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CARLESON MEASURE, ATOMIC DECOMPOSITION AND FREE INTERPOLATION FROM BLOCH SPACE

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Abstract. Several characterizations, Carleson measures and atomic decomposition for the Bloch space B are given. For their applications, free interpolations from B are also discussed.

1. Introduction

Let $D = \{ z : |z| < 1 \}$ be the unit disk in the finite complex plane $\mathbf{C}$ and $dm_\alpha(z) = \left(1 - |z|^2\right)^\alpha dm(z)$ the two-dimensional Lebesgue measure with weight $\left(1 - |z|^2\right)^\alpha$, $\alpha > -1$. Denote by A and H^∞ the sets of functions analytic and boundedly analytic on D, respectively. For $f \in A$ we say $f \in B$ if

$$\|f\|_B = \left|f(0)\right| + \sup_{z\in D}\left(1 - |z|^2\right)\left|f'(z)\right| < \infty; \tag{1.1}$$

also $f \in A^1_\alpha$ if

$$\|f\|_{1,\alpha} = \int_D \left|f(z)\right| dm_\alpha(z) < \infty. \tag{1.2}$$

B and A^1_α are the so-called Bloch space and the Bergman space weighted by $\left(1 - |z|^2\right)^\alpha$, [6], [15].

It is well known that the dual space of A^1_0 is identified with B under the following inner product:

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{\pi}\lim_{t\to 1}\int_{tD} f(z)\overline{g(z)}\, dm(z)\\ &= \frac{1}{\pi}\int_D (\vee f)(z)\left(1 - |z|^2\right)\overline{g'(z)}\, dm(z) + f(0)\overline{g(0)}\end{aligned} \tag{1.3}$$

for $f \in A^1_0$ and $g \in B$, where $t \in (0,1)$, $tD = \{ z : |z| < t \}$ and $(\vee f)(z) = \left[f(z) - f(0)\right]/z$; see [3].

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In [11] we discussed the atomic decomposition and the free interpolation on the Bergman space A_0^1. Since $(A_0^1)^* = B$, it is very natural to consider similar problems on the Bloch space. As far as we know, these questions have not been thoroughly dealt with yet ([8], [13]), which is what we try to do in this paper. First, in Section 2, we give several characterizations of B as well as relations between B and Carleson measure. Next, in Section 3, we obtain an atomic decomposition of B by means of the pseudohyperbolic metric. Finally, in Section 4, we study the free interpolations by functions from B by means of the direct construction and the operator theory.

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2. Bloch space and Carleson measure

There are many works on the Bloch space, [1], e.g. [2], [7], [10]. Here we will give several interesting characterizations, some of which are new.

For z and w in D, let $\varphi_w(z) = (w-z)/(1-\overline{w}z)$, $\varrho(w,z) = |\varphi_w(z)|$ and $d(w,z) = \frac{1}{2}\log\{[1+\varrho(w,z)]/[1-\varrho(w,z)]\}$. Here $\varrho(\,\cdot\,,\,\cdot\,)$ and $d(\,\cdot\,,\,\cdot\,)$ are called the pseudohyperbolic and hyperbolic distances, respectively. Also, denote the measure of set $E\subset D$, relative to $dm_\alpha(z)$, by $m_\alpha(E) = \int_E dm_\alpha(z) = \int_E (1-|z|^2)^\alpha\, dm(z)$. Then we have the following result.

Theorem 2.1. *Let $f\in A$. Then the following statements are equivalent:*

(i) $f\in B$;

(ii) $\sup_{w,z\in D}|f(w)-f(z)|/d(w,z)<\infty$;

(iii) *there is a constant $C>0$ such that*

$$\sup_{w\in D}\int_D \exp\bigl[C\bigl|(f\circ\varphi_w)(z)-f(w)\bigr|\bigr]\, dm_\alpha(z)<\infty.$$

Proof. We will show this fact according to (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

Firstly, (i) $\Longrightarrow$ (ii). Let $f\in B$, and $g_w(\lambda) = (f\circ\varphi_w)(\lambda) - f(w)$, $\lambda, w\in D$. Then $g_w(0)=0$ and $\|g_w\|_B = \sup_{z\in D}(1-|z|^2)|f'(z)| \le \|f\|_B<\infty$. Further,

$$|g_w(\lambda)| = \left|\int_0^\lambda g_w'(\zeta)\,d\zeta\right| \le \tfrac{1}{2}\|f\|_B \log\frac{1+|\lambda|}{1-|\lambda|}.$$

Setting $z=\varphi_w(\lambda)$ we obtain

$$|f(z)-f(w)| \le \tfrac{1}{2}\|f\|_B\log\frac{1+\varrho(z,w)}{1-\varrho(z,w)} = \|f\|_B d(z,w),$$

i.e., (ii) holds.

