_0$ also. This proves Sublemma 2.

5.8. *Proof of Sublemma* 3. Recall that there is a point $x_1 \in S^2 \setminus \{x_0\}$ such that the half orbit G^+x_1 is not dense in S^2. Suppose that $x_2 \in S^2$ and that Gx_2 is dense in S^2. Since $Gx_0 = \{x_0\}$, we have $x_2 \neq x_0$. We may or may not have $x_2 = x_1$. Since Gx_2 is dense, we have $a_j = g^{k_j}(x_2) \to x_1$ while $a_j \neq x_1$ for each j, for some sequence k_j with $k_j \to \infty$ or $k_j \to -\infty$. Similarly, given any $x_3 \in S^2$, we have $b_j = g^{\ell_j}(x_2) \to x_3$ and $b_j \neq x_3$, for some sequence ℓ_j. Choose x_3 so that $x_3 \neq x_0$ and so that G^+x_1 fails to intersect some neighbourhood U of x_3. For each j, choose $q_j = n_{p(j)} - k_j + \ell_j$ where $p(j)$ is chosen so that $p(j+1) > p(j)$ and $q_j \to \infty$. Since $g^{n_{p(j)}} \to \mathrm{Id}$, we have $g^{q_j}(a_j) = g^{n_{p(j)}}(b_j) \to x_3$.

By passing to a subsequence and using the definition of a convergence group together with Sublemma 2, we may assume that $g^{q_j} \to h$ uniformly on S^2, where

h is a homeomorphism, or that $g^{q_j} \to x_0$ locally uniformly in $S^2 \setminus \{x_0\}$. The latter alternative cannot occur since $a_j \to x_1 \neq x_0$ and $g^{q_j}(a_j) \to x_3 \neq x_0$. Thus $g^{q_j} \to h$, which implies that $g^{q_j}(x_1) \to h(x_1)$ and that $g^{q_j}(a_j) \to h(x_1)$. Hence $h(x_1) = x_3$ and so $x_3 \in \overline{G^+ x_1}$, which gives a contradiction. It follows that Gx is not dense for any x and Sublemma 3 is proved.

5.9. *Proof of Sublemma* 4. If U is as in Sublemma 4, pick $x \in U$ and note that $g^{m_j}(x) \in U$ and $g^{m_j}(x) \to x_0$. Thus $x_0 \in \overline{U}$ and so $x_0 \in \partial U$. If U has only the components $U_1, \dots, U_n$, note that g maps each U_i onto some U_j. Thus the U_i can be partitioned into cycles of components with the following property. If $U_1, \dots, U_k$, for example, is such a cycle, and if the components have been renumbered in a suitable way, then $g^{i-1}(U_1) = U_i$ for $1 \le i \le k$, and $g^k(U_1) = U_1$.

If $x_0 \notin \overline{U}_1$ then

$$x_0 \notin \overline{U}_1 \cup \dots \cup \overline{U}_k = \bigcup_{j=-\infty}^{\infty} g^j(\overline{U}_1),$$

which is impossible since $g^{m_j} \to x_0$ in some disk contained in U_1. Thus $x_0 \in \partial U_1$ and so $x_0 \in \partial U_i$ for $1 \le i \le k$. The same argument applies to all the other cycles, and we deduce that $x_0 \in \partial U_i$ for $1 \le i \le n$. This proves Sublemma 4.

5.10. *Proof of Sublemma* 5. The definition of ϱ readily implies that V is equal to the right hand expression in (5.2), and this representation of V shows that $g(V) = V$. Thus (5.2) holds. Let the components of V be denoted by V_i for $i \ge 1$, and let V_1 be the component of V containing E.

Since $g(V) = V$, it follows that g maps any V_i onto some V_k. Clearly any V_i is contained in, and thus coincides with, $g^k(V_1)$ for some k. Recall that $g^{\pm n_j} \to \mathrm{Id}$. As V_1 is open and $g^{\pm n_j} \to \mathrm{Id}$, if n_j is large enough, then $g^{\pm n_j}(V_1)$ intersects V_1 and thus $g^{\pm n_j}(V_1) = V_1$. Therefore V has only finitely many components, say $V_1, \dots, V_N$, which we may number so that $V_j = g^{j-1}(V_1)$ for $1 \le j \le N$ while $g^N(V_1) = V_1$. Note that if q is the smallest positive integer such that $g^{-q}(V_1) = V_1$, then the components $V_1, g^{-1}(V_1), \dots, g^{-(q-1)}(V_1)$ are distinct, and $g^q(V_1) = g^q\bigl(g^{-q}(V_1)\bigr) = V_1$. Thus $q = N$, and the components $g^{-1}(V_1), g^{-2}(V_1), \dots, g^{-(q-1)}(V_1)$ coincide with $g^{N-1}(V_1), g^{N-2}(V_1), \dots, g(V_1)$.

By Sublemma 4, we have $x_0 \in \partial V_i$ for $1 \le i \le N$.

Suppose that $x \in \partial V \setminus \{x_0\}$. Then $x \notin V$ so that $\varrho(x) \le r$. Since $x \in \partial V$, there are points $w_n \in V$ tending to x with $\varrho(w_n) > r$. For each n, there is g^{k_n} such that $q\bigl(x_0, g^{k_n}(w_n)\bigr) > r$. We pass to a subsequence, apply Sublemma 2, and find that $g^{k_n} \to h$ uniformly on S^2, where h is a homeomorphism, since we cannot have $g^{k_n} \to x_0$ uniformly in a neighbourhood of x. Thus $g^{k_n}(w_n) \to h(x)$ and so $q\bigl(x_0, h(x)\bigr) \ge r$. Therefore $q\bigl(x_0, g^{k_n}(x)\bigr) \to r$, and consequently $\varrho(x) \ge r$. Hence $\varrho(x) = r$, as required. This proves Sublemma 5.

5.11. *Proof of Sublemma* 6. Clearly $W = S^2 \setminus \overline{V}$ is open. To show that W is nonempty, it suffices to prove that $x_1 \in W$. Since $\varrho(x_1) < r$, we have $x_1 \notin V$. By the last statement of Sublemma 5, we also have $x_1 \notin \partial V$. Thus $x_1 \in W$.

Since $g(V) = V$ by Sublemma 4, we have $g(\overline{V}) = \overline{V}$ and so $g(W) = W$. Thus any g^p maps a component of W onto another such component. Let W_1 be a component of W. Since $g^{n_j} \to \mathrm{Id}$ and W_1 is open, we have $g^{n_j}(W_1) \cap W_1 \neq \emptyset$ and thus $g^{n_j}(W_1) = W_1$ for all large j. Hence there is a smallest positive integer M such that $g^M(W_1) = W_1$, and then the components $W_1, g(W_1), \ldots, g^{M-1}(W_1)$ are distinct. Applying Sublemma 4 to

$$U = \bigcup_{i=1}^{M} g^{i-1}(W_1),$$

we deduce that $x_0 \in \partial g^{i-1}(W_1)$ for $1 \leq i \leq M$. In particular, $x_0 \in \partial W_1$ for any component W_1 of W.

Since V has N components, the domain W_1 is of connectivity at most N. Let $\Gamma_1, \ldots, \Gamma_k$ be the components of ∂W_1, where $k \leq N$ and $x_0 \in \Gamma_1$. If $\Gamma_1 = \{x_0\}$ then $k \geq 2$ and, since any iterate of g^M maps each Γ_i onto some Γ_j, it follows that

$$g^{jM}(\Gamma_2 \cup \cdots \cup \Gamma_k) = \Gamma_2 \cup \cdots \cup \Gamma_k$$

for each $j \geq 1$. But $\Gamma_2 \cup \cdots \cup \Gamma_k$ is a compact subset of $S^2 \setminus \{x_0\}$ on which $g^{Mk_j} \to x_0$ for some sequence $k_j \to \infty$. (For if $m_j = Mk_j + \ell_j$ where $0 \leq \ell_j < M$ then, since $g^{m_j} \to x_0$, we also have $g^{Mk_j} = g^{m_j} \circ g^{-\ell_j} \to x_0$.) This contradiction shows that Γ_1 is a continuum. For the same reason, it follows that $\Gamma_2 \cup \cdots \cup \Gamma_k = \emptyset$, so that $\Gamma_1 = \partial W_1$, and W_1 is simply connected. Obviously $\partial W_1 \subset \partial W \subset \partial V$. This proves Sublemma 6.

5.12. *Proof of Sublemma* 7. Since $\overline{W}_1$ and $\overline{\mathscr{V}}$ are connected, it follows that $\mathscr{V}$ and $\mathscr{W}$ are simply connected, being components of the complement of a connected closed set. Let us write $\Gamma = \partial \mathscr{V}$. We have $x_0 \in \Gamma$ since $x_0 \in \partial V_1$ (by Sublemma 5) and since $x_0 \in \overline{W}_1$ (by Sublemma 6) so that $x_0 \notin \mathscr{V}$. Clearly $\Gamma \subset \partial W_1 = \Gamma_1$.

Since $g^M(W_1) = W_1$ by Sublemma 6, we have $g^M(S^2 \setminus \overline{W}_1) = S^2 \setminus \overline{W}_1$. Thus g^M maps $\mathscr{V}$ onto a component of $S^2 \setminus \overline{W}_1$, and so does g^{kM} for any $k \geq 1$. But $V_1 \subset \mathscr{V}$ and $g^N(V_1) = V_1$ by Sublemma 5 so that $g^{MN}(V_1) = V_1$ also. It follows that $g^{MN}(\mathscr{V}) = \mathscr{V}$. Therefore $g^{MN}(\Gamma) = \Gamma$, as required.

Since $g^{MN}(\mathscr{V}) = \mathscr{V}$, $g^M(W_1) = W_1$, and $W_1 \subset \mathscr{W}$, we see in the same way that $g^{MN}(\mathscr{W}) = \mathscr{W}$. Clearly $\partial \mathscr{W} \subset \partial \mathscr{V} = \Gamma$. We claim that $\partial \mathscr{W} = \Gamma$. If not, pick $z_0 \in \Gamma \setminus \partial \mathscr{W}$. Then z_0 has a connected neighbourhood U with $U \cap \overline{\mathscr{W}} = \emptyset$ while $U \cap \mathscr{V} \neq \emptyset$.

Since $\Gamma \subset \partial W_1$, we have $z_0 \in \partial W_1$ and so $U \cap W_1 \neq \emptyset$. But $W_1 \subset \mathscr{W}$ so that $U \cap \mathscr{W} \neq \emptyset$. Since also $U \cap \overline{\mathscr{W}} = \emptyset$, we get a contradiction. It follows that $\partial \mathscr{W} = \Gamma$, as asserted.

Since $\mathscr{V}$ and $\mathscr{W}$ are simply connected domains with $\partial\mathscr{V} = \partial\mathscr{W} = \Gamma$, they are distinct components of $S^2 \setminus \Gamma$. Now Lemma 6 implies that Γ has no cut points. We clearly have $\Gamma = \partial\mathscr{V} \subset \partial W_1 = \Gamma_1$. By Sublemma 6, we further have $\partial W_1 \subset \partial V$, and by Sublemma 5, we have $\varrho(x) = r$ for all $x \in \partial V \setminus \{x_0\}$ and hence for all $x \in \Gamma \setminus \{x_0\}$. This proves Sublemma 7.

5.13. *Proof of Sublemma* 8. We apply Lemma 14 with $D = \mathscr{V}$ and note that by Sublemma 7, the continuum Γ has no cut points. Thus Lemma 5 implies that Γ is a Jordan curve provided that Γ is locally connected at each of its points. To prove this, it suffices, by [22, Statement 10.4, p. 14], to show that Γ is locally connected at each point of $\Gamma \setminus \{x_0\}$. To get a contradiction, we now assume that $z_1 \in \Gamma \setminus \{x_0\}$ and that Γ is not locally connected at z_1.

We may assume that $x_0 = \infty \in \overline{\mathbf{C}} = S^2$. Choose $\varepsilon \in \bigl(0, \frac{1}{2}q(x_0, z_1)\bigr)$ and set

$$W_0 = \bigl\{y \in S^2 : q(z_1, y) < \varepsilon\bigr\}.$$

By Lemma 4, there are parallel lines L_1 and L_2, which we may assume to be vertical, intersecting W_0, and distinct components C_i of $S \cap \Gamma$, where S is the open strip domain between L_1 and L_2, with the properties described in Lemma 4. All the $\overline{C}_i$ as well as their limit continuum $\mathscr{C}$ are contained in a fixed compact subset E_1 of $S^2 \setminus \{x_0\} = \mathbf{C}$. We may and will take E_1 to be the cap

$$E_1 = \bigl\{y \in S^2 : q(x_0, y) \ge \varepsilon_1\bigr\}$$

where ε_1 is sufficiently small and in particular, $0 < \varepsilon_1 < \varrho(x_1)/2 < r/2$. We further take ε_1 so small that the $\overline{C}_i$ and $\mathscr{C}$ are contained in the smaller set $\bigl\{y \in S^2 : q(x_0, y) \ge 2\varepsilon_1\bigr\}$.

We define the set H_1, not necessarily a group, by

$$H_1 = \bigl\{g^n, g^{-n} : n \in \mathbf{Z} \text{ and } q\bigl(x_0, g^n(z)\bigr) \ge \tfrac{1}{2}\varepsilon_1 \text{ for some } z \in E_1\bigr\} \subset G.$$

We claim that there is $r_2 > 0$ such that if $h \in H_1$ and $z_1 \in E_1$ then $q\bigl(x_0, h(z_1)\bigr) \ge r_2$. If not, then there are $g^{k_j} \in H_1$ and $z_j \in E_1$ such that $z_j \to w \in E_1$ and $q\bigl(x_0, g^{k_j}(z_j)\bigr) \to 0$. Since for each j there is $w_j \in E_1$ such that $q\bigl(x_0, g^{k_j}(w_j)\bigr) \ge \frac{1}{2}\varepsilon_1$ or $q\bigl(x_0, g^{-k_j}(w_j)\bigr) \ge \frac{1}{2}\varepsilon_1$, we cannot have $g^{k_j} \to x_0$ and $g^{-k_j} \to x_0$ uniformly on E_1 even if we pass to a subsequence. Thus, by Sublemma 2, we may assume that $g^{k_j} \to h$ uniformly on S^2 where h is a homeomorphism. But $h(x_0) = x_0 = h(w)$, which gives a contradiction. Thus there is a number r_2 with the required property. We may assume that $r_2 \le \varepsilon_1/2$.

