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# WEIGHTED SOBOLEV SPACES AND CAPACITY

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**Abstract.** We discuss the role of capacity in the pointwise definition of functions in Sobolev spaces involving weights of Muckenhoupt's $A_p$-class. In particular, it is shown that Sobolev functions possess Lebesgue points quasieverywhere with respect to an appropriate capacity.

## Introduction

Let $\Omega$ be an open set in $\mathbf{R}^n$ and $1 < p < \infty$. In this paper we consider the theory of weighted Sobolev spaces $H^{1,p}$ with weight function in Muckenhoupt's $A_p$-class. Our main purpose is to provide a coherent exposition of the behavior of functions in weighted Sobolev spaces and this leads us to use a concept of capacity. The motivation arises from the theory of partial differential equations, see e.g. [F], [HKM]. Most of the results we present are probably not new but according to our knowledge they have not yet appeared in printed form.

We define the weighted Sobolev space $H^{1,p}(\Omega; w)$ to be the completion of $C^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|\varphi\|_{1,p,w} = \left(\int_\Omega |\varphi|\, w(x)\, dx\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|\, w(x)\, dx\right)^{1/p}.$$

This definition is useful when one studies degenerate elliptic partial differential equations [FKS], [HKM]. Another approach, for example followed by Kufner [K], is to define the weighted Sobolev space $W^{1,p}(\Omega; w)$ as the class of functions $u$ such that both $u$ and its distributional gradient $\nabla u$ belong to $L^p(\Omega; w)$. Since $w$ is an $A_p$-weight, $w^{1/(1-p)}$ is locally integrable, and hence $W^{1,p}(\Omega; w)$ is a Banach space under the norm $\|\cdot\|_{1,p,w}$. We show that these two definitions result in the same space (Theorem 2.5):

$$W^{1,p}(\Omega; w) = H^{1,p}(\Omega; w).$$

According to the definition the functions in $H^{1,p}(\Omega; w)$ are defined a.e. However, a more accurate description of the pointwise behavior of them is often needed. It is well known in the unweighted case that the a.e. equivalence can be refined by

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means of $(1,p)$-capacity: unweighted Sobolev functions possess Lebesgue points except on a set of $(1,p)$-capacity zero [MK], [Z]. That the corresponding refinement can also be made in the weighted theory seems to belong to the folklore, but we have not been able to find these results in the existing literature. In the present paper we prove, for instance, that each $u \in H^{1,p}(\Omega;w)$ has a representative that possesses Lebesgue points (with respect to Lebesgue measure or to the weighted measure) everywhere except possibly on a set of $(1,p,w)$-capacity zero; the Hausdorff dimension of a set of $(1,p,w)$-capacity zero depends on the weight $w$, the dimension $n$ and $p$. It is always strictly less than $n-1$.

The paper is organized as follows. In Section 1 we list the required prerequisites from the theory of weights. Basic properties of the weighted Sobolev spaces $H^{1,p}(\Omega;w)$ and capacities are discussed in Section 2. Section 3 is devoted to the refinement of the a.e. equivalence in Sobolev spaces—a special attention is paid to Lebesgue points.

## 1. Preliminaries

Throughout this paper we be assume that $\Omega$ is an open subset of $\mathbf{R}^n$, $n \ge 2$, and $1 < p < \infty$.

We start by recalling that $w$ is a *weight in Muckenhoupt's $A_p$-class*, or an *$A_p$-weight*, if $w$ is a nonnegative, locally (Lebesgue) integrable function in $\mathbf{R}^n$ (not identically zero) such that

$$\sup\biggl(\fint_B w(x)\,dx\biggr)\biggl(\fint_B w(x)^{1/(1-p)}\,dx\biggr)^{p-1} = c_{w,p} < \infty, \tag{1.1}$$

where the supremum is taken over all balls $B$ in $\mathbf{R}^n$. Here, and throughout, the barred integral sign means the integral average

$$\fint_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu,$$

where $\mu$ is a measure and $0 < \mu(E) < \infty$.

If $w$ is an $A_p$-weight, we write $w \in A_p$ and call the constant $c_{w,p}$ in (1.1) the *$A_p$-constant of $w$*. It follows from the Hölder inequality that $c_{w,p} \ge 1$ and that $w \in A_q$ whenever $q \ge p$. A more intriguing fact is that there exists an $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$, as well. For these and other properties of $A_p$-weights we refer to the monographs [GCRF], [T], and [HKM].

As an example, we have that the function $w(x) = |x|^\gamma$ is an $A_p$-weight if and only if $-n < \gamma < n\,(p-1)$. Moreover, positive superharmonic functions in $\mathbf{R}^n$, and more generally positive supersolutions to certain quasilinear elliptic equations, belong to $A_p$ for all $p > 1$ (see [HKM, Theorem 3.59]).

We shall identify the weight $w$ with the measure

$$w(E) = \int_E w(x)\,dx;$$

for instance, the integral of $f$ with respect to the measure $w$ is written as $\int f\,dw$.

It is well known that, for an $A_p$-weight $w$, the corresponding measure is *doubling*, i.e. $w(2B) \le c\,w(B)$ for all balls $B = B(x,r)$; here the constant $c$ depends only on $p$ and $c_{p,w}$ and $2B$ stands for the enlarged ball $B(x,2r)$. Moreover, $w$ and the Lebesgue measure are mutually absolutely continous.