Secondly, (ii) $\Longrightarrow$ (iii). Suppose that

$$0 < \|f\|'_B = \sup_{w,z\in D} \big|f(w) - f(z)\big|/d(w,z) < \infty;$$

then for $t \geq 0$,

$$\begin{aligned} &\{\, z : z \in D,\, |g_w(z)| > t \,\} \\ &\qquad \subset \Big\{\, z : z \in D,\ |z| > \Big[\exp\Big(\frac{2t}{\|f\|'_B}\Big) - 1\Big] \Big/ \Big[\exp\Big(\frac{2t}{\|f\|'_B}\Big) + 1\Big] \,\Big\}. \end{aligned}$$

Moreover, when $0 < C < \big[2(\alpha+1)/\|f\|'_B\big]$,

$$\begin{aligned} \int_D \exp\big[C|g_w(z)|\big]\, dm_\alpha(z) &= C \int_0^\infty (\exp Ct) \cdot m_\alpha\big(\{\, z : z \in D,\ |g_w(z)| > t \,\}\big)\, dt \\ &\leq C \int_0^\infty (\exp Ct) \frac{4\pi}{\alpha+1} \exp\Big(-\frac{2(\alpha+1)t}{\|f\|'_B}\Big)\, dt \\ &= \frac{4\pi C \cdot \|f\|'_B}{(\alpha+1)\big[2(\alpha+1) - C\|f\|'_B\big]}. \end{aligned}$$

Thirdly, (iii) $\Longrightarrow$ (i). Let

$$\|f\|''_B = \sup_{w\in D} \int_D \exp\big[C|(f\circ\varphi_w)(z) - f(w)|\big]\, dm_\alpha(z) < \infty$$

for some constant $C > 0$. Then

$$\big\|(f\circ\varphi_w)(z) - f(w)\big\|_{1,\alpha} \leq \frac{\|f\|''_B}{C} < \infty.$$

Since g_w has a Taylor series $\sum_n a_n z^n$ which converges uniformly on tD $(0 < t < 1)$, a simple calculation gives

$$a_1 = g'_w(0) = \frac{(\alpha+1)(\alpha+2)}{1 - \big[1 + (\alpha+1)t^2\big](1-t^2)^{\alpha+1}} \int_{tD} g_w(z)\overline{z}\, dm_\alpha(z).$$

By letting $t \to 1$ we get

$$\big|g'_w(0)\big| \leq (\alpha+1)(\alpha+2) \int_D \big|g_w(z)\big|\, dm_\alpha(z),$$

i.e.,

$$\big(1 - |w|^2\big)\big|f'(w)\big| \leq \Big[\frac{(\alpha+1)(\alpha+2)}{C}\Big] \cdot \|f\|''_B.$$

So, $f \in B$.

Remark. This theorem tells us that B is Lipschiz's class, relative to the hyperbolic metric $d(\cdot,\cdot)$. However, we know that B can be identified with the Zygmund class (see [1], [5]). Hence our result is much clearer than the one in [1].

In what follows we characterize connection between the Bloch space and Carleson measure.

For $w \in D$ let $D(w,r) = \{ z : z \in D,\ \varrho(w,z) < r \}$, $r \in (0,1)$. $D(w,r)$ is called the pseudohyperbolic disk. It is more convenient to use $D(w,r)$ (not Carleson square) for discussing Borel measure on the Bergman space A^1_α; see [6], [12]. Similarly, we have the following theorem.

Theorem 2.2. *Let $p \in (0,\infty)$ and $r \in (0,1)$, and let μ be a nonnegative Borel measure on D. Then the following statements are equivalent:*

(i) $\displaystyle \sup_{\substack{w\in D\\ 0\not\equiv f\in B}} \left[\frac{1}{\|f\|_B^p}\int_D \big|f(z)-f(w)\big|^p \frac{\big(1-|w|^2\big)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}}\,d\mu(z)\right]^{1/p} < \infty;$

(ii) $\displaystyle \sup_{w\in D}\left[\frac{\mu\big(D(w,r)\big)}{m_\alpha\big(D(w,r)\big)}\right] < \infty;$

(iii) $\displaystyle \sup_{w\in D}\left[\int_D \frac{\big(1-|w|^2\big)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}}\,d\mu(z)\right] < \infty.$

Proof. (ii) $\Leftrightarrow$ (iii) has been derived in [13], so we only need to claim (i)$\Leftrightarrow$(ii).