We set $E_2 = \bigl\{y \in S^2 : q(x_0, y) \ge r_2\bigr\}$ and claim that H_1 is equicontinuous on E_2 with respect to the chordal metric q. If not, there are sequences $a_i, b_i \in E_2$ and $g^{k_i} \in H_1$, and a positive number ε_2, such that $a_i \to b \in E_2$ and $b_i \to b$ as $i \to \infty$ while $q\bigl(g^{k_i}(a_i), g^{k_i}(b_i)\bigr) \ge \varepsilon_2$ for all i. By passing to a subsequence, we may assume that $g^{k_i} \to h$ where h is a homeomorphism, since we cannot have $g^{k_i} \to x_0$ and $g^{-k_i} \to x_0$ uniformly even on E_1. But then $g^{k_i}(a_i) \to h(b)$ and $g^{k_i}(b_i) \to h(b)$, which gives a contradiction. Hence H_1 is equicontinuous on E_2.

So if δ is a given positive number, there is $\delta' > 0$ such that

$$q\bigl(\varphi(z), \varphi(w)\bigr) < 10^{-2} \min\{\varepsilon_1, \delta\} \tag{5.4}$$

whenever $z, w \in E_2$, $\varphi \in H_1$ and $q(z, w) \le \delta'$.

Let L_3 be the vertical line halfway between L_1 and L_2. Choose $\delta_0 > 0$ so small that any two of the sets $L_i \cap E_2$ for $1 \le i \le 3$ have chordal distance not less than $2\delta_0$ from each other.

We now choose δ so small that

$$q\bigl(\varphi(z), \varphi(w)\bigr) < 10^{-2} \min\{\varepsilon_1, \delta_0\} \tag{5.5}$$

whenever $z, w \in E_2$, $\varphi \in H_1$ and $q(z, w) \le \delta$, and then we pick δ' accordingly so that $\delta' < \varepsilon_1$ and so that also (5.4) holds. We further assume that $\delta + r < 2$, the diameter of S^2 in the chordal metric.

5.14. Suppose that $\zeta_1, \zeta_3 \in L_i \cap E_1$ and $\zeta_2 \in L_j \cap E_1$, where $i \ne j$ and $i, j \in \{1, 2, 3\}$. Suppose that $g^{k_j} \in H_1$ and that $g^{k_j} \to h$ uniformly on S^2 as $j \to \infty$, where h is a homeomorphism. We claim that

$$q\bigl(h(\zeta_1), h(\zeta_2)\bigr) \ge \delta. \tag{5.6}$$

If, in addition, $q(\zeta_1, \zeta_3) \le \delta'$, then

$$q\bigl(h(\zeta_1), h(\zeta_3)\bigr) \le 10^{-2} \min\{\varepsilon_1, \delta\} = \delta_1, \tag{5.7}$$

say.

To prove (5.6), suppose that it does not hold. Then $q\bigl(g^{k_j}(\zeta_1), g^{k_j}(\zeta_2)\bigr) < \delta$ for all large j. Since $\zeta_1, \zeta_2 \in E_1$, we have $g^{k_j}(\zeta_i) \in E_2$ for $i = 1, 2$. Since $g^{k_j} \in H_1$, we also have $g^{-k_j} \in H_1$, by the definition of H_1. Now (5.5), applied to $z = g^{k_j}(\zeta_1)$, $w = g^{k_j}(\zeta_2)$ and $\varphi = g^{-k_j}$, gives $q(\zeta_1, \zeta_2) < 10^{-2}\delta_0$, which contradicts the definition of δ_0. This proves (5.6).

If $q(\zeta_1, \zeta_3) \le \delta'$, then by (5.4), we have

$$q\bigl(g^{k_j}(\zeta_1), g^{k_j}(\zeta_3)\bigr) < \delta_1$$

for all j. This gives (5.7) as $j \to \infty$.

5.15. Fix i. There is a component Δ_i of $S \setminus (C_{i-1} \cup C_{i+1})$ containing C_i. The domain Δ_i is simply connected, being a component of the complement of the connected set $L_1 \cup L_2 \cup \overline{C_{i-1}} \cup \overline{C_{i+1}}$. Also Δ_i is bounded since C_i separates C_{i-1} from C_{i+1} in S.

Since $C_i \subset \Delta_i$ and $\overline{C}_i \cap L_j \neq \emptyset$, we have $L_j \cap \partial\Delta_i \neq \emptyset$ for $j = 1, 2$. We claim that for all large i, we have

$$q(L_j \cap \partial\Delta_i) < \delta' \qquad \text{for } j = 1, 2, \tag{5.8}$$

where $q(E)$ denotes the diameter of the set E in the chordal metric. Suppose that $j = 1$. By Lemma 4 and symmetry, we may assume, without loss of generality, that there are points $\zeta_i \in \overline{C}_i \cap L_1$ with y-coordinate y_{i1} strictly decreasing to y_1 as $i \to \infty$, where y_1 is the y-coordinate of a point in $\mathscr{C} \cap L_1$. Then y_1 is the maximal y-coordinate of any point in $\mathscr{C} \cap L_1$ (since $\mathscr{C}$ and the $\overline{C}_i$ are disjoint and contained in $\overline{S}$ and since $\overline{C}_i \to \mathscr{C}$). Furthermore, if t_i is the maximal y-coordinate of any point in $\overline{C}_i \cap L_1$ then $t_i \to y_1$ also as $i \to \infty$, since $\overline{C}_i \to \mathscr{C}$. Now if u_i is the minimal y-coordinate of any point in $\overline{C}_i \cap L_1$, then by the above, we have $t_{i-1} - u_{i+1} \to 0$ as $i \to \infty$. Clearly $L_1 \cap \partial\Delta_i$ is contained in the closed interval of L_1 lying between the points of L_1 with y-coordinates t_{i-1} and u_{i+1}. This proves (5.8) for $j = 1$, and the proof for $j = 2$ is similar.

If $j \geq i + 2$, then $\Delta_i \cap \Delta_j = \emptyset$. For if not, then $\Delta_i \cup \Delta_j$ is connected so that C_i and C_j can be joined in $\Delta_i \cup \Delta_j$ and hence in $S \setminus C_{i+1}$. This is impossible since C_{i+1} separates C_i from C_j in S, by Lemma 4. Thus $\Delta_i \cap \Delta_j$ is empty.

5.16. Pick a point $z_i \in C_i \cap L_3$, so that $z_i \in \Delta_i$. Since $z_i \in \Gamma$, we have $\varrho(z_i) = r$ by Sublemma 7. Choose a sequence g^{k_j} such that $q\big(x_0, g^{k_j}(z_i)\big) \to r$. By Sublemma 2, we may assume that $g^{k_j} \to h_i$ and $g^{-k_j} \to h_i^{-1}$ uniformly on S^2, where h_i is a homeomorphism. Then $q\big(x_0, h_i(z_i)\big) = r$ while $q\big(x_0, h_i(z)\big) \leq r$ for all $z \in \Gamma$. Since $z_i \in E_1$ and $r > \varepsilon_1/2$, we have $g^{k_j} \in H_1$ for all large j.

Fix i so large that (5.8) holds. For $j = 1, 2$, there is $\zeta_j \in L_j \cap \partial\Delta_i$ with $\zeta_j \in \overline{C}_i$. Hence $\zeta_j \in \Gamma \cap E_1$ also. Suppose that $\zeta_j' \in L_j \cap \partial\Delta_i$. Then $q(\zeta_j, \zeta_j') < \delta'$ by (5.8). Also $\zeta_j' \in E_1$ since $\delta' < \varepsilon_1$. By (5.7), we obtain

$$q\big(h_i(\zeta_j), h_i(\zeta_j')\big) \leq \delta_1 \leq 10^{-2}\delta. \tag{5.9}$$

Next, by (5.6) we have

$$q\big(h_i(z_i), h_i(\zeta_j')\big) \geq \delta \tag{5.10}$$

for any such point ζ_j', and in particular if $\zeta_j' = \zeta_j$. We define

$$E_3 = \big\{y \in S^2 : q(x_0, y) \leq r \text{ or } q\big(h_i(\zeta_1), y\big) \leq \delta_1 \text{ or } q\big(h_i(\zeta_2), y\big) \leq \delta_1\big\}.$$

Then $h_i(z_i) \in \partial E_3$ by (5.9) and (5.10). Since $\partial\Delta_i \setminus (L_1 \cup L_2) \subset \overline{C_{i-1}} \cup \overline{C_{i+1}} \subset \Gamma$, we further have $h_i(\partial\Delta_i) \subset E_3$.

Recall that $\delta + r < 2 = q(x_0, x_2)$ and define $E_4 = \{y \in S^2 : q(x_2, y) < 2 - \delta - r\}$. Then $E_4 \subset S^2 \setminus E_3$. Since $h_i(\partial\Delta_i) \subset E_3$, we thus have either $S^2 \setminus E_3 \subset h_i(\Delta_i)$ or $(S^2 \setminus E_3) \cap h_i(\Delta_i) = \emptyset$. Since $h_i(z_i) \in h_i(\Delta_i) \cap \partial E_3$, the open set $h_i(\Delta_i)$ must intersect $S^2 \setminus E_3$, and so $S^2 \setminus E_3 \subset h_i(\Delta_i)$. We conclude that the set E_4, which is independent of i, is contained in $h_i(\Delta_i)$, and therefore $h_i^{-1}(E_4) \subset \Delta_i \subset E_1$ for all large i.

For each i, there is some iterate of g, say g^{p_i}, such that $q\big(g^{p_i}(y), h_i^{-1}(y)\big) < 2^{-i}$ for all $y \in S^2$. In particular, $g^{p_i}(E_4) \subset E_2$ for all large i. By passing to a subsequence and using Sublemma 2, we may assume that $g^{p_i} \to \psi$ uniformly on S^2, where ψ is a homeomorphism. Then $h_{i_\ell}^{-1} \to \psi$ uniformly on S^2, for some sequence i_ℓ that tends to infinity as $\ell \to \infty$. By passing to a further subsequence, we may assume that $i_{\ell+1} - i_\ell \geq 2$ for all ℓ so that Δ_{i_ℓ} and Δ_{i_m} are disjoint for all distinct ℓ and m. In particular, if $U_\ell = h_{i_\ell}^{-1}(E_4)$ then $U_\ell \cap U_m = \emptyset$ whenever $\ell \neq m$.

We claim that $\psi(x_2) \in U_\ell$ for all large ℓ. This gives a contradiction and completes the proof of Sublemma 8. If the claim is false, we may pass to another subsequence and assume that $\psi(x_2) \notin U_\ell$ for any ℓ.

Recall that $x_0 = \infty$, and let γ be a circle in $E_4 \subset \mathbf{C}$ centred at x_2. Then

$$\delta_2 = \min\big\{q\big(\psi(x_2), \psi(y)\big) : y \in \gamma\big\} > 0,$$

and so by uniform convergence,

$$\min\big\{q\big(\psi(x_2), h_{i_\ell}^{-1}(y)\big) : y \in \gamma\big\} > \delta_2/2$$

for all large ℓ. Thus the Jordan curve $\gamma_\ell = h_{i_\ell}^{-1}(\gamma)$ then lies outside the disk

$$E_5 = \big\{y \in S^2 : q\big(\psi(x_2), y\big) < \delta_3\big\} \subset \mathbf{C}$$

if $0 < \delta_3 < \delta_2/2$ and $\delta_3 < q\big(\psi(x_2), x_0\big)$. We note that $\psi(x_2) \neq \psi(x_0) = x_0 = h_{i_\ell}^{-1}(x_0) \notin \gamma_\ell$.

Let E_6 be the open disk containing x_2 with $\partial E_6 = \gamma$. Then $\psi(x_2) \notin h_{i_\ell}^{-1}(E_6) \equiv U'_\ell \subset U_\ell$. Since U'_ℓ coincides with one of the components of $S^2 \setminus \gamma_\ell$, we deduce that $U'_\ell \cap E_5 = \emptyset$. Thus $h_{i_\ell}^{-1}(x_2) \notin E_5$ for all large ℓ. Since $h_{i_\ell}^{-1}(x_2) \to \psi(x_2)$ as $\ell \to \infty$, we obtain a contradiction. Hence $\psi(x_2) \in U_\ell$ for all large ℓ, as asserted. This proves Sublemma 8, and also completes the proof of Lemma 2. Thus the proof of Theorem 1 is also complete.

6. Proof of Theorem 2

6.1. Let G be an abelian nondiscrete convergence group on S^2. We denote by $\overline{G}$ the *closure* of G in $H(S^2)$ so that $h \in \overline{G}$ if and only if there is a sequence $g_n \in G$ such that g_n tends to the homeomorphism h uniformly on S^2. Clearly $\overline{G}$ is a group, and it is nondiscrete if G is. As in [5, Lemma 5, p. 93], we see that $\overline{G}$ is a convergence group if G is. It clearly suffices to find a function f such that $f^{-1} \circ \overline{G} \circ f$ is a Möbius group. On the other hand, if $f^{-1} \circ G \circ f$ is a Möbius group, then so is $f^{-1} \circ \overline{G} \circ f$, so that we are not putting any extra restrictions on f by considering $\overline{G}$. Therefore we may and will assume that G is closed, that is, $G = \overline{G}$. In view of Theorem 1 and the above, we may assume that G is not contained in the closure of a cyclic group.