On the one hand, if (ii) is true, it follows by Theorem 2.1 that

$$\left[\int_D \big|(f\circ\varphi_w)(z)-f(w)\big|^p\,dm_\alpha(z)\right]^{1/p} \le C\|f\|_B$$

for $f \in B$, where $C > 0$ is a constant independent of f. Further, by [15], [12] and [6] it yields another constant C_0 depending on the condition (ii) such that

$$\begin{aligned}
&\left[\int_D \big|f(z)-f(w)\big|^p \frac{\big(1-|w|^2\big)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}}\,d\mu(z)\right]^{1/p}\\
&\quad\le C_0\left[\int_D \big|f(z)-f(w)\big|^p \frac{\big(1-|w|^2\big)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}}\,dm_\alpha(z)\right]^{1/p}\\
&\quad= C_0\left[\int_D \big|(f\circ\varphi_w)(z)-f(w)\big|^p\,dm_\alpha(z)\right]^{1/p} \le C_0C\|f\|_B.
\end{aligned}$$

On the other hand, let (i) hold. Taking $f_0(z) = \big[1/(1-\overline{w}_0 z)\big]-1$ for $w_0 = \big(-\frac12(r+1)+w\big)/(1-\frac12(r+1)\cdot\overline{w})$, $\frac12(r+1) \ne w \in D$, $r \in (0,1)$, we get

$\|f_0\|_B = |w_0|/\big(1-|w_0|^2\big)$, $\big|f_0(z)-f_0(w_0)\big| = |w_0||z-w_0|/|1-\overline{w}_0 z|\big(1-|w_0|^2\big)$ and

$$\begin{aligned}\sup_{z\in D(w,r)} |1-\overline{w}_0 z| &= \sup_{\lambda\in rD}\left|1-\overline{\Big(\frac{w-\frac{1}{2}(r+1)}{1-\overline{w}\frac{1}{2}(r+1)}\Big)}\cdot\Big(\frac{w-\lambda}{1-\overline{w}\lambda}\Big)\right| \\ &\le \frac{(2+r^2+r)\big(1-|w|^2\big)}{(1-r)^2}.\end{aligned}$$

Also, there are two constants $C_1 > 0$ and $C_2 > 0$ depending only on α and r such that (see [14])

$$C_1\cdot\big(1-|w|^2\big)^{2+\alpha} \le m_\alpha\big(D(w,r)\big) \le C_2\cdot\big(1-|w|^2\big)^{2+\alpha}.$$

We also have

$$\begin{aligned}\infty &> \sup_{\substack{\lambda\in D\\ 0\ne f\in B}}\left[\frac{1}{\|f\|_B^p}\cdot\int_D \big|f(z)-f(\lambda)\big|^p\cdot\frac{\big(1-|\lambda|^2\big)^{2+\alpha}}{|1-\overline{\lambda}z|^{4+2\alpha}}\,d\mu(z)\right]^{1/p} \\ &\ge \left[\frac{1}{\|f_0\|_B^p}\cdot\int_D \big|f_0(z)-f_0(w_0)\big|^p\cdot\frac{\big(1-|w_0|^2\big)^{2+\alpha}}{|1-\overline{w}_0 z|^{4+2\alpha}}\,d\mu(z)\right]^{1/p} \\ &\ge \left[\Big(\frac{1-|w_0|^2}{|w_0|}\Big)^p\cdot\int_{D(w,r)}\Big(\frac{|w_0|}{1-|w_0|^2}\Big)^p\big[\varrho(z,w_0)\big]^p\cdot\frac{(1-r)^{4(2+\alpha)}}{(2+r+r^2)^{4+2\alpha}}\right. \\ &\qquad\left.\cdot\frac{1}{\big(1-|w|^2\big)^{2+\alpha}}\,d\mu(z)\right]^{1/p} \\ &\ge \left[\frac{(1-r)^{4(2+\alpha)}}{4^{4+2\alpha}}\cdot\int_{D(w,r)}\big[\varrho(w_0,w)-\varrho(z,w)\big]^p\cdot\frac{1}{(1-|w|^2)^{2+\alpha}}\,d\mu(z)\right]^{1/p} \\ &\ge \frac{(1-r)^{(4\alpha+8+p)/p}}{2^{(4\alpha+8+p)/p}}C_2^{1/p}\cdot\left[\frac{\mu\big(D(w,r)\big)}{m_\alpha\big(D(w,r)\big)}\right]^{1/p}.\end{aligned}$$

Therefore

$$\sup_{w\in D}\left[\frac{\mu\big(D(w,r)\big)}{m_\alpha\big(D(w,r)\big)}\right] < \infty. \ \square$$

The measure μ satisfying one of the three statements in Theorem 2.2 is said to be α-Carleson measure. The following fact is interesting.

Theorem 2.3. *Let $f\in A$. Then the following statements are equivalent:*

(i) $f\in B$;

(ii) $\big|f'(z)\big|^2\big(\log 1/|z|\big)^2\,dm(z)$ *is* 0*-Carleson measure;*

(iii) $\big|f'(z)\big|^2\big(1-|z|^2\big)^2\,dm(z)$ *is* 0*-Carleson measure.*

Proof. We will give the whole claim in accordance with the order (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

First of all, (i) $\Longrightarrow$ (ii). Under $f \in B$, we consider the integral below:

$$I_1 = \int_D \frac{\left(1-|w|^2\right)^2}{|1-\overline{w}z|^4} \cdot \left|f'(z)\right|^2 \left(\log \frac{1}{|z|}\right)^2 dm(z) = \left(\int_{\{|z|>\frac{1}{4}\}} + \int_{\{|z|\le\frac{1}{4}\}}\right)\{\cdots\}\, dm(z).$$

Since $\log(1/|z|) \le C_1\left(1-|z|^2\right)$ when $|z| > \frac{1}{4}$,

$$\begin{aligned}\int_{\{|z|>\frac{1}{4}\}} \{\cdots\}\, dm(z) &\le C_1^2 \int_{\{|z|>\frac{1}{4}\}} \frac{\left|f'(z)\right|^2\left(1-|w|^2\right)^2\left(1-|z|^2\right)^2}{|1-\overline{w}z|^4}\, dm(z) \\ &\le C_1^2 \cdot \|f\|_B^2 \int_D \frac{\left(1-|w|^2\right)^2}{|1-\overline{w}z|^4}\, dm(z) \le \pi C_1^2 \|f\|_B^2,\end{aligned}$$

where $C_1 > 0$ is an absolute constant. At the same time

$$\begin{aligned}\int_{\{|z|\le\frac{1}{4}\}} \{\cdots\}\, dm(z) &\le \left(\frac{16}{15}\right)^2 \|f\|_B^2 \int_{\{|z|\le\frac{1}{4}\}} \frac{\left(1-|w|^2\right)^2}{|1-\overline{w}z|^4}\left(\log\frac{1}{|z|}\right)^2 dm(z) \\ &\le \left(\frac{16}{15}\right)^2 \cdot \frac{4^4}{3^4}\|f\|_B^2 \int_{\{|z|\le\frac{1}{4}\}} \left(\log\frac{1}{|z|}\right)^2 dm(z) = C_2\|f\|_B^2,\end{aligned}$$

where $C_2 > 0$ is an absolute constant. Consequently

$$I_1 \le (\pi C_1^2 + C_2)\|f\|_B^2.$$

So, from Theorem 2.2 (iii) we see that $\left|f'(z)\right|^2\left(\log(1/|z|)\right)^2 dm(z)$ is 0-Carleson measure.

Next (ii) $\Longrightarrow$ (iii). This is obvious, since $\left(1-|z|^2\right)^2 \le 4\left(\log(1/|z|)\right)^2$ for all $z \in D$.

Finally (iii) $\Longrightarrow$ (i). Assuming that $\left|f'(z)\right|^2\left(1-|z|^2\right)^2 dm(z)$ is 0-Carleson measure, we have

$$I_2 = \sup_{w\in D} \int_D \left|f'(z)\right|^2 \cdot \frac{\left(1-|z|^2\right)^2\left(1-|w|^2\right)^2}{|1-\overline{w}z|^4}\, dm(z) < \infty,$$

and obviously $\infty > I_2 \ge \int_D \left|f'(z)\right|^2\left(1-|z|^2\right)^2 dm(z)$. Moreover,

$$\left(f'(w)\right)^2 = \frac{3}{\pi}\int_D \left(f'(\lambda)\right)^2 \cdot \frac{\left(1-|\lambda|^2\right)^2}{(1-w\overline{\lambda})^4}\, dm(\lambda).$$