We write $\operatorname{fix}(g)$ for the set of fixed points of $g \in G$. Since $g \circ h = h \circ g$ and h^{-1} exists for $g, h \in G$, it is easily seen that for all $g, h \in G$,

$$\operatorname{fix}(g) = h\big(\operatorname{fix}(g)\big). \tag{6.1.1}$$

We divide the argument into two cases as follows:

I. There is a parabolic element g in G; and
II. There is no parabolic element in G.

6.2. *Case* I. We may assume that $\operatorname{fix}(g) = \{\infty\}$, identifying S^2 and $\overline{\mathbf{C}}$. If $h \in G$ then h fixes ∞ by (6.1.1). Let H denote the index one or two subgroup of sense-preserving elements of G. If $h \in G$ and if h has at most two fixed points, then since $\operatorname{fix}(h) = g\big(\operatorname{fix}(h)\big)$ by (6.1.1), it follows that $\operatorname{fix}(h) = \{\infty\}$ also. By Corollary 1, this is true whenever $h \neq \mathrm{Id}$ and h is not conjugate to $c/\bar{z}$ where $|c| = 1$, and in particular whenever $h \in H \setminus \{\mathrm{Id}\}$. Thus each function in $H \setminus \{\mathrm{Id}\}$ is parabolic. So if $h_1, h_2 \in H$ and $h_1(z) = h_2(z)$ for some $z \in \mathbf{C}$ then $h_1^{-1} \circ h_2 \in H$ and $(h_1^{-1} \circ h_2)(z) = z$ so that $h_1^{-1} \circ h_2 = \mathrm{Id}$, and hence $h_1 = h_2$. We shall often make use of this fact.

If $h_1, h_2 \in G \setminus H$ then $h_1 \circ h_2^{-1} \in H$. Thus either $G = H$ or

$$G = H \cup \big\{h \circ h_1 : h \in H\big\} \tag{6.2.1}$$

for a fixed but arbitrary $h_1 \in G \setminus H$.

First we prove that H is topologically conjugate to a Möbius group. Clearly H is nondiscrete and locally compact. Also H contains no small subgroups, that is, Id has a neighbourhood U in H such that $\{\mathrm{Id}\}$ is the only subgroup of H contained in U. For example, we may take

$$U = \big\{h \in H : d(h, \mathrm{Id}) < 1\big\},$$

where the metric d is as defined in Subsection 2.2, noting that the diameter of S^2 in the chordal metric q is equal to 2. For if $J \neq \{\mathrm{Id}\}$ is a subgroup of H and

$h \in J \setminus \{\mathrm{Id}\}$, then h is parabolic with fixed point at infinity so that for all large n,

$$d(h^n, \mathrm{Id}) \geq q\big(h^n(0), 0\big) > 1.$$

Thus $h^n \notin U$, so that J is not contained in U.

By [10, Theorem 6, p. 95], the group H contains a nontrivial one-parameter family of elements. That is, corresponding to each real number t there is a function $h_t \in H$ such that $h_t \circ h_u = h_{t+u}$ for all real t and u, and the mapping that takes t onto h_t is a continuous function of the real axis $\mathbf{R}$ into $H(S^2)$. Also, $h_t \neq \mathrm{Id}$ for some t. Obviously, $h_0 = \mathrm{Id}$ and $h_{-t} = (h_t)^{-1}$.

We now claim that $h_t \neq \mathrm{Id}$ whenever $t \neq 0$, which then immediately implies that $h_t \neq h_u$ whenever $t \neq u$. For suppose that $h_t = \mathrm{Id}$ for some $t \neq 0$. Then $(h_{t/n})^n = h_t = \mathrm{Id}$. Thus $h_{t/n}$ cannot be parabolic, and so $h_{t/n} = \mathrm{Id}$. It follows that $h_{rt} = \mathrm{Id}$ for each rational number r. The continuity of the map $u \mapsto h_u$ now implies that $h_u = \mathrm{Id}$ for all u, which is a contradiction. Thus $h_t \neq \mathrm{Id}$ for $t \neq 0$ and $h_t \neq h_u$ if $t \neq u$.

We next note that if $g_n \in H$ and $g_n \to x_0$ and $g_n^{-1} \to y_0$ locally uniformly in $\overline{\mathbf{C}} \setminus \{y_0\}$ and $\overline{\mathbf{C}} \setminus \{x_0\}$, respectively, then $x_0 = y_0 = \infty$. Since each g_n fixes ∞, it follows that at least one of x_0 and y_0 is ∞. Replacing g_n by g_n^{-1}, if necessary, we may assume that $x_0 = \infty$. Suppose that $y_0 \neq \infty$. Then if $t \neq 0$, $\varepsilon > 0$ and ε is small enough we have $g_n \to \infty$ uniformly in $h_t\big(B(y_0, \varepsilon)\big)$ since $h_t(y_0) \neq y_0$. But since for $z \in B(y_0, \varepsilon)$, we have

$$h_t\big(g_n(z)\big) = g_n\big(h_t(z)\big) \to \infty \qquad \text{as } n \to \infty\ ,$$

it follows that $g_n \to \infty$ uniformly in $B(y_0, \varepsilon)$. But $g_n^{-1} \to y_0$ uniformly in $B(y_0, \varepsilon)$. Thus

$$\mathrm{Id} = g_n \circ g_n^{-1} \to \infty \qquad \text{as } n \to \infty$$

uniformly in $B(y_0, \varepsilon)$, which is a contradiction. Thus $y_0 = \infty$, as required, and so $x_0 = y_0 = \infty$.

6.3. Next we verify that the h_t satisfy a continuity condition considered by Kerékjártó [12]. For $z \in \mathbf{C}$, define $\Gamma(z) = \big\{h_t(z) : t \in \mathbf{R}\big\}$. If $\Gamma(z) \cap \Gamma(w) \neq \emptyset$ then $h_t(z) = h_u(w)$ for some $t, u \in \mathbf{R}$, so that $h_s(z) = h_{s+u-t}(w)$ for all $s \in \mathbf{R}$. Thus $\Gamma(z) \subset \Gamma(w)$ and similarly $\Gamma(w) \subset \Gamma(z)$ so that $\Gamma(z) = \Gamma(w)$. So two sets $\Gamma(z)$ are disjoint unless they coincide.

Suppose that $z_n \to w \in \mathbf{C}$ as $n \to \infty$. We want to prove that then $\Gamma(z_n) \to \Gamma(w)$ uniformly in the chordal metric. Then it will follow from a result of Kerékjártó ([12, p. 115]) that there is a homeomorphism f_1 of $\overline{\mathbf{C}}$ such that $f_1^{-1} \circ h_t \circ f_1 = T_t$ for all real t, where $T_t(z) = z + t$ for all complex t and z. Obviously, $f_1(\infty) = \infty$.

Set

$$\delta_n = \sup\Big(\big\{q\big(x, \Gamma(w)\big) : x \in \Gamma(z_n)\big\} \cup \big\{q\big(x, \Gamma(z_n)\big) : x \in \Gamma(w)\big\}\Big)$$

where $q\bigl(x,\Gamma(w)\bigr) = \inf\bigl\{q(x,y) : y \in \Gamma(w)\bigr\}$. We claim that $\delta_n \to 0$ as $n \to \infty$. If not, we may pass to a subsequence and assume that there are real numbers t_n such that

$$q\bigl(h_{t_n}(z_n), y\bigr) \geq \varepsilon > 0 \tag{6.3.1}$$

for all n and for all $y \in \Gamma(w)$, or that

$$q\bigl(h_{t_n}(w), y_n\bigr) \geq \varepsilon > 0 \tag{6.3.2}$$

for all n and for all $y_n \in \Gamma(z_n)$, and further that the sequence t_n converges, possibly to ∞ or $-\infty$. If $t_n \to u \in \mathbf{R}$, we have $h_{t_n} \to h_u$ uniformly on $\overline{\mathbf{C}}$ so that $h_{t_n}(z_n) \to h_u(w) \in \Gamma(w)$. This contradicts (6.3.1). If $|t_n| \to \infty$, write $t_n = k_n + \theta_n$ where $k_n \in \mathbf{Z}$ and $0 \leq \theta_n < 1$. We may assume that $\theta_n \to \theta \in [0,1]$ so that $h_{\theta_n} \to h_\theta$. Since $h_{k_n} = h_1^{k_n}$ and h_1 fixes ∞ and is conjugate to the translation $z \mapsto z+1$, we have $h_{k_n}(z) \to \infty$ and so $h_{t_n}(z) \to \infty$ locally uniformly in $\mathbf{C}$. But then $q\bigl(h_{t_n}(z_n), h_{t_n}(w)\bigr) \to 0$, which contradicts (6.3.1) since $h_{t_n}(w) \in \Gamma(w)$. We deal with (6.3.2) in the same way. This proves that $\delta_n \to 0$ as $n \to \infty$.

6.4. Define $H_1 = f_1^{-1} \circ H \circ f_1$ and $\mathscr{T}_1 = \{T_t : t \in \mathbf{R}\} \subset H(S^2)$. Then $\mathscr{T}_1 \subset H_1$. If $H_1 = \mathscr{T}_1$, it follows that H is topologically conjugate to a group of translations. So we may and will now assume that $H_1 \neq \mathscr{T}_1$. We say that $g_1, g_2 \in H_1$ are equivalent if $g_1 = T_t \circ g_2$ for some real t. Clearly this defines an equivalence relation, and the equivalence classes form an abelian group Γ. The group Γ acts on $\overline{\mathbf{R}}$ as follows. If $h \in \Gamma$, $h' \in H_1$ and h' is a representative of h, we set $h(t) = \operatorname{Im} h'(it)$ for $t \in \mathbf{R}$, and $h(\infty) = \infty$. Clearly h is then a well-defined function on $\overline{\mathbf{R}}$, continuous on $\mathbf{R}$. If $\operatorname{Im} h'(it) = \operatorname{Im} h'(iu)$, and if $s = \operatorname{Re}\bigl\{h'(it) - h'(iu)\bigr\}$, then, since H_1 is abelian, we have

$$h'(iu+s) = h'(iu) + s = \operatorname{Re} h'(it) + i \operatorname{Im} h'(it) = h'(it)$$

so that $iu + s = it$ since h' is a homeomorphism. Thus $s = 0$ and hence $h'(it) = h'(iu)$ and so $t = u$. Therefore h is one-to-one. We have $|h(t)| \to \infty$ as $t \to \pm\infty$ since

$$\mathbf{C} = h'(\mathbf{C}) = \bigl\{h'(it+u) : t, u \in \mathbf{R}\bigr\} = \bigl\{h'(it) + u : t, u \in \mathbf{R}\bigr\}.$$

Thus h defines a homeomorphism of $\overline{\mathbf{R}}$ onto itself, and we denote the group of such homeomorphisms by J. Since $H_1 \neq \mathscr{T}_1$, the group J is not trivial.

Alternatively, we may note that each $g_1 \in H_1$ commutes with T_t for all real t and therefore maps each horizontal line onto another such line. Identifying every horizontal line with its y-coordinate, we obtain the group action of Γ on $\mathbf{R}$. Two elements of H_1 give rise to the same element of the resulting group J if and only if they differ by T_t for some real t.

We claim that J is a convergence group on $\overline{\mathbf{R}}$, which we identify with S^1. For if $g_n \in J$, we may choose the corresponding sequence of representatives $g_{1n} \in H_1$ of the classes in Γ that give rise to the g_n, so that $\operatorname{Re} g_{1n}(0) = 0$, and pass to a subsequence such that $g_{1n} \to \hat{g} \in H_1$ uniformly on $\overline{\mathbf{C}}$, or $g_{1n} \to \infty$ and $g_{1n}^{-1} \to \infty$ locally uniformly in $\mathbf{C}$ as $n \to \infty$, since H has the corresponding property and $f_1(\infty) = \infty$. In the former case, $g_n \to \widetilde{g}$ given by $\widetilde{g}(t) = \operatorname{Im} \hat{g}(it)$, and in the latter case, $g_n \to \infty$, locally uniformly on $\mathbf{R}$. The same argument shows that J is closed.

If $h \in J$ and $h(t) = t$ for some $t \in \mathbf{R}$ then there is $h' \in H_1$ with $h'(it) = a + it$ for some real a. Then

$$it = (T_{-a} \circ h')(it)$$

so that $T_{-a} \circ h' = \mathrm{Id}$, the only element of H_1 that has a finite fixed point. Thus $h' = T_a$ so that $\operatorname{Im} h'(it) = t$ and $h(t) = t$ for all real t. Hence $h = \mathrm{Id}$, and so the elements of $J \setminus \{\mathrm{Id}\}$ have no finite fixed points.

6.5. Now J is an abelian convergence group on $\overline{\mathbf{R}}$ with parabolic elements only (note that J may or may not be discrete), so that by [5, Theorem 1, p. 88], there is a homeomorphism F_2 of $\overline{\mathbf{R}}$ fixing infinity such that $F_2^{-1} \circ J \circ F_2$ is a group of translations. Now define a homeomorphism f_2 of $\overline{\mathbf{C}}$ by

$$f_2(t + iu) = t + iF_2(u) \qquad \text{when } t, u \in \mathbf{R}$$

and by $f_2(\infty) = \infty$. If $g_1 \in H_1$ then

$$(f_2^{-1} \circ g_1 \circ f_2)(t+iu) = t + \operatorname{Re} g_1(iu) + i(F_2^{-1} \circ h \circ F_2)(u) = t + \operatorname{Re} g_1(iu) + i(u+a),$$

say, where $h \in J$ corresponds to the equivalence class of g_1 and $F_2^{-1} \circ h \circ F_2 = T_a$. Thus if $H_2 = f_2^{-1} \circ H_1 \circ f_2$, then the action of each element of H_2 in the direction of the imaginary axis amounts to a translation. Note that

$$f_2^{-1} \circ T_t \circ f_2 = T_t \qquad \text{for all real } t,$$

so that $\mathscr{T}_1 \subset H_2$.

Since $J_2 = F_2^{-1} \circ J \circ F_2$ is a closed group of translations, it is cyclic unless it is equal to $\mathscr{T}$, the group of all translations of $\mathbf{R}$. In either case, if $g_1, g_2 \in H_2$ correspond to the same element of J_2 then $g_2 = T_t \circ g_1$ for some real t. Thus each element of H_2 can be uniquely represented in the form $T_t \circ \varphi_u$ for some suitable $t \in \mathbf{R}$ and $\varphi_u \in H_2$, where the map $x \mapsto x + t$ lies in J_2, $\operatorname{Re} \varphi_u(0) = 0$, and $\operatorname{Im} \varphi_u(0) = u$, hence $\operatorname{Im} \varphi_u(z) = u + \operatorname{Im} z$ for all z. We have $\varphi_u \circ \varphi_v = T_t \circ \varphi_{u+v}$ where $t = t(u,v) \in \mathbf{R}$ depends on u and v only.