Hence

$$\left(1-|w|^2\right)^2\left|f'(w)\right|^2 \le \frac{3}{\pi}\int_D \left|f'(\lambda)\right|^2 \cdot \frac{\left(1-|\lambda|^2\right)^2\left(1-|w|^2\right)^2}{|1-\overline{w}\lambda|^4}\, dm(\lambda) \le I_2 < \infty,$$

i.e., $f \in B$. □

Supposing $g_D(z,w) = \log|(1-\overline{w}z)/(w-z)|$ (the Green's function on D), we just have

Corollary 2.4. *Let $f \in A$. Then $f \in B$ if and only if*

$$\sup_{w\in D}\int_D \left|f'(z)\right|^2 g_D^2(z,w)\,dm(z) < \infty. \tag{2.1}$$

Proof. This fact is readily derived from the equivalence between $\|f\circ\varphi_w\|_B$ and $\|f\|_B$, and Theorem 2.3 (ii). Nevertheless, the result can also be shown by Theorem 2.1 and 2.3. □

3. Atomic decomposition

To begin with, we let 1^1 and 1^∞ stand for the usual sequence spaces as follows:

$$1^1 = \Big\{ \{c_n\} : \{c_n\} \subset \mathbf{C},\ \|\{c_n\}\|_1 = \sum_n |c_n| < \infty \Big\}, \tag{3.1}$$

$$1^\infty = \{ \{c_n\} : \{c_n\} \subset \mathbf{C},\ \|\{c_n\}\|_\infty = \sup_n |c_n| < \infty \}. \tag{3.2}$$

Both are Banach spaces. Also, suppose that $\{z_n\}$ is a sequence of points on D. A sequence of points $\{z_n\}$ is called δ-weakly separated if $\delta = \inf_{m\neq n}\varrho(z_m,z_n) > 0$ and η-uniformly separated if $\eta = \inf_n \prod_{m\neq n}\varrho(z_m,z_n) > 0$. Clearly an η-uniformly separated sequence must be δ-weakly separated. A sequence of points $\{z_n\}$ is said to be ε-dense if $D = \cup_n D(z_n,\varepsilon)$, where $D(z_n,\varepsilon) = \{ z : z \in D,\ \varrho(z_n,z) < \varepsilon\}$ and $\varepsilon \in (0,1)$.

Luecking [6] and Xiao [12] proved the quasi-atomic decomposition theorem of A^1_α as follows.

Lemma 3.1. *Let $\{z_n\}$ be a sequence of points on D, $\alpha > -1$ and $f \in A^1_\alpha$. If $\{z_n\}$ is δ-weakly separated, there is a constant $C_1 > 0$ depending only on δ and α so that*

$$\|f\|_{1,\alpha} \geq C_1 \cdot \sum_n \big(1-|z_n|^2\big)^{2+\alpha}\big|f(z_n)\big|. \tag{3.3}$$

Furthermore, there are an $\varepsilon_0 > 0$ and a constant $C_2 > 0$ depending only on δ and α so that

$$\|f\|_{1,\alpha} \leq C_2 \sum_n \big(1-|z_n|^2\big)^{2+\alpha}\big|f(z_n)\big| \tag{3.4}$$

if $\{z_n\}$ is also ε-dense with $0 < \varepsilon \leq \varepsilon_0$.

After the above lemma, we can state an atomic decomposition theorem on the Bloch space.

Theorem 3.2. *Let* $\{z_n\}$ *be a sequence of points on* D. *If* $\{z_n\}$ *is* δ*-weakly separated, the function of the form* (3.5) *is in* B *for any* $\{c_n\} \in 1^\infty$

$$f(z) = \sum_n c_n \cdot \left(\frac{1-|z_n|^2}{1-\overline{z}_n z}\right)^2. \tag{3.5}$$

Moreover, there is an $\varepsilon_0 > 0$ *such that every* $f \in B$ *has the form* (3.5) *for some* $\{c_n\} \in 1^\infty$ *if* $\{z_n\}$ *is also* ε*-dense with* $0 < \varepsilon \le \varepsilon_0$.

Proof. Let $\{z_n\}$ be δ-weakly separated. Then $\mathbf{T}$, defined as follows, is a bounded linear operator from A_0^1 to 1^1,

$$\mathbf{T}f = \{(\mathbf{T}f)_n\} = \{\left(1-|z_n|^2\right)^2 f(z_n)\}, \tag{3.6}$$

in that (3.3) holds under $\{z_n\}$ being δ-weakly separated. Thus $\mathbf{T}^*$, the adjoint operator of $\mathbf{T}$ given by (3.7), is a bounded linear operator from 1^∞ $(= (1^1)^*)$ to B $(= A_0^1)^*)$,

$$\langle \mathbf{T}f, y\rangle = \langle f, \mathbf{T}^* y\rangle, \qquad f \in A_0^1, \quad y \in 1^\infty, \tag{3.7}$$

where the left $\langle \cdot,\cdot\rangle$ is just the usual inner product between 1^1 and 1^∞.

To compute $\mathbf{T}^*$, we take

$$y = e_n, \qquad (e_n)_m = \begin{cases} 1, & m = n \\ 0, & m \ne n, \end{cases}$$

so

$$\langle \mathbf{T}f, e_n\rangle = (\mathbf{T}f)_n = \left(1-|z_n|^2\right)^2 f(z_n) = \left(1-|z_n|^2\right)^2 \langle f, K_{z_n}\rangle,$$

where $K_{z_n}(z) = 1/(1-\overline{z}_n z)^2$ is the reproducing kernel for A_0^1. Hence

$$\mathbf{T}^* e_n = \left(1-|z_n|^2\right)^2 K_{z_n}(z)$$

and

$$\mathbf{T}^* y = \sum_n c_n \cdot \frac{\left(1-|z_n|^2\right)^2}{(1-\overline{z}_n z)^2} \qquad \text{for } y = \{c_n\} \in 1^\infty,$$

i.e., the function in the form (3.5) is in B. Indeed, it is easy to derive $\mathbf{T}^* y \in B$ by means of the direct computation.

Now we turn to showing the second part of Theorem 3.2. In fact, it is only necessary to claim $\mathbf{T}^*$ to be surjective. However, $\mathbf{T}^*$ is onto if and only if T is bounded below. By Lemma 3.1, there exists an $\varepsilon_0 > 0$ such that $\mathbf{T}$ is bounded below if $\{z_n\}$ is ε-dense with $0 < \varepsilon \le \varepsilon_0$. That is to say, there is an $\varepsilon_0 > 0$ such that every $f \in B$ has the form (3.5) for some $\{c_n\} \in 1^\infty$ as $\{z_n\}$ is ε-dense with $0 < \varepsilon \le \varepsilon_0$. Therefore the proof is completed. ◻

4. Free interpolation

As is well-known, a given sequence of points $\{z_n\}$ on D is called an H^∞-interpolating sequence if for any $\{c_n\} \in 1^\infty$ there exists $f \in H^\infty$ satisfying $f(z_n) = c_n$ for all n. Carleson stated in [4] that $\{z_n\}$ is an H^∞-interpolating sequence if and only if $\{z_n\}$ is η-uniformly separated. Here we want to extend this fact to the Bloch space. Yet, it is unfortunate that the η-uniformly separated property is only a sufficient condition for B. A sequence of points $\{z_n\}$ is said to be a B-interpolating sequence if there is $f \in B$ such that $f(z_n) = c_n$ for all n and any $\{c_n\} \in 1^\infty$.

Theorem 4.1. *Let $\{z_n\}$ be a sequence of points on D. If $\{z_n\}$ is a B-interpolating sequence, $\{z_n\}$ is δ-weakly separated. Conversely, if $\{z_n\}$ is δ-weakly separated and* (4.1) *or* (4.2) *is true, then $\{z_n\}$ is a B-interpolating sequence where*

$$\sup_n \sum_{m \neq n} \frac{\big(1-|z_m|^2\big)\big(1-|z_n|^2\big)}{|1-\overline{z}_n z_m|^2} < \infty, \tag{4.1}$$

$$\sup_n \sum_{m \neq n} \frac{\big(1-|z_m|^2\big)^2}{|1-\overline{z}_n z_m|^2} < 1. \tag{4.2}$$

Proof. Firstly, if $\{z_n\}$ is a B-interpolating sequence, then $1^\infty \subset \mathbf{T}_\infty B$, where $\mathbf{T}_\infty f = \big\{f(z_n)\big\}$. Since B is a Banach space, relative to $\|\cdot\|_B$, it follows from the open mapping theorem that there is a uniform constant $C_1 > 0$ and $f \in B$ so that $\|f\|_B \le C_1$ with $f(z_n) = w_n$ for all n and $\big\|\{w_n\}\big\|_\infty \le 1$. Picking $w_m = 0$, $m \neq n$; $w_m = 1$, $m = n$, there exist $f_n \in B$, $\|f_n\|_B \le C_1$ satisfying $f_n(z_n) = 1$; $f_n(z_m) = 0$, $m \neq n$. Theorem 2.1 yields

$$\frac{\big|f_n(z_n) - f_n(z_m)\big|}{d(z_n, z_m)} \le C_1, \qquad m \neq n,$$

and so $\inf_{m\neq n} d(z_n, z_m) \ge 1/C_1 > 0$, i.e.,

$$\delta = \inf_{m \neq n} \varrho(z_m, z_n) \ge (e^{2/C_1} - 1)/(e^{2/C_1} + 1) > 0.$$