6.6. Since φ_u commutes with each T_t and increases the imaginary part of the variable by u, we may write

$$\varphi_u(x+iy) = x + \psi_u(y) + i(y+u) \tag{6.6.1}$$

where the real-valued function ψ_u is continuous in y with $\psi_u(0)=0$. We look for a homeomorphism f_3 of $\overline{\mathbf{C}}$ fixing ∞ of the form

$$f_3(x+iy) = x + Q(y) + iy, \tag{6.6.2}$$

where Q is a real-valued continuous function of y, that would conjugate H_2 to a group of translations. Clearly $f_3^{-1}\circ T_t \circ f_3 = T_t$ for all $t\in\mathbf{R}$, no matter how Q is chosen.

We have

$$\{f_3^{-1}\circ\varphi_u\circ f_3\}(x+iy) = x + Q(y) + \psi_u(y) - Q(y+u) + i(y+u),$$

which is a translation if and only if

$$Q(y+u) - Q(y) = \psi_u(y) + K(u) \tag{6.6.3}$$

where $K(u)$ depends on u only and not on y.

If $J_2 \neq \mathscr{T}$, we may assume that J_2 is generated by the map $x\mapsto x+1$. Then $\varphi_n = T_{t(n)}\circ\varphi_1^n$ for all $n\in\mathbf{Z}$ where $t(n)\in\mathbf{R}$, so that H_2 is generated $\mathscr{T}_1$ and φ_1. We choose $K(1)=0$ and define $Q(y)=\psi_1(0)y$ for $0\le y\le 1$. Then the condition

$$Q(y+1) - Q(y) = \psi_1(y) \qquad \text{for all } y\in\mathbf{R}$$

extends Q to a continuous function on $\mathbf{R}$. We conclude that H_2 is conjugate to a group of translations in this case.

6.7. Suppose that $J_2 = \mathscr{T}$. The following argument, which is perhaps simpler than our original one for this case, was suggested by the referee. We first claim that for any $g\in H_2$ there is a unique $h\in H_2$ with $h^2=g$. For if $g\in H_2$ then g is given by $g(t+iu) = t + \operatorname{Re} g_1(iu) + i(u+a)$ for some real a and some $g_1\in H_1$ that are related as in Subsection 6.5. Since $J_2=\mathscr{T}$, there is $g_2\in H_1$ such that $F_2^{-1}\circ\eta\circ F_2 = T_{a/2}$ where $\eta\in J$ corresponds to the equivalence class of g_2. We define $h\in H_2$ by $h(t+iu) = t+b+\operatorname{Re} g_2(iu) + i(u+\frac{1}{2}a)$ where the real parameter b is still at our disposal. We find that

$$h^2(t+iu) = t + 2b + \operatorname{Re} g_2(iu) + \operatorname{Re} g_2\big(i(u+\tfrac{1}{2}a)\big) + i(u+a).$$

We choose b so that

$$\operatorname{Re} g_1(0) = 2b + \operatorname{Re} g_2(0) + \operatorname{Re} g_2(\tfrac{1}{2}ai).$$

Then $h^2(0) = g(0)$ so that $g^{-1} \circ h^2 \in H_2$ fixes the origin. But then $g^{-1} \circ h^2 = \mathrm{Id}$ since H_2 is a topological conjugate of H and H has this property, and so $g = h^2$, as required.

Next, if $h_1^2 = g$ also for some $h_1 \in H_2$ then the translation in J_2 corresponding to h_1 must be $T_{a/2}$, the same as for h. Thus there is a real number c such that the maps h and $T_c \circ h_1$ agree at the origin and hence everywhere in the plane. Thus $g = h^2 = T_{2c} \circ h_1^2 = T_{2c} \circ g$, and so $c = 0$ and $h = h_1$. Hence the solution $h \in H_2$ to $g = h^2$ is unique.

As we noted after (6.2.1), H is locally compact and contains no small subgroups, so that the topological conjugate H_2 of H has the same properties. We shall show that any neighbourhood of $\{\mathrm{Id}\}$ in H_2 contains an element not in $\mathscr{T}_1$. Then by [10, Lemma 40, p. 129] (see [10, Theorem 7, p. 95], for the definition of the set K in Lemma 40), there is a (compact) neighbourhood U of $\{\mathrm{Id}\}$ in H_2 with the following property: the set of points of the form $X(1)$, where $X(t)$ defines a one-parameter subgroup of H_2 such that $\{X(t) : |t| \le 1\} \subset U$, is a (closed) neighbourhood of $\{\mathrm{Id}\}$ in H_2. Choosing such a point $X(1) \in U \setminus \mathscr{T}_1$, we find a nontrivial one-parameter family $X(t)$ in H_2 that $\{X(t) : t \in \mathbf{R}\} \cap \mathscr{T}_1 = \{\mathrm{Id}\}$. For if $t \ne 0$ and $X(t) = T_u$ then by the existence and uniqueness of square roots just proved we see that $X(rt)$ is a translation in $\mathscr{T}_1$ for all dyadic rational numbers r. By continuity, this is then true for all real r, which gives a contradiction when $r = 1/t$.

Now to show that any neighbourhood of $\{\mathrm{Id}\}$ in H_2 contains a function not in $\mathscr{T}_1$, suppose, to get a contradiction, that some neighbourhood U of $\{\mathrm{Id}\}$ in H_2 is contained in $\mathscr{T}_1$. Since $\mathscr{T}_1 \ne H_1$ and hence $\mathscr{T}_1 \ne H_2$, there is $h \in H_2 \setminus \mathscr{T}_1$. By the above, for each $k \ge 1$, we can define $h_k \in H_2$ with $h_k^{2^k} = h$. Since $h \notin \mathscr{T}_1$, we have $h_k \notin \mathscr{T}_1$. It now suffices to show that some subsequence of h_k tends to Id as $k \to \infty$. Choose a subsequence h_{k_j} satisfying the definition of a convergence group.

Recall that

$$h_k(x + iy) = \big(x + b_k(y)\big) + i(y + 2^{-k}a)$$

where $a \in \mathbf{R}\setminus\{0\}$ is fixed (and depends on h only) and b_k is a continuous function of y. Now in any case, $b_{k_j}(y)$ tends to a limit (either a finite-valued function or the constant $\pm\infty$) locally uniformly in y as $j \to \infty$. In case of a finite limit function, say $\beta(y)$, we have $h_k(x+iy) \to (x+iy) + \beta(y) \equiv \kappa(x+iy)$, and since $\kappa \in H_2$ and $\kappa(0) = T_{\beta(0)}(0)$, we have $\kappa = T_{\beta(0)}$ so that $\beta(y) = \beta(0)$ for all y. Suppose that $b_{k_j}(0) \to b$ where $b \in [-\infty, \infty]$. With $a_j = b_{k_j}(0)$, we have $(h_{k_j} \circ T_{-a_j})(0) \to 0$ and hence $g_{k_j} \equiv h_{k_j} \circ T_{-a_j}$ tends to a homeomorphism in H_2 fixing the origin. Thus $g_{k_j} \to Id$. Note that $g_k(x + iy) = \big(x + b_k(y) - b_k(0)\big) + i(y + 2^{-k}a)$. Now $h_{k_j} = T_{a_j} \circ g_{k_j}$. Suppose that $b \ne 0$. (We may have $b = \pm\infty$.) Then

$$h(0) = h_{k_j}^{2^{k_j}}(0) = (T_{2^{k_j} a_j} \circ g_{k_j}^{2^{k_j}})(0) = 2^{k_j} a_j + g_{k_j}^{2^{k_j}}(0).$$

For $0 \le p < 2^{k_j}$, the points $g_{k_j}^p(0)$ have their y-coordinates between 0 and a. When j is large enough, $|b_{k_j}(y) - b_{k_j}(0)| < \min\{1, |b|/3\}$ for $|y| \le |a|$ since $g_{k_j} \to \mathrm{Id}$, while furthermore $|a_j| > \min\{2, 2|b|/3\}$ and a_j will have the same sign as b. It follows that

$$|\operatorname{Re} g_{k_j}^{2^{k_j}}(0)| < 2^{k_j} \min\{1, |b|/3\}$$

so that $|\operatorname{Re} h(0)| = |\operatorname{Re} h_{k_j}^{2^{k_j}}(0)| > 2^{k_j} \min\{1, |b|/3\} \to \infty$ as $j \to \infty$, which is a contradiction. Thus $b = 0$, so that $\beta(y)$ is finite and $\beta(y) = \beta(0) = b = 0$ for all y. This shows that $h_{k_j} \to \mathrm{Id}$ as $j \to \infty$. This also completes the proof of the existence of the subgroup $X(t)$.

Write h_t instead of $X(t)$. We have $h_1 \notin \mathcal{T}_1$ so that $a = \operatorname{Im} h_1(0) \ne 0$. By the argument at the beginning of Subsection 6.7 and by continuity, $\operatorname{Im} h_t(0) = at$ for all real t. By reparametrization, we may assume that $a = 1$. Thus if $g \in H_2$, there are real t and u such that $g(0) = T_t \circ h_u(0)$ and so $g = T_t \circ h_u$. Hence (cf. the end of Subsection 6.5) each element of H_2 can be uniquely represented in the form $T_t \circ h_u$ for some real t and u. We may write

$$h_r(x + iy) = x + \chi(r, y) + i(y + r).$$

The condition $h_{r+s} = h_r \circ h_s$ implies that

$$\chi(r + s, y) = \chi(r, y) + \chi(s, r + y) \tag{6.7.1}$$

for all real y, r, s.

We now define a function f of $\mathbf{R}^2$ into itself by

$$f(x + iy) = (h_y \circ T_x)(0) = h_y(x) = x + \chi(y, 0) + iy.$$

Clearly f is a homeomorphism of $\mathbf{R}^2$ onto itself (and hence extends to a homeomorphism of S^2). The map f commutes with each T_t. We further have

$$(h_u \circ f)(x + iy) = h_u\bigl(x + \chi(y, 0) + iy\bigr) = x + \chi(y, 0) + \chi(u, y) + i(y + u).$$

We define T_{iu} by $T_{iu}(x + iy) = x + i(y + u)$. We have

$$\begin{aligned}(f \circ T_{iu})(x + iy) &= f\bigl(x + i(y + u)\bigr) = x + \chi(u + y, 0) + i(u + y)\\ &= (h_u \circ f)(x + iy)\end{aligned}$$

since $\chi(u + y, 0) = \chi(y, 0) + \chi(u, y)$ as we see by replacing r, s, y by $y, u, 0$ in (6.7.1). Hence $f^{-1} \circ T_t \circ f = T_t$ and $f^{-1} \circ h_u \circ f = T_{iu}$ so that $f^{-1} \circ H_2 \circ f$ is a group of translations, in fact the group of all translations of $\mathbf{C}$. We have now proved in all cases (arising when considering Case I) that H is topologically conjugate to a group of translations containing $\mathcal{T}_1$.

6.8. We remark that an alternative proof in the case $J_2 = \mathscr{T}$ can be given along the following lines. This, in fact, was our original proof, and will appear in [7]. In this case consider $\psi_u(v)$ defined by (6.6.1). Since $\varphi_u \circ \varphi_v = \varphi_v \circ \varphi_u$, we have

$$\psi_u(s) + \psi_v(s+u) = \psi_v(s) + \psi_u(s+v) \tag{6.8.1}$$

for all $u, v, s \in \mathbf{R}$, as a calculation shows. Taking $s = 0$ in (6.8.1) and recalling that $\psi_u(0) = 0$, we get

$$\psi_u(v) = \psi_v(u).$$

Furthermore, for any $C > 0$ there is $M > 0$ such that $|\psi_u(v)| \le M$ whenever $|u| \le C$ and $|v| \le C$; and $\psi_{u_n}(v_n) \to 0$ whenever u_n, v_n are real sequences with $u_n \to 0$ and $v_n \to 0$ as $n \to \infty$.

Under these assumptions, it is shown in [7] that $\psi_u(v)$ is of the form

$$\psi_u(v) = Q(u+v) - Q(u) - Q(v)$$

for some continuous function Q with $Q(0) = 0$. We may then use Q in the definition of f_3, and (6.6.3) is obviously satisfied.

6.9. Let us now set $f_4 = f_1$ when $H_1 = \mathscr{T}_1$ and $f_4 = f_1 \circ f_2 \circ f_3$ otherwise so that $H_4 = f_4^{-1} \circ H \circ f_4$ is a Möbius group containing $\mathscr{T}_1$. Set $G_4 = f_4^{-1} \circ G \circ f_4$ so that G_4 contains H_4. If $G = H$, there is nothing else to prove. Otherwise, G is given by (6.2.1), so that

$$G_4 = H_4 \cup \{h \circ g_0 : h \in H_4\} \tag{6.9.1}$$

for some $g_0 \in G_4 \setminus H_4$.

Recall that $\mathscr{T}_1 = \{T_s : s \in \mathbf{R}\}$ and write $\mathscr{T}_2 = \{T_s : s \in \mathbf{C}\}$. Since H_4 is closed, it is clearly equal to $\mathscr{T}_1$ or $\mathscr{T}_2$, or is generated by $\mathscr{T}_1$ and $T_{i\alpha}$ for some positive α. If g_0 fixes $\beta \in \mathbf{C}$ then, since H_4 is abelian, g_0 fixes $T_s(\beta)$ whenever $T_s \in H_4$.

If $H_4 = \mathscr{T}_2$ then $g_0(s) = g_0(s+0) = s + g(0)$ for all $s \in \mathbf{C}$ so that g_0 is sense-preserving, which is a contradiction. Thus $H_4 \ne \mathscr{T}_2$.

By Corollary 1, the function g_0 is conjugate to $\bar{z} + 1$ or $2\bar{z}$ or $c/\bar{z}$ where $|c| = 1$. Thus, if $\operatorname{fix}(g_0)$ contains at least three points, then $\operatorname{fix}(g_0)$ is a Jordan curve. So if $H_4 = \mathscr{T}_1$ then either $\operatorname{fix}(g_0) = \{\infty\}$ and g_0 is conjugate to $\bar{z}+1$, or $\operatorname{fix}(g_0)$ is a horizontal line and g_0 is conjugate to $1/\bar{z}$, hence to $\bar{z}$. If $H_4 \ne \mathscr{T}_1$ and $\operatorname{fix}(g_0) \ne \{\infty\}$, then $\operatorname{fix}(g_0)$ contains infinitely many horizontal lines, which is impossible. Thus if $H_4 \ne \mathscr{T}_1$ then $\operatorname{fix}(g_0) = \{\infty\}$ and g_0 is conjugate to $\bar{z}+1$.