Conversely, let $\{z_n\}$ be δ-weakly separated. If (4.1) is true, $\{z_n\}$ is η-uniformly separated and hence $1^\infty = \mathbf{T}_\infty H^\infty \subset \mathbf{T}_\infty B$ since H^∞ is a proper subspace of B. Furthermore, if (4.2) holds, we consider the linear operator $\mathbf{T}^*$, given by $\mathbf{T}^*\big(\{c_n\}\big) = \sum_n c_n \cdot \big((1-|z_n|^2)/(1-\overline{z}_n z)\big)^2$, $\{c_n\} \in 1^\infty$. Clearly, $\mathbf{T}^*$ is bounded from 1^∞ to B (by Theorem 3.2), while

$$\begin{aligned}\big\|(\mathbf{T}_\infty \mathbf{T}^* - \mathbf{I})\{c_n\}\big\|_\infty &= \sup_n \Big| \sum_{m \neq n} c_m \cdot \Big(\frac{1-|z_m|^2}{1-\overline{z}_m z_n}\Big)^2 \Big| \\ &\le \big\|\{c_n\}\big\|_\infty \cdot \sup_n \sum_{m\neq n} \Big(\frac{1-|z_m|^2}{|1-\overline{z}_n z_m|}\Big)^2.\end{aligned}$$

So, $\|(\mathbf{T}_\infty\mathbf{T}^* - \mathbf{I})\| < 1$, where $\mathbf{I}$ is the identify operator, i.e., $\mathbf{T}_\infty\mathbf{T}^*$ has an inverse, denoted by $(\mathbf{T}_\infty\mathbf{T}^*)^{-1}$. Further, $\mathbf{T}_\infty$ has a right inverse $\mathbf{T}^*(\mathbf{T}_\infty\mathbf{T}^*)^{-1}$, that is to say, $\mathbf{T}_\infty\big(\mathbf{T}^*(\mathbf{T}_\infty\mathbf{T}^*)^{-1}\big) = \mathbf{I}$, and thus $1^\infty \subset \mathbf{T}_\infty B$. So, $\{z_n\}$ is a B-interpolating sequence. □

Note that $\mathbf{T}_\infty H^\infty \subsetneq \mathbf{T}_\infty B$. In general, it is necessary to take into consideration the generic free interpolation problem from B. That is, for which $\{w_n\} \subset \mathbf{C}$ there is $f \in B$ satisfying $\{f(z_n)\} = \mathbf{T}_\infty f = \{w_n\}$. For this we obtain the following fact.

Theorem 4.2. *Let $\{z_n\}$ be a δ-weakly separated sequence of points on D. If $\{f(z_n)\} = \mathbf{T}_\infty f = \{w_n\}$ is solvable in B for $\{w_n\} \subset \mathbf{C}$, the following assertions* (i) *and* (ii) *hold:*

(i) *there are a constant $C_1 > 0$ and a function $\beta(z)$ such that*

$$\sup_{z\in D} \sum_n \big[1 - \varrho^2(z, z_n)\big]^\gamma \exp\big[C_1|w_n - \beta(z)|\big] < \infty \tag{4.3}$$

for $\gamma > 1$;

(ii) *there is a constant $C_2 > 0$ such that*

$$\sup_{z\in D} \sum_n \big[1 - \varrho^2(z, z_n)\big]^\gamma \exp\big[C_1|w_n - h(z)|\big] < \infty \tag{4.4}$$

for $\gamma > 1$, where $h(z) = \big\{\sum_n w_n\big[1 - \varrho^2(z, z_n)\big]^\gamma\big\} / \sum_n \big[1 - \varrho^2(z, z_n)\big]^\gamma$.

Conversely, if (i) *or* (ii) *holds for $\gamma = 1$, then $\{f(z_n)\} = \mathbf{T}_\infty f = \{w_n\}$ is solvable in B.*

Proof. First we consider the case (i). If $\{f(z_n)\} = \mathbf{T}_\infty f = \{w_n\}$ is solvable in B, then Theorem 2.1 yields

$$\sup_{z\in D} \int_D \exp\big[C_1|g_z(w)|\big]\, dm_\alpha(w) < \infty$$

for $C_1 < 2(\alpha + 1)/\|f\|_B$ ($\|f\|_B > 0$ is naturally assumed), where $g_z(w) = (f \circ \varphi_z)(w) - f(z)$. The above statement means that $\exp(C_1 g_z)$ is in A^1_α. Consequently, by Lemma 3.1,

$$\sup_{z\in D} \sum_n \exp\big[C_1|g_z(\tilde z_n)|\big]\big(1 - |\tilde z_n|^2\big)^{\alpha+2} \le C \sup_{z\in D} \|\exp C_1 g_z\|_{1,\alpha} < \infty, \tag{4.5}$$

where $\{\tilde z_n\} = \{\varphi_z(z_n)\}$, $C > 0$ is a constant independent of g_z, and $\{\tilde z_n\}$ is also a δ-weakly separated sequence of points on D since $\{z_n\}$ is such a sequence. Thus (4.5) means that (i) holds for $\gamma = \alpha + 2 > 1$ and $\beta(z) = f(z)$.

Now let us consider (ii).

Because $\{z_n\}$ is δ-weakly separated, we get $\sum_n \big[1-\varrho^2(z,z_n)\big]^{\alpha+2} < \infty$ by Lemma 3.1. By (4.5) we further have

$$\sum_{\{n:|w_n-f(z)|>t\}} \big[1-\varrho^2(z,z_n)\big]^{\alpha+2} \le C_2 \exp(-C_1 t)$$

for $t \ge 0$, where C_1 and C_2 are constants with $C_1 < 2(\alpha+1)/\|f\|_B$, $\|f\|_B > 0$, and f is the interpolating function for $\mathbf{T}_\infty f = \{w_n\}$ in B.

Thus, for $\gamma = \alpha + 2 > 1$

$$\begin{aligned}
\big|h(z)-f(z)\big| &\le \frac{1}{\sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma} \sum_n \big|w_n - f(z)\big|\big[1-\varrho^2(z,z_n)\big]^\gamma \\
&= \frac{1}{\sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma} \int_0^\infty \Bigg\{ \sum_{\{n:|w_n-f(z)|>t\}} \big[1-\varrho^2(z,z_n)\big]^\gamma \Bigg\}\, dt \\
&\le \frac{1}{\sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma} \int_0^\infty \min\Bigg\{ \sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma, C_2 \exp(-C_1 t) \Bigg\}\, dt \\
&\le \frac{1}{C_1} \Bigg\{ 1 + \log \frac{C_2}{\sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma} \Bigg\}
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&\sup_{z\in D} \sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma \exp\Big[\frac{C_1}{2}\big|w_n - h(z)\big|\Big] \\
&\quad\le \sup_{z\in D} \sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma \exp\Big[\frac{C_1}{2}\big|w_n - f(z)\big|\Big] \exp\Big[\frac{C_1}{2}\big|f(z) - h(z)\big|\Big] \\
&\quad\le \sup_{z\in D} \Bigg\{ \Bigg\{ \sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma \exp\big[C_1|w_n - f(z)|\big] \Bigg\}^{1/2} \cdot \Bigg\{ \sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma \Bigg\}^{1/2} \\
&\qquad\quad \cdot \big(\exp\tfrac{1}{2}\big) \Bigg\{ \frac{C_2}{\sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma} \Bigg\}^{1/2} \Bigg\} \\
&\quad= \sqrt{C_2 e} \cdot \Bigg\{ \sup_{z\in D} \sum_n \big[1-\varrho^2(z,z_n)\big]^\gamma \exp\big[C_1|w_n - f(z)|\big] \Bigg\}^{1/2} < \infty.
\end{aligned}$$

That is to say, (ii) is true for $\gamma = \alpha + 2 > 1$.

Next we show the contrary assertion. If (i) or (ii) holds for $\gamma = 1$, then it follows from (4.4) that

$$\sup_{z\in D} \sum_n \big[1-\varrho^2(z,z_n)\big] < \infty.$$

This, together with $\{z_n\}$ being δ-weakly separated, shows that $\{z_n\}$ is η-uniformly separated. Also, it follows by [9] that there is $f \in \mathrm{BMOA}(\partial D)$ to make $\mathbf{T}_\infty f = \{f(z_n)\} = \{w_n\}$ for $\{w_n\} \subset \mathbf{C}$ which is satisfied with (4.3) or (4.4) for $\gamma = 1$. Since $\mathrm{BMOA}(\partial D) \subsetneq B$, there exists $f \in B$ such that $\mathbf{T}_\infty f = \{f(z_n)\} = \{w_n\}$ under the previous assumption. Thus the theorem is proved. □

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