6.10. If $H_4 = \mathscr{T}_1$ and $\mathrm{fix}(g_0)$ is a horizontal line, we may assume that $\mathrm{fix}(g_0) = \overline{\mathbf{R}}$. Set $U = \{z : \operatorname{Im} z > 0\}$ and $L = \{z : \operatorname{Im} z < 0\}$. Since g_0 is sense-reversing, g_0 interchanges U and L. Also $g_0(z+t) = g_0(z)+t$ for all $z \in \mathbf{C}$ and $t \in \mathbf{R}$. So if we define

$$f_5(z) = z \qquad \text{when } z \in L \cup \overline{\mathbf{R}}$$

and

$$f_5(z) = g_0(\bar{z}) \qquad \text{when } z \in U\ ,$$

then f_5 is a homeomorphism of $\overline{\mathbf{C}}$ that fixes ∞. Since g_0 is conjugate to $\bar{z}$, we have $g_0^2 = \mathrm{Id}$. Thus $f_5^{-1}(z) = z$ for $z \in L \cup \overline{\mathbf{R}}$ and $f_5^{-1}(z) = \overline{g_0(z)}$ for $z \in U$. Now one can verify that

$$(f_5^{-1} \circ g_0 \circ f_5)(z) = \bar{z}$$

and

$$(f_5^{-1} \circ T_s \circ f_5)(z) = z + s$$

for all $z \in \overline{\mathbf{C}}$ and all $s \in \mathbf{R}$. Hence $f_5^{-1} \circ G_4 \circ f_5$ is a Möbius group, and so is $f_6^{-1} \circ G \circ f_6$ where $f_6 = f_4 \circ f_5$.

6.11. If g_0 is conjugate to $\bar{z} + 1$, set

$$\Gamma_t = \bigl\{g_0(z) : \operatorname{Re} z = t\bigr\} \qquad \text{for } t \in \mathbf{R}$$

so that the Γ_t are disjoint open Jordan arcs whose union is $\mathbf{C}$. Since g_0 commutes with T_t, it follows that

$$\Gamma_t = \bigl\{z + t : z \in \Gamma_0\bigr\} \qquad \text{for all real } t.$$

Thus Γ_0 has a unique point of intersection with each horizontal line, and the map $t \mapsto \operatorname{Im} g_0(it)$ is a homeomorphism of $\mathbf{R}$ onto itself. Since g_0 is sense-reversing and the curves Γ_t move to the right as t increases, the map $\operatorname{Im} g_0(it)$ is a decreasing function of t and therefore has a fixed point. Hence there are $t, u \in \mathbf{R}$ such that $g_0(it) = u + it$. But then $T_{-u} \circ g_0 \in G_4 \setminus H_4$ fixes it and hence fixes each point on a line and is conjugate to $\bar{z}$. We may use (6.9.1) with g_0 replaced by any element of $G_4 \setminus H_4$, in particular by $T_{-u} \circ g_0$. Now the argument above shows that G_4 is conjugate to a Möbius group when $H_4 = \mathscr{T}_1$.

Finally, if $H_4 \neq \mathscr{T}_1$ but $G_4 \neq H_4$, we have found an element of $G_4 \setminus H_4$ that fixes each point on a line, which contradicts what was said about the case $H_4 \neq \mathscr{T}_1$ at the end of Subsection 6.9.

This completes the proof of Theorem 2 in Case I.

6.12. *Case* II. We reduce Case II to Case I by means of a suitable transformation. We essentially conjugate G by the exponential function but we need to be careful about the technical details. Let H be the index one or two subgroup of G consisting of the sense-preserving elements of G. Note that H is a closed nondiscrete abelian convergence group. We first consider the problem of conjugating H to a Möbius group. Pick $g_0 \in H \setminus \{\mathrm{Id}\}$. Then g_0 is conjugate to $2z$ or cz where $|c| = 1$, and has therefore exactly two fixed points, which we may assume to be 0 and ∞. We claim that each $g \in H$ fixes 0 and ∞. If not, then by (6.1.1), g interchanges 0 and ∞. Thus $\mathrm{fix}(g) = \{a, b\}$ where $a, b \notin \{0, \infty\}$. Now any $h \in H \setminus \{\mathrm{Id}\}$ fixes or interchanges each of the pairs a, b and $0, \infty$. Since a hyperbolic or loxodromic map fixes two points and interchanges no pairs of points, h must be elliptic. Since h^2 fixes each of $0, \infty, a$ and b, and h^2 is elliptic, we have $h^2 = \mathrm{Id}$. If g_n is a sequence of distinct functions in H with $g_n \to \mathrm{Id}$ uniformly on S^2 then for all large n, we cannot have $g_n(0) = \infty$. Thus g_n fixes 0 and ∞. Similarly, g_n fixes a and b. But since g_n is elliptic, we have $g_n = \mathrm{Id}$. This contradiction shows that each function in $H \setminus \{\mathrm{Id}\}$ fixes both 0 and ∞ and hence fixes no other point.

We say that an element of $H \setminus \{\mathrm{Id}\}$ is elliptic if it can be topologically conjugated to an elliptic Möbius transformation, which is therefore of finite order unless it is an irrational rotation. An element is hyperbolic (loxodromic, respectively) if it can be topologically conjugated to a hyperbolic (loxodromic, respectively) Möbius transformation. Thus each element of $H \setminus \{\mathrm{Id}\}$ is elliptic or hyperbolic or loxodromic.

We note that if $g_1, g_2 \in H \setminus \{\mathrm{Id}\}$ are elliptic then also g_1^{-1} and $g_1 \circ g_2$ are elliptic (or $g_1 \circ g_2 = \mathrm{Id}$). This is clear for g_1^{-1}. Concerning $g_1 \circ g_2$, we may assume that a preliminary conjugation has been performed which takes g_1 to an elliptic Möbius transformation, and so we assume that g_1 actually is an elliptic Möbius transformation. If $g_1^n = g_2^m = \mathrm{Id}$ then $(g_1 \circ g_2)^{mn} = \mathrm{Id}$ since H is abelian so that $g_1 \circ g_2$ is elliptic. So we may assume that one of g_1 and g_2, say g_1, is an irrational rotation, say $g_1(z) = cz$ where $c = e^{2\pi i \beta}$ for some irrational number β. Since g_1 and g_2 commute, it follows as in (3.1) that g_2, and hence $g_1 \circ g_2$, maps every circle centred at the origin onto itself. Thus $g_1 \circ g_2$ cannot be hyperbolic, loxodromic (or parabolic, which is ruled out by membership in H anyway), and is therefore elliptic. Thus the elliptic elements together with Id form a subgroup of H, which we denote by E.

Suppose that $H = E$. We shall show that then H contains a function g that is topologically conjugate to an irrational rotation. Suppose that this is not the case. Then every element of $H \setminus \{\mathrm{Id}\}$ is of finite order. Since H is nondiscrete and hence infinite, it follows that the orders of the elements of H must be unbounded. To see this, it suffices to show that for any positive integer N, the subgroup H_N of H generated by all the elements of H of order N, is finite. We note that by the commutativity on H, if $g = g_1 \circ \ldots \circ g_k \in H_N$ where $g_j^N = \mathrm{Id}$, then $g^N = g_1^N \circ \ldots \circ g_k^N = \mathrm{Id}$ so that the order of g is at most N. Now to get a contradiction, suppose that H_N is infinite. Then there is a sequence h_n of distinct elements of H_N satisfying the definition of a convergence group. If $h_n \to h$ where h is a homeomorphism, then the functions $g_n = h_{n+1} \circ h_n^{-1} \in H_N$ tend to Id uniformly on S^2 while each g_n has order at most N. But by a theorem of Newman [18], there is $\varepsilon = \varepsilon(N) > 0$ such that for each n, we have $q\bigl(x_n, g_n(x_n)\bigr) > \varepsilon$ for some $x_n \in S^2$. This gives a contradiction. Thus there must be points $x_0, y_0 \in S^2$ such that $h_n \to x_0$ and $h_n^{-1} \to y_0$ locally uniformly in $S^2 \setminus \{y_0\}$ and $S^2 \setminus \{x_0\}$, respectively. Since each h_n fixes 0 and ∞, we have $\{x_0, y_0\} = \{0, \infty\}$, say $x_0 = \infty$ and $y_0 = 0$. However, $h_n(1)$ is bounded. To see this, note that by the uniform convergence on $\{z : |z| = 1\}$, if $h_n(1) \to \infty$, then $h_n(\{z : |z| = 1\}) \to \infty$. Thus, as $h_n(0) = 0$, $h_n(\infty) = \infty$, for all sufficiently large n, the image $h_n\bigl(B(1)\bigr)$ contains the unit disk $B(1)$ as a relatively compact subset. So if μ_n is a homeomorphism fixing 0 and ∞ such that $\mu_n^{-1} \circ h_n \circ \mu_n = v_n$ is a rotation about the origin, say $v_n(z) = cz$ where c is a root of unity, we see that there is a compact set $K_n = \mu_n\bigl(\overline{B(1)}\bigr)$, which is the closure of a Jordan domain containing the origin, such that K_n is contained in the interior of $v_n(K_n)$. This is seen to be impossible, considering any point of K_n which is at a maximal distance from the origin among all points of K_n. So $h_n(1)$ is bounded, which contradicts the fact that also $h_n(1) \to x_0 = \infty$. We conclude that H_N is finite, as asserted, and so the orders of the elements of H are unbounded.

Choosing elements h_n of H of order at least n and replacing h_n by a suitable iterate of h_n that has the same order as h_n, if necessary, without changing notation, we find a sequence h_n such that if h_n is topologically conjugate to a rotation around the origin by the angle $2\pi\alpha_n$ then $\alpha_n \to \alpha$ as $n \to \infty$ where α is an irrational number. Then $h_n(1)$ is bounded away from $\{0, \infty\}$. This follows by considering rotations $\mu_n^{-1} \circ h_n \circ \mu_n = v_n$ as in the previous paragraph. Therefore by passing to a subsequence we may assume that h_n converges to a limit, uniformly on the entire sphere, which will necessarily be a homeomorphism g and $g \in H = E$ as H is closed. Now g must be elliptic and not of finite order. To see this last claim we let us suppose that g is elliptic of finite order, say N. Because of the uniform convergence it is clear that $g_n = h_n^N \to \mathrm{Id}$ as $n \to \infty$. Now each g_n is the topological conjugate of a finite order rotation through angle $2N\pi\alpha_n \to 2N\pi\alpha \not\equiv 0 \pmod{2\pi}$. Choose $\eta \in (0,1)$ so that $\eta \equiv N\alpha \pmod 1$. Choose the smallest integer m so that $m\eta > 1$. Then for all n the points $\bigl\{h_n^j(1) : 0 \le j \le m\bigr\}$ lie on an invariant Jordan curve for h_n winding once around the origin (for instance the image of a round circle under the topological conjugacy) and as $m\eta > 1$ the sum of the angular displacements

$$\angle\bigl(1, 0, g_n(1)\bigr) + \angle\bigl(g_n(1), 0, g_n^2(1)\bigr) + \ldots + \angle\bigl(g_n^{m-1}(1), 0, g_n^m(1)\bigr) > 2\pi.$$

Thus at least one of the angular displacements is greater than $2\pi/m$.

We have shown that there is a number m such that for every n there is an integer j with $0 \le j \le m$ such that the angular displacement $\angle\bigl(g_n^j(1), 0, g_n^{j+1}(1)\bigr)$ is at least $2\pi/m$. But the uniform convergence to the identity mapping should imply that this displacement tends uniformly to 0, which is not the case. Hence g, as claimed above, has infinite order.

We may then assume, in view of Theorem 1, that a preliminary conjugation has been performed so that $g(z) = c_0 z$ where $c_0 = e^{2\pi i\alpha}$ for some irrational number α. Since H is closed, it follows that the map $\mathscr{R}_c$ given by $\mathscr{R}_c(z) = cz$ belongs to H for every complex number c with $|c| = 1$. Now if $h \in H$ then, since $g \circ h = h \circ g$, we have, as in (3.1), $h(cz) = ch(z)$ whenever $|c| = 1$. Thus h maps each circle $S(r)$ centred at the origin onto some such circle $S\bigl(u(r)\bigr)$. Here $u(r)$ is a homeomorphism of $(0,\infty)$ onto itself so that u is strictly increasing since h fixes 0 and ∞. If $u(r) > r$ or $u(r) < r$ for some $r > 0$ then clearly we cannot have $h^{n_j} \to \mathrm{Id}$ for any sequence of integers $n_j \to \infty$, so that h cannot be elliptic. Thus $u(r) = r$ for all $r > 0$. It follows that $|h(z)| = |z|$ and $h(z) = zh(|z|)/|z|$ for all $z \in S^2 \setminus \{0,\infty\}$. In particular, if $h(1) = a$ then $|a| = 1$ so that the map $\mathscr{R}_{\overline{a}}$ belongs to H. But $(\mathscr{R}_{\overline{a}} \circ h)(1) = 1$ so that $\mathscr{R}_{\overline{a}} \circ h = \mathrm{Id}$, and hence $h = \mathscr{R}_a$, so that h is a Möbius transformation. Thus, after the preliminary conjugation, we have made H into a Möbius group, as required. This completes our treatment of the case $H = E$. For future reference, we note that the above argument in this paragraph also shows that if E contains all rotations $\mathscr{R}_c$ then E contains no other functions.

Suppose then that $H \ne E$. Suppose first that E is not a finite cyclic group. We claim that E contains a function topologically conjugate to an irrational rotation. If not, then E contains functions of arbitrarily large finite order, by the argument that we used in the case $H = E$. Choosing elements h_n of E of order at least n and replacing h_n by a suitable iterate of h_n that has the same order as h_n, if necessary, without changing notation, we find a sequence h_n such that if h_n is topologically conjugate to a rotation around the origin by the angle $2\pi\alpha_n$ then $\alpha_n \to \alpha$ as $n \to \infty$ where α is an irrational number. As before, we see that $h_n(1)$ is bounded away from 0 and ∞. By passing to a subsequence we may therefore assume that $h_n \to h \in H$ where h is a homeomorphism. Now it is seen by the same argument as in the case $H = E$ above, that if $h \in E$ then h is topologically conjugate to an irrational rotation. If $h \notin E$ then h is loxodromic and $h^k(1) \to \infty$, say, as $k \to \infty$. For each k there is $n = n(k)$ such that $q\bigl(h_n^k(1), h^k(1)\bigr) < 1/k$ so that $h_{n(k)}^k(1) \to \infty$. But $h_{n(k)}^k \in E$ is conjugate to a rotation around the origin (by the angle $2\pi k\alpha_{n(k)}$). Hence, by the same argument as before, we see that $h_{n(k)}^k(1)$ remains bounded away from 0 and ∞, which gives a contradiction. This shows that indeed $h \in E$.

Thus we may perform a preliminary conjugation of H and assume that H contains an irrational rotation so that E contains and therefore, as noted before, coincides with the group of all rotations.

Suppose that H also contains a sequence of distinct loxodromic elements tending to the identity. Now H contains no small subgroups. For if $g \in H \setminus \{\mathrm{Id}\}$ is not elliptic then g fixes 0 and ∞ and g is topologically conjugate to the Möbius transformation $z \mapsto 2z$, by the theorem of Kerékjártó [14] referred to in the introduction. Hence $g^n(1) \to \infty$ as $n \to \infty$ or as $n \to -\infty$, and so the subgroup of H generated by g cannot be contained in the neighbourhood $U = \bigl\{h \in H : d(h, \mathrm{Id}) < 1/2\bigr\}$ of the identity. If g is elliptic then g is an elliptic Möbius transformation fixing 0 and ∞ so that again it is clear that there is some fixed neighbourhood U of Id independent of g that does not contain the group generated by g. Now by [10, Theorem 6, p. 95], H contains a nontrivial one-parameter family h_t, and by [10, Theorem 5, p. 93], we may in fact assume that some nontrivial h_t is loxodromic. Then it clearly follows that h_t is loxodromic whenever $t \neq 0$.

Since $h_n = h_1^n$, the points $h_n(1)$ cluster to both 0 and ∞, so that $\bigl\{|h_t(1)| : t \in \mathbf{R}\bigr\} = \{t : t > 0\}$. We claim that H is generated by E and the maps h_t for $t \in \mathbf{R}$. For if $h \in H$ then there is c with $|c| = 1$ and $t \in \mathbf{R}$ such that $h(1) = ch_t(1)$. Since $h^{-1} \circ \mathscr{R}_c \circ h_t \in H$ fixes the point $1 \notin \{0, \infty\}$, it follows that $h = \mathscr{R}_c \circ h_t$.

By the last statement of Theorem 4, there is a homeomorphism f of S^2 onto itself that conjugates each rotation $\mathscr{R}_c$ onto itself, and conjugates each h_t to the dilation $z \mapsto e^t z$. Hence $f^{-1} \circ H \circ f$ is a Möbius group.

Suppose that H does not contain a sequence of distinct loxodromic elements tending to the identity. (We are still assuming that $H \neq E$ and that E is the group of all rotations.) Now if $h \in H \setminus E$ then, since $g \circ h = h \circ g$, we have, as in (3.1), $h(cz) = ch(z)$ whenever $|c| = 1$. Thus h maps each circle $S(r)$ centred at the origin onto some such circle $S\bigl(u(r)\bigr)$. Here $u(r) = u_h(r)$ is a homeomorphism of $(0, \infty)$ onto itself so that u is strictly increasing since h fixes 0 and ∞. For different choices of h we get different values for $u_h(1)$ and at least one of h and h^{-1} gives a value of $u_h(1) > 1$. Some $h = h_0$ gives the smallest value of $u_h(1) > 1$ as otherwise there is a sequence of distinct loxodromic elements tending to the identity. For the same reason the set of values of $u_h(1) > 1$ has no finite limit point. We further note that if $h, k \in H$ and if there is $r > 0$ such that $u_h(r) = u_k(r)$ then $h = \mathscr{R}_c \circ k$ for some c with $|c| = 1$ so that $u_h = u_k$. Otherwise, we have either $u_h(r) < u_k(r)$ for all $r > 0$, or $u_h(r) > u_k(r)$ for all $r > 0$. Similarly, if $h \in H \setminus E$, we have $u_h(r) > r$ for all $r > 0$, or $u_h(r) < r$ for all $r > 0$.

Denote the map u_h for $h = h_0$ by u_0. We shall now prove that every possible value of $u_h(1)$ is of the form $u_0^n(1)$ for some integer n where u_0^n denotes the n^{th} iterate of u_0. Concerning the values of $u_h(1) > 1$, suppose that the first N (where $N \geq 1$) have been proved to be of the form $u_0^n(1)$ where $1 \leq n \leq N$. Let the next value be $u_g(1)$ where $g \in H$. By definition, $u_0^N(1) < u_g(1) \leq u_0^{N+1}(1)$. If $u_g(1) < u_0^{N+1}(1)$, note that $u_0^N(r) < u_g(r) < u_0^{N+1}(r)$ for all $r > 0$, and consider

$\kappa = g \circ h_0^{-N} \in H$. We have $u_\kappa = u_g \circ u_0^{-N}$, and u_g and u_0 commute. Hence $1 < u_\kappa(1) < u_0(1)$, which is a contradiction. We deduce that $u_g(1) = u_0^{N+1}(1)$, as required.

We claim that every element h of H is of the form $\mathscr{R}_c \circ h_0^n$ for some integer n. Choose n so that $u_h(1) = u_0^n(1)$ and hence $u_h = u_0^n$. As noted above, there is c with $|c| = 1$ such that $h(1) = ch_0^n(1)$. Consequently, $h = \mathscr{R}_c \circ h_0^n$, as required.

There is an increasing homeomorphism k of $(0,\infty)$ onto itself such that $(k^{-1} \circ u_0 \circ k)(x) = ax$ where $a > 0$ and $a \neq 1$ (we may take $a = 2$, in fact). Define $f(re^{i\theta}) = k(r)e^{i\theta}$, and let f fix 0 and ∞. Then f is a homeomorphism of S^2 onto itself that conjugates each $\mathscr{R}_c$ to itself and satisfies $(f^{-1} \circ h_0 \circ f)(z) = acz$ for some c with $|c| = 1$. Thus $f^{-1} \circ H \circ f$ is a Möbius group.

Suppose then that $H \neq E$ and that E is a finite cyclic group. Now clearly H contains no small subgroups (and is locally compact), and so, by [10, Theorem 6, p. 95], H contains a nontrivial one-parameter family h_t. Since H is nondiscrete, it must contain a sequence of distinct loxodromic elements tending to the identity. Now by [10, Theorem 5, p. 93], we may in fact assume that some nontrivial h_t is loxodromic. Then it clearly follows that h_t is loxodromic whenever $t \neq 0$. By our Theorem 4, still to be proved, we may perform a preliminary conjugation of H and assume that $h_t(z) = e^t z$ for all real t. However, we prefer to change notation and assume that for all real positive t, the mapping $h_t(z) = tz$ belongs to H.

Suppose now that $g \in H \setminus \{h_t : t > 0\}$. Since $g(tz) \equiv tg(z)$ for all $t > 0$, it follows that g maps rays (from 0 to ∞) onto rays. Hence g determines a homeomorphism k of the circle S^1 onto itself via $k(e^{i\theta}) = e^{i\psi}$ if and only if $g(\{te^{i\theta} : t > 0\}) = \{te^{i\psi} : t > 0\}$. We claim that the mappings of S^1 so obtained when g goes through all elements of H, form a convergence group on S^1. (Note that if $g = h_t$ then $k = \mathrm{Id}$.) Clearly the set K of these maps k is a group. Note that if k fixes a point so that g maps some ray onto itself then there is $z \neq 0, \infty$ such that $g(z) = tz$ for some $t > 0$. Then $t^{-1}g \in H$ fixes z so that $t^{-1}g = \mathrm{Id}$ and $g = h_t$. Thus $k = \mathrm{Id}$. So every element of $K \setminus \{\mathrm{Id}\}$ is elliptic.

Suppose that $k_n \in K$ and that $g_n \in H$ gives rise to k_n. We may replace g_n by tg_n for any $t > 0$ without changing k_n. Thus we may choose t depending on n so that $|g_n(1)| = 1$. Since H is a convergence group, we may pass to a subsequence without changing notation and assume that g_n converges to g. Since g_n fixes 0 and ∞ and $|g_n(1)| = 1$, it follows that the limit function g must be a homeomorphism to which g_n converges uniformly on the sphere, and then $g \in H$ since H is closed. Thus g gives rise to $k \in K$ and clearly $k_n \to k$ uniformly on S^1. Every element of K is topologically conjugate to a Möbius transformation by [5], for example, and as we have seen above, every such Möbius transformation must be elliptic or the identity. Since H is abelian, so is K. Thus K is nondiscrete or finite, and in both cases there is a homeomorphism p of S^1 onto itself such that $K' = p^{-1} \circ K \circ p$ is again a group of elliptic Möbius transformations, by [5]. Since H and K are abelian, so is K'. Thus all the elements of K', when viewed as Möbius transformations of the unit disk, have the same fixed point in the disk, and by choosing p in an appropriate way (replacing p by $p \circ M$ for a suitable Möbius transformation M of the unit disk, if necessary), we may assume that this fixed point is the origin. Hence each element k' of K' is of the form $k'(e^{i\theta}) = ce^{i\theta}$ where $|c| = 1$. The group K', being a closed abelian group of rotations, is either a finite cyclic group, or contains every rotation $\mathscr{R}_{\exp(2\pi i\alpha)}$ given by $\mathscr{R}_{\exp(2\pi i\alpha)}(z) = e^{2\pi i\alpha}z$.

We claim that K' is a finite cyclic group on the basis that E is a finite cyclic group. To get a contradiction, suppose that K' contains all rotations around the origin. Suppose that every element of E has order at most N, and consider an element k' of K' of order $2N$, say. Then the corresponding element k of K has order $2N$, and if $g \in H$ corresponds to k, then g^{2N} fixes every ray so that $g^{2N} = h_t$ for some $t > 0$. Set $h(z) = g(z)/t^{1/(2N)}$ so that $h \in H$. Then, since H is abelian, we have $h^{2N} = \mathrm{Id}$ while clearly $h^j \neq \mathrm{Id}$ if $1 \le j < 2N$. Thus h is elliptic of order $2N$, which is a contradiction. It follows that K', and hence K, is a finite cyclic group.

Now define a homeomorphism f_1 of the sphere onto itself fixing 0 and ∞ by $f_1(re^{i\theta}) = rp(e^{i\theta})$. Then $f_1^{-1} \circ h_t \circ f_1 = h_t$ for all $t > 0$. Further, if $g \in H$ gives rise to $k \in K$ and if $(p^{-1} \circ k \circ p)(e^{i\theta}) = ce^{i\theta}$ where $|c| = 1$, then

$$(f_1^{-1} \circ g \circ f_1)(re^{i\theta}) = cre^{i\theta}\big|g\big(p(e^{i\theta})\big)\big|. \tag{6.12.1}$$

Now $H' = f_1^{-1} \circ H \circ f_1$ contains every h_t, and each element of H' fixes 0 and ∞, and no element of $H' \setminus \{\mathrm{Id}\}$ has any other fixed points.

Suppose then that K' is a finite cyclic group generated by $\mathscr{R}_{\exp(2\pi i/n)}$ corresponding to a function $g' = f_1^{-1} \circ g \circ f_1 \in H'$ where $g \in H$. Then $c = e^{2\pi i/n}$ in (6.12.1). We note that now H is generated by g and the dilations h_t for $t > 0$. For if $h \in H$ then the action of h on rays is the same as for some g^j. Thus $g^{-j} \circ h$ maps each ray onto itself so that $g^{-j} \circ h = h_t$ for some $t > 0$. Next, g^n fixes each ray so that $g^n = h_t$ for some $t > 0$. Replacing g by $g \circ h_u$ and hence replacing g' by $g' \circ h_u$ where $u^n = 1/t$, we may assume that $g^n = (g')^n = \mathrm{Id}$.

We define $f(re^{i\theta}) = F(\theta)re^{i\theta}$ and choose $F(\theta) > 0$ for $0 \le \theta \le 2\pi/n$ so that f will define a homeomorphism of the sector $\{re^{i\theta} : r > 0,\ 0 \le \theta \le 2\pi/n\}$ onto itself satisfying

$$\big|g'\big(rF(0)\big)\big| = \big|g'\big(f(r)\big)\big| = rF(0)\big|g\big(p(1)\big)\big| = rF(2\pi/n)$$

for all $r > 0$. That is, we require that $F(2\pi/n) = F(0)\big|g\big(p(1)\big)\big|$. We use the equation

$$F(\theta + 2\pi/n) = F(\theta)\big|g\big(p(e^{i\theta})\big)\big|$$

to extend the definition of F to all θ. This leads to the requirement that

$$F(0) = F(2\pi) = F(0)\prod_{j=0}^{n-1}\big|g\big(p(e^{2\pi ij/n})\big)\big|,$$

which reads

$$1 = \prod_{j=0}^{n-1} \bigl|g + \bigl(p(e^{2\pi ij/n})\bigr)\bigr|. \tag{6.12.2}$$

Applying (6.12.1) n times starting with $re^{i\theta} = 1$, we obtain (6.12.2) since $(g')^n = \mathrm{Id}$. Now f conjugates each h_t onto itself while $f^{-1} \circ g' \circ f = \mathscr{R}_{\exp(2\pi i/n)}$. Thus we find an f such that $f^{-1} \circ H' \circ f$ is a Möbius group. This completes the proof that we can conjugate H to a Möbius group.

6.13. *Proof of Theorem* 4. Let the assumptions of Theorem 4 be satisfied. We may assume that every h_t fixes 0 and ∞, and then for any $t \neq 0$, the function h_t has no fixed points in $\mathbf{C} \setminus \{0\}$. For any $z \in \mathbf{C} \setminus \{0\}$, all the points $h_t(z)$ are distinct. For if not, then there are real distinct t and u with $h_t(z) = h_u(z)$. But then h_{u-t} fixes z, a contradiction. Hence $\{h_t(z) : t \in \mathbf{R}\}$ is an open Jordan arc. We claim that this arc has the endpoints 0 and ∞. As $n \to \infty$, the points $h_n(z) = h_1^n(z)$ tend to 0 or ∞, say to ∞. We want to show that $\lim_{t\to\infty} h_t(z) = \infty$. Suppose that $t(j) = n(j) + \varepsilon(j) \to \infty$ where $n(j) \in \mathbf{Z}$ and $0 \le \varepsilon(j) < 1$. To get a contradiction, suppose that $h_{t(j)}(z)$ does not tend to ∞. By passing to a subsequence, we may assume that $h_{t(j)}(z) \to \alpha \in \mathbf{C}$ and that $\varepsilon(j) \to \varepsilon \in [0,1]$. By the assumptions of Theorem 4, we have $h_{\varepsilon(j)} \to h_\varepsilon$ uniformly on S^2. Hence

$$h_{t(j)}(z) = h_{\varepsilon(j)}\bigl(h_{n(j)}(z)\bigr) \to h_\varepsilon\bigl(\lim_{j\to\infty} h_{n(j)}(z)\bigr) = h_\varepsilon(\infty) = \infty$$

as $j \to \infty$, which gives a contradiction. It follows that $\lim_{t\to\infty} h_t(z) = \infty$. Similarly, $\lim_{n\to-\infty} h_n(z) = 0$ and so $\lim_{t\to-\infty} h_t(z) = 0$.

Now define a family of mappings on $\mathbf{C}$ as follows. Suppose that $z \in \mathbf{C}$. Then the points $h_t(e^z)$ for $t \in \mathbf{R}$ define a Jordan arc through $e^z = h_0(e^z)$. There is a unique arc of points $k_t(z)$ in the plane such that $k_0(z) = z$ and $e^{k_t(z)} = h_t(e^z)$ for all real t. This defines $k_t(z)$. Let m be any integer. Then the curves $h_t(e^z)$ and $h_t(e^{z+2\pi im})$ are the same, and it follows from the definition above that $k_t(z + 2\pi im) = k_t(z) + 2\pi im$ for all t. It easily follows from the definition that for any fixed real t, the map k_t is continuous in z. The map k_t is one-to-one, for if $k_t(z) = k_t(w)$ then $h_t(e^z) = h_t(e^w)$ so that $e^z = e^w$ and so $z = w + 2\pi im$ for some integer m. But then $k_t(z) = k_t(w) + 2\pi im$ so that $m = 0$ and so $z = w$, as required. Also $k_t(\mathbf{C}) = \mathbf{C}$. For if $w \in \mathbf{C}$ then there is $\zeta \in \mathbf{C} \setminus \{0\}$ such that $h_t(\zeta) = e^w$. Choose any $z \in \mathbf{C}$ with $e^z = \zeta$. Then

$$\bigl\{k_t(z + 2\pi im) : m \in \mathbf{Z}\bigr\} = \{w + 2\pi in : n \in \mathbf{Z}\}$$

so that $w \in k_t(\mathbf{C})$, as required. Thus each k_t is a homeomorphism of $\mathbf{C}$ onto itself, and clearly $k_0 = \mathrm{Id}$.

If $t \neq 0$, $z \in \mathbf{C}$, $m \in \mathbf{Z}$, and $k_t(z) = z + 2\pi i m$, then by the definition above, we have

$$h_t(e^z) = e^{k_t(z)} = e^{z+2\pi i m} = e^z,$$

which is a contradiction. In particular, taking $m = 0$, we see that k_t has no finite fixed points when $t \neq 0$. Further if t, u are distinct real numbers and $z \in \mathbf{C}$ then $k_t(z) \neq k_u(z)$. For if $k_t(z) = k_u(z)$ then $h_t(e^z) = h_u(e^z)$, a contradiction. Hence the points $k_t(z)$ for $t \in \mathbf{R}$ form a Jordan arc $\gamma(z)$.

We claim that $k_{t+u} = k_t \circ k_u$ for all real t and u. For this purpose, pick $z \in \mathbf{C}$, denote $\gamma(z)$ by γ, and write

$$\Gamma(e^z) = \bigl\{h_t(e^z) : t \in \mathbf{R}\bigr\} = \bigl\{e^{k_t(z)} : t \in \mathbf{R}\bigr\}.$$

Now $k_u(z) \in \gamma$, and $h_u(e^z) = e^{k_u(z)} \in \Gamma(e^z)$. But now $\Gamma\bigl(h_u(e^z)\bigr) = \Gamma(e^z)$. Hence $k_t\bigl(k_u(z)\bigr)$ is obtained from $k_u(z)$ by lifting an inverse image (under e^z) of $\Gamma\bigl(h_u(e^z)\bigr)$ going through $k_u(z)$. But this must be the arc γ itself. Thus $k_t\bigl(k_u(z)\bigr)$ is the unique point w on γ with $e^w = h_t\bigl(e^{k_u(z)}\bigr)$. But $k_{t+u}(z) \in \gamma$ and

$$e^{k_{t+u}(z)} = h_{t+u}(e^z) = h_t\bigl(h_u(e^z)\bigr) = h_t\bigl(e^{k_u(z)}\bigr).$$

It follows that $k_{t+u}(z) = k_t\bigl(k_u(z)\bigr)$, and so $k_{t+u} = k_t \circ k_u$, as claimed. This shows that the set $G = \{k_t : t \in \mathbf{R}\}$ is a group, and in fact, an abelian group.

If $t_n \in \mathbf{R}$ and $t_n \to \infty$ or $t_n \to -\infty$, it is now seen that $k_{t_n} \to \infty$ locally uniformly in $\mathbf{C}$. If $t_n \to t \in \mathbf{R}$ then clearly $k_{t_n} \to k_t$ locally uniformly in $\mathbf{C}$. Thus G is an abelian closed convergence group, and every element of $G \setminus \{\mathrm{Id}\}$ is parabolic. Furthermore, since $k_t(z + 2\pi i m) \equiv k_t(z) + 2\pi i m$, it follows that the group G' generated by G and the mappings $z \mapsto z + 2\pi i m$ for $m \in \mathbf{Z}$, is also an abelian closed convergence group, and by the property $k_t(z) \neq z + 2\pi i m$ (for $m \in \mathbf{Z}$ and $t \neq 0$) established above, every element of $G' \setminus \{\mathrm{Id}\}$ is parabolic.

It cannot be the case that $\operatorname{Re} k_t(z) = \operatorname{Re} z$ for some $t \neq 0$ and all z with $\operatorname{Re} z = r$ for some fixed real r. For suppose that this identity holds. Then $|h_t(e^z)| = |e^{k_t(z)}| = e^{\operatorname{Re} k_t(z)} = e^{\operatorname{Re} z} = |e^z|$ for this t and all such z. Then $|h_t(w)| = |w|$ whenever $|w| = e^r$. This contradicts the assumption that h_t is loxodromic so that the orbit of any point e^z under the iterates of h_t clusters to 0 and ∞. This proves our assertion. In particular, if $t \neq 0$, we cannot have $k_t(z) \equiv z + 2\pi i m$ for any integer m.

For any $z \in \mathbf{C}$, we have $\bigl\{\operatorname{Re} k_t(z) : t \in \mathbf{R}\bigr\} = \mathbf{R}$. By continuity, this follows as soon as $\bigl\{\operatorname{Re} k_t(z) : t \in \mathbf{R}\bigr\}$ is unbounded above and below. This is the case since $e^{\operatorname{Re} k_n(z)} = |h_n(e^z)|$ and since $h_n(e^z)$ for $n \in \mathbf{Z}$ clusters to 0 and ∞.

By the part of Theorem 2 for groups with only parabolic elements, which we have proved already without any reference to Theorem 4, there is a homeomorphism f_1 of $\overline{\mathbf{C}}$ fixing infinity such that each element of $G^0 = f_1^{-1} \circ G' \circ f_1$, other than the identity, is a parabolic Möbius transformation fixing infinity, that is, a translation. Since $k_{t+u} = k_t \circ k_u$ for all real t and u, we may clearly choose f_1 so that $f_1^{-1} \circ k_t \circ f_1 = T_t$ for all real t where $T_t(z) = z + t$ as before. Write $g = f_1^{-1} \circ T_{2\pi i} \circ f_1 \in G^0$. Then g commutes with T_t for all real t while g^m is not equal to T_t for any real t, for any nonzero integer m. Thus $g = T_a$ where a is a nonreal complex number.

We now define $f_2(z) = az/(2\pi i)$, $f = f_1 \circ f_2$, and $G'' = f_2^{-1} \circ G^0 \circ f_2 = f^{-1} \circ G' \circ f$. Then $f^{-1} \circ k_t \circ f = T_{2\pi it/a}$ while $f^{-1} \circ T_{2\pi i} \circ f = T_{2\pi i}$.

Thus G'' contains the maps $z \mapsto z + 2\pi im$ for all $m \in \mathbf{Z}$, and f satisfies $f(z + 2\pi im) = f(z) + 2\pi im$ for all $z \in \mathbf{C}$ and all $m \in \mathbf{Z}$. We define $a_t \in \mathbf{C}$ by $(f^{-1} \circ k_t \circ f)(z) \equiv z + a_t$, so that $a_t = 2\pi it/a$. Then $a_{t+u} = a_t + a_u$ for all real t and u. We have $a_t \equiv ct$ where the constant c is not purely imaginary.

Now define a homeomorphism F of $\mathbf{C} \setminus \{0\}$ onto itself by $F(z) = e^{f(\log z)}$. This is well-defined, no matter which value of $\log z$ is used, and is continuous by local considerations. It clearly maps $\mathbf{C} \setminus \{0\}$ onto itself. If $F(z) = F(w)$ then $f(\log z) = f(\log w) + 2\pi im = f(\log w + 2\pi im)$ for some $m \in \mathbf{Z}$. But then $\log z = \log w + 2\pi im$ so that $z = w$. Thus F is one-to-one, and so F defines a homeomorphism of $\mathbf{C} \setminus \{0\}$ onto itself. Clearly F extends to a homeomorphism of S^2 onto itself.

Suppose that $z \in \mathbf{C} \setminus \{0\}$. We have

$$\begin{aligned}(F^{-1} \circ h_t \circ F)(z) &= \exp\Big\{f^{-1}\big(\log\big[h_t\big(e^{f(\log z)}\big)\big]\big)\Big\} \\ &= e^{f^{-1}(k_t(f(\log z)))} = e^{\log z + a_t} = e^{a_t} z = e^{ct} z.\end{aligned}$$

Write $c = c_1 + ic_2$ where c_1 and c_2 are real. We replace $F(z)$ by the function F_1 defined by $F(z) = F_1(z)$ when $z \in \{0, \infty\}$ and $F_1(re^{i\theta}) = F(r^{c_1} e^{i(\theta + \beta(r))})$ for $z = re^{i\theta} \in \mathbf{C} \setminus \{0\}$. Here $\beta(r)$ can be any function satisfying $\beta(e^t r) - \beta(r) \equiv c_2 t$ $(\operatorname{mod} 2\pi)$ whenever $r > 0$ and t is real. One possible choice is $\beta(r) = c_2 \log r$. It is now easy to verify that then $(h_t \circ F_1)(z) = F_1(e^t z)$ so that $(F_1^{-1} \circ h_t \circ F_1)(z) = e^t z$.

Suppose finally that each h_t commutes with every rotation around the origin. We claim that then each k_t commutes with the translation T_{iu} for every real u. To see this, note that if $\gamma = e^{iu}$ for a fixed real u and if $h_t(\gamma z) = \gamma h_t(z)$ for all z, then for any fixed $z \in \mathbf{C}$, we have

$$e^{k_t(z) + iu} = \gamma e^{k_t(z)} = \gamma h_t(e^z) = h_t(\gamma e^z) = h_t(e^{z+iu}) = e^{k_t(z+iu)}$$

so that $\big(k_t(z) + iu\big) - k_t(z + iu) = 2\pi im$ for some integer m. For $t = 0$ this holds with $m = 0$ so that by continuity with respect to t, for fixed z and u, the same identity holds with $m = 0$ for all real t. This shows that k_t commutes with T_{iu}. If now $\operatorname{Re} k_t(z) = \operatorname{Re} z$ for some $t \neq 0$ and some z, then $\operatorname{Re} k_t(z + iu) = \operatorname{Re}\big(k_t(z) + iu\big) = \operatorname{Re}(z + iu)$ for all real u. As we have seen, this leads to a contradiction. Hence $T_{iu} \circ k_t$ has no finite fixed points when $(t, u) \neq (0, 0)$.

Next we may, in the above proof, replace G' by the group generated by the k_t and the T_{iu} for all real t and u. We still find a homeomorphism f_1 such that $G^0 = f_1^{-1} \circ G' \circ f_1$ is a group of translations, while $f_1^{-1} \circ k_t \circ f_1 = T_t$ for all t. We pass to G'' as before using a map f_2, and set $f = f_1 \circ f_2$. The mappings $f^{-1} \circ T_{iu} \circ f = q_u$ form a one-parameter family of translations so that they are given by $q_u = T_{bu}$ for some nonzero complex number b. Since $q_{2\pi} = T_{2\pi i}$, we have $b = i$. Hence $q_u = T_{iu}$ and so $f^{-1} \circ T_{iu} \circ f = T_{iu}$. Defining F as before, we find that for any $\gamma = e^{iu}$ with $|\gamma| = 1$ we have

$$(F^{-1} \circ \mathscr{R}_\gamma \circ F)(z) = e^{\log z + iu} = \gamma z.$$

If a further transformation is needed and $F_1(re^{i\theta}) = F(r^{c_1} e^{i(\theta + \log r)})$, then

$$(\mathscr{R}_\gamma \circ F_1)(z) = F(r^{c_1} e^{i(\theta + u + \log r)}) = F_1(re^{i(\theta+u)}) = F_1(\gamma z),$$

as required for F_1 to conjugate each $\mathscr{R}_\gamma$ onto itself. This completes the proof of Theorem 4.

6.14. We return to the proof of Theorem 2 in Case II. We have shown that the subgroup H of G consisting of the sense-preserving elements of G can be topologically conjugated to a Möbius group. Let us therefore assume, without changing notation, that H, in fact, is a Möbius group consisting of mappings of the form $z \mapsto cz$ where $c \in \mathbf{C} \setminus \{0\}$. Then there is a closed nondiscrete group J' of translations T_a of $\mathbf{C}$, containing $T_{2\pi i n}$ for all integers n, such that $H = \{z \mapsto e^a z : T_a \in J'\}$.

Since J' is closed and nondiscrete, it contains a one-parameter family of translations. In fact, the set $E = \{\widetilde{g}(0) : \widetilde{g} \in J'\}$ is a closed set in $\mathbf{C}$ and E is an additive group which contains points arbitrarily close to 0 as well as all the points $2\pi i n$ for $n \in \mathbf{Z}$. We claim that if $E \neq \mathbf{C}$ then

$$E = \{tc_1 + nc_2 : t \in \mathbf{R},\ n \in \mathbf{Z}\} \tag{6.14.1}$$

where $c_1, c_2 \in \mathbf{C}$ with $c_1 \neq 0$ and $\operatorname{Re}(\bar{c}_1 c_2) = 0$. For suppose that $a_n \in E \setminus \{0\}$ and $a_n \to 0$. If a_n belongs to line L_n through the origin, we may assume that the L_n tend to a limit line L. If $w \in L$ and $\varepsilon > 0$, there is a line L_n such that the distance of w from L_n is less than $\varepsilon/2$ and such that $|a_n| < \varepsilon/2$. Since $ma_n \in E$ for all $m \in \mathbf{Z}$, there is $\beta \in E$ with $|\beta - w| < \varepsilon$. Since ε was arbitrary and E is closed, we have $w \in E$, and so $L \subset E$ (cf. an argument of Baker in [1, p. 285]). We may write $L = \{tc_1 : t \in \mathbf{R}\}$ where $c_1 \neq 0$. If $L \neq E \neq \mathbf{C}$ and $c_2 \in E \setminus L$ then $nc_2 + tc_1 \in E$ for all $n \in \mathbf{Z}$ and $t \in \mathbf{R}$. We may thus only consider those $c_2 \in E \setminus L$ that are perpendicular to L as vectors in $\mathbf{R}^2$, that is, those with $\operatorname{Re}(\bar{c}_1 c_2) = 0$. These elements c_2 of E on the line M through 0 orthogonal to L either form a discrete set of the form $\{mc_2' : m \in \mathbf{Z}\}$ for some $c_2' \neq 0$, or $M \subset E$, in which case $E = \mathbf{C}$. Thus (6.14.1) holds. Note that if $E = L$ then (6.14.1) holds with $c_2 = 0$.

Thus H consists of mappings M_c where $M_c(z) = cz$, such that if V is defined by $H = \{M_c : c \in V\}$ then either $V = \mathbf{C} \setminus \{0\}$ or

(6.14.2) $$V = \{c_3^n e^{c_1 t} : t \in \mathbf{R},\ n \in \mathbf{Z}\}$$

where $c_3 = e^{c_2} \neq 0$.

6.15. If $G \neq H$ then G is generated by H and some $h \in G \setminus H$. Also G is abelian. If $V = \mathbf{C} \setminus \{0\}$ then

(6.15.1) $$h(cz) = ch(z) \quad \text{for all } z \in \mathbf{C} \quad \text{and} \quad c \in V,$$

and taking $z = 1$ we see that h is sense-preserving, which is a contradiction.

Let V be as in (6.14.2). Suppose first that $|c_3| = |e^{c_1}| = 1$. Then

$$V = \{c \in \mathbf{C} : |c| = 1\}$$

since $c_1 \neq 0$. Now (6.15.1) is the same as (3.1), so that the proof of Lemma 1 shows that G is topologically conjugate to a Möbius group.

Suppose then that $|c_3| \neq 1$ or $|e^{c_1}| \neq 1$. Now (6.14.2) and (6.15.1) with $z = 1$ show that h fixes 0 and ∞. If $c_2 = 0$ then, since $2\pi i \in E$, we have $c_1 = ib$ where b is real. But then $|c_3| = |e^{c_1}| = 1$, which is a contradiction. Thus $c_2 \neq 0$, and in view of the structure of the set E, we may assume first that c_1 and c_2 are orthogonal vectors in $\mathbf{R}^2$ and then, since c_1 can be multiplied by a nonzero real number without changing E, that $c_1 = ic_2$. Since $2\pi i \in E$, there are $t_0 \in \mathbf{R}$ and $n_0 \in \mathbf{Z}$ such that $(n_0 + it_0)c_2 = 2\pi i$, that is,

$$c_2 = 2\pi(t_0 + in_0)(t_0^2 + n_0^2)^{-1}.$$

Replacing c_2 by $-c_2$, if necessary, we may assume that $n_0 \geq 0$. A calculation shows that either $n_0 \neq 0$ and

(6.15.2) $$V = \{\omega^k \exp[t(n_0 - it_0)] : t \in \mathbf{R},\ k \in \mathbf{Z} \text{ and } 0 \leq k < n_0\}$$

where $\omega = \exp\{2\pi i/n_0\}$, or $n_0 = 0 \neq t_0$ and

(6.15.3) $$V = \{\exp[u(n + it)] : n \in \mathbf{Z},\ t \in \mathbf{R}\}$$

where $u = 2\pi/t_0$. In the latter case, V contains the unit circle (take $n = 0$), so that (6.15.1) and the proof of Lemma 1 show that G is topologically conjugate to a Möbius group. (In fact, it may be that (6.15.3) implies some contradiction but we need not be concerned about that.) More precisely, to get (6.15.2), note that every $\alpha \in E$ is of the form $\alpha = (n + it)c_2 = 2\pi i(n + it)/(n_0 + it_0)$. Choose $k \in \mathbf{Z}$ with $0 \leq k < n_0$ so that $k \equiv n \pmod{n_0}$. Then choose $u = 2\pi(nt_0 - tn_0)/\big(n_0(n_0^2 + t_0^2)\big)$, write e^α in terms of k and u, and denote u again by t. This gives (6.15.2).

6.16. So we may assume that V is as in (6.15.2). Since h is sense-reversing and fixes 0 and ∞, it follows that the image $h\bigl(S(1)\bigr)$ of the unit circle covered in the positive direction, when projected radially by the map $z \mapsto z/|z|$ onto a path Γ on the unit circle, is homotopic to the circle covered once in the negative direction. On the other hand, if Γ starts at $\alpha = h(1)/|h(1)|$, then it goes through $\omega\alpha, \omega^2\alpha, \dots, \omega^{n_0-1}\alpha$ in this order since $h(\omega^k) = \omega^k h(1)$ by (6.15.1) and (6.15.2). The part of Γ from α to $\omega\alpha$ is homotopic to $S(1)$ covered m times plus an arc of angular measure $2\pi/n_0$ covered in the positive direction. Since $h(\omega^k z) = \omega^k h(z)$, it follows by calculating the winding number of Γ around the origin that $-1 = 1 + n_0 m$, that is, $n_0 m = -2$, so that $n_0 = 1$ or $n_0 = 2$.

We have $h^2 \in H$, so $h^2 = M_c$ for some $c \in V$. There is $d \in V$ with $d^2 = \pm c$. Replacing h by $h \circ M_{1/d}$ if necessary, we may assume that $h^2 = \mathrm{Id}$ or $h^2 = M_{-1}$, the latter case occurring at most when $n_0 = 2$.

6.17. Suppose that $h^2 = \mathrm{Id}$, and for $r > 0$, set $D_0(r) = B(r) \cup h\bigl(B(r)\bigr)$, let $U(r)$ be the unbounded component of $\overline{\mathbf{C}} \setminus \overline{D_0(r)}$, and set $D(r) = \mathbf{C} \setminus \overline{U(r)}$. Then $D(r)$ is a Jordan domain and $\Gamma(r) = \partial D(r)$ is a Jordan curve. This is clear if $S(r)$ and $h\bigl(S(r)\bigr)$ have at most one point of intersection, and otherwise follows from a theorem of Kerékjártó ([11, Hilfssatz I, p. 87]; see also [19, Example, p. 168]) since $\Gamma(r)$ is the boundary of one of the components of

$$\overline{\mathbf{C}} \setminus \bigl\{S(1) \cup h\bigl(S(1)\bigr)\bigr\}.$$

Also h maps $U(r)$ and hence $\Gamma(r)$ and $D(r)$ onto itself. The map $h \mid \Gamma(r)$ is sense-reversing and has exactly two fixed points, say $a(r)$ and $b(r)$, that divide $\Gamma(r)$ into two arcs that are interchanged by h (this follows since h is sense-reversing and fixes 0 and ∞).

We may write the elements of V as $e^{t(1-it_0)}$ or $\pm e^{t(1-it_0/2)}$ depending on if $n_0 = 1$ or $n_0 = 2$. In each case, we may take

$$a(r) = rc(r)a(1) \qquad \text{and} \qquad b(r) = rc(r)b(1) \tag{6.17.1}$$

where $c(r) = \exp\bigl\{-(\log r)it_0\bigr\}$ or $c(r) = \exp\bigl\{-(\log r)it_0/2\bigr\}$ so that $|c(r)| = 1$. If $n_0 = 2$, we further have $b(r) = -a(r)$ for all r, since $h(-z) = -h(z)$ and so $D(r) = \{-z : z \in D(r)\}$. More precisely, if $c \in V$ then by (6.15.1) we have $h\bigl(B(|c|r)\bigr) = \bigl\{cz : z \in h\bigl(B(r)\bigr)\bigr\}$ and so $D(|c|r) = \bigl\{cz : z \in D(r)\bigr\}$, hence $\Gamma(|c|r) = \bigl\{cz : z \in \Gamma(r)\bigr\}$. This together with (6.15.1) gives (6.17.1).

We now define a homeomorphism f_3 of $\overline{\mathbf{C}}$ fixing 0 and ∞ as follows. Let F be a sense-preserving homeomorphism of $\Gamma(1)$ onto $S(1)$ with $F\bigl(a(1)\bigr) = 1$ and $F\bigl(b(1)\bigr) = -1$, for example, the boundary value map of a suitable conformal mapping of $D(1)$ onto $B(1)$. Set $\Gamma_1 = F^{-1}\bigl(S(1) \cap \{w : \operatorname{Im} w \ge 0\}\bigr)$ and $\Gamma_2 = \Gamma(1) \setminus \Gamma_1$. Suppose that $z \in \Gamma(r)$. If $n_0 = 1$ there is a unique $c \in V$ with $|c| = r$. If $n_0 = 2$, there are two points, say $\pm c$, that belong to V and have modulus r. Then we choose $c = \exp\bigl\{(\log r)(1 - it_0/2)\bigr\}$. Now $z/c \in \Gamma(1)$. If $z/c \in \Gamma_1$, we set $f_3(z) = |c|F(z/c)$. If $z/c \in \Gamma_2$, we set $f_3(z) = \overline{f_3\bigl(h(z)\bigr)}$. Thus $f_3 \mid \Gamma(r)$ is a homeomorphism of $\Gamma(r)$ onto $S(r)$. We have $f_3(z) = \overline{f_3\bigl(h(z)\bigr)}$ also when $z/c \in \Gamma_1$ since then $z = h(w)$ where $w/c \in \Gamma_2$ and since $h^2 = \mathrm{Id}$. Thus $(f_3 \circ h \circ f_3^{-1})(z) = \bar{z}$ for all z.

Clearly the curves $\Gamma(r)$ are disjoint and their union is $\mathbf{C} \setminus \{0\}$. Thus f_3 is one-to-one and onto, and continuous at 0 and at ∞. For $c_0, c \in V$ and $z/c \in \Gamma_1$, we have $f_3(z) = |c|F(z/c)$ and

$$f_3(c_0 z) = |cc_0| F\bigl\{(c_0 z)/(c_0 c)\bigr\} = |cc_0| F(z/c) = |c_0| f_3(z).$$

If $z/c \in \Gamma_2$ then $f_3(z) = \overline{f_3\bigl(h(z)\bigr)}$ while

$$f_3(c_0 z) = \overline{f_3\bigl(h(c_0 z)\bigr)} = \overline{f_3\bigl(c_0 h(z)\bigr)} = \overline{|c_0| f_3\bigl(h(z)\bigr)} = |c_0|\, \overline{f_3\bigl(h(z)\bigr)}.$$

Thus $f_3(c_0 z) = |c_0| f_3(z)$ for all z and so

$$f_3 \circ M_{c_0} \circ f_3^{-1} = M_{|c_0|}.$$

It follows that f_3 conjugates G to a Möbius group.

It follows easily from the definitions that f_3 is continuous on $\mathbf{C} \setminus \{0\}$ also and hence on $\overline{\mathbf{C}}$. Since $\overline{\mathbf{C}}$ is compact, f_3 is a homeomorphism. Thus G is topologically conjugate to a Möbius group.

6.18. If $h^2 = M_{-1}$ then $z = h^2(z) = -z$ for any fixed point z of h so that $\operatorname{fix}(h) = \{0, \infty\}$. But we may proceed as above and define

$$D_0(r) = \bigcup_{n=0}^{3} h^n\bigl(B(r)\bigr) = D_0'(r) \cup h^2\bigl(D_0'(r)\bigr)$$

where $D_0'(r) = B(r) \cup h\bigl(B(r)\bigr)$. Let the unbounded component of $\overline{\mathbf{C}} \setminus \overline{D_0'(r)}$ be $U_1(r)$ so that $U_1(r)$ is a Jordan domain as above, and set $D_1(r) = \mathbf{C} \setminus \overline{U_1(r)}$ and $D_2(r) = D_1(r) \cup h^2\bigl(D_1(r)\bigr)$. Then the unbounded components of $\overline{\mathbf{C}} \setminus \overline{D_0(r)}$ and $\overline{\mathbf{C}} \setminus \overline{D_2(r)}$ coincide. Thus, if that component is denoted by $U(r)$, then $U(r)$ is a Jordan domain and $\Gamma(r) = \partial U(r)$ is a Jordan curve contained in $\mathbf{C} \setminus \{0\}$. Since $h^4 = \mathrm{Id}$, the map h takes each of $D_0(r)$, $U(r)$ and $\Gamma(r)$ onto itself. Now $h \mid \Gamma(r)$ is sense-reversing and has two fixed points so that h has two fixed points in $\mathbf{C} \setminus \{0\}$, which is a contradiction. Hence the case $h^2 = M_{-1}$ cannot occur.

This proves Theorem 2 in Case II. Thus the proof of Theorem 2 is complete.

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