A celebrated theorem of Muckenhoupt states that the $A_p$-weights can be characterized as those weights for which the Hardy–Littlewood maximal operator is a bounded operator from $L^p(\mathbf{R}^n;w)$ into itself. Here $L^p(E;w)$ is the Banach space of measurable functions $u$ on $E$ with

$$\|u\|_{L^p(E;w)} = \left(\int_E |u|^p\,dw\right)^{1/p} < \infty.$$

More precisely, for a locally integrable function $f$ let

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f|\,dy.$$

Then (see e.g. [GCRF]):

**1.2. Theorem** (Muckenhoupt). *Suppose that $w$ is a nonnegative locally integrable function in $\mathbf{R}^n$. If $w \in A_p$, then there is a constant $c > 0$ depending only on $c_{p,w}$ such that*

$$\|Mf\|_{L^p(\mathbf{R}^n;w)} \le c\,\|f\|_{L^p(\mathbf{R}^n;w)} \tag{1.3}$$

*whenever $f \in L^p(\mathbf{R}^n;w)$.*

*Conversely, if (1.3) holds for all $f \in L^p(\mathbf{R}^n;w)$ with $c$ independent of $f$, then $w \in A_p$.*

We apply Theorem 1.2 and show that certain convolution operators also are bounded from $L^p(\mathbf{R}^n;w)$ into itself. We first need a lemma (see [S, Theorem III.2.2]).

**1.4. Lemma.** *Suppose that $\eta \in C_0^\infty(\mathbf{R}^n)$ is nonnegative with $\int_{\mathbf{R}^n} \eta\,dx = 1$. Suppose, furthermore, that $\eta$ is radial and decreasing, i.e. $\eta(x) = \eta(y) \ge \eta(z)$ if $|x| = |y| \le |z|$. Let $f$ be a locally integrable function in $\mathbf{R}^n$. Then*

$$|\eta * f| \le Mf$$

*a.e. in* $\mathbf{R}^n$, *where*

$$\eta * f(x) = \int_{\mathbf{R}^n} \eta(x-y) f(y)\, dx.$$

For example, if

$$\eta(x) = \begin{cases} a \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where the constant $a$ is chosen such that $\int_{\mathbf{R}^n} \eta\, dx = 1$, then $\eta$ satisfies the hypotheses of Lemma 1.4. Also the standard mollifiers $\eta_j(x) = j^n \eta(jx)$, $j = 1, 2, \ldots$, have the properties of $\eta$ in Lemma 1.4.

**1.5. Lemma.** *If* $w \in A_p$ *and* $f \in L^p(\mathbf{R}^n; w)$, *then* $\eta_j * f \to f$ *in* $L^p(\mathbf{R}^n; w)$.

*Proof.* First note that the assertion is trivial if $f$ is continuous. Moreover, it follows from Lemma 1.4 and Theorem 1.2 that there is a constant $c$ depending only on the $A_p$-constant of $w$ such that

$$\|\eta_j * g\| \leq \|Mg\| \leq c\,\|g\| \tag{1.6}$$

for all $g \in L^p(\mathbf{R}^n; w)$; here $\|u\|$ denotes the $L^p(\mathbf{R}^n; w)$-norm of $u$. To complete the proof, fix $\varepsilon > 0$ and let $h$ be a continuous function with $\|h - f\| < \varepsilon$. Choosing $j$ such that $\|h - \eta_j * h\| < \varepsilon$ we have

$$\begin{aligned} \|f - \eta_j * f\| &\leq \|f - h\| + \|h - \eta_j * h\| + \|\eta_j * h - \eta_j * f\| \\ &< \varepsilon + \varepsilon + c\,\|h - f\| < (c+2)\varepsilon, \end{aligned}$$

where we used (1.6) with $g = h - f$. The lemma follows.

**1.7. Corollary.** *The set* $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega; w)$.

**1.8. Remark.** In the unweighted case Lemma 1.5 is usually established by the aid of the fact that

$$\lim_{y \to 0} \int_\Omega |f(x+y) - f(x)|^p\, dx = 0$$

whenever $f \in L^p(\Omega; dx)$. Unfortunately, in general the corresponding continuity assertion for $L^p$-functions is not true in the weighted $L^p$-classes. To display a particular example, let $0 < \gamma < n(p-1)$ and $w(x) = |x|^\gamma$. Then $w$ is in $A_p$ and the function

$$f(x) = \begin{cases} |x|^{-n/p} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

belongs to $L^p(\mathbf{R}^n;w)$. However, for all $y\neq 0$

$$\int_{\mathbf{R}^n} |f(x+y)|^p\, w(x)\, dx = \infty,$$

and hence

$$\int_{\mathbf{R}^n} |f(x+y)-f(x)|^p\, w(x)\, dx = \infty$$

whenever $y\neq 0$.

## 2. Weighted Sobolev spaces

In this section we define weighted Sobolev spaces, where the weight function belongs to Muckenhoupt's $A_p$-class. We refer to [HKM, Chapter 1], where a general theory of weighted Sobolev spaces of the first order is presented. Here we prove some results that do not hold for a general class of weights and therefore are not included in the discussion in [HKM]; for instance, we establish a weighted version of the celebrated $H=W$ theorem. For simplicity, we consider the first order spaces only. In the end of the section we define weighted capacities and record some of their properties. Throughout this paper the number $1<p<\infty$ is fixed and $w$ is an $A_p$-weight.

For $\varphi\in C^\infty(\Omega)$ we write

$$\|\varphi\|_{1,p,w} = \|\varphi\|_{1,p,w,\Omega} = \left(\int_\Omega |\varphi|\, dw\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|\, dw\right)^{1/p},$$

where $\nabla\varphi = (\partial_1\varphi,\partial_2\varphi,\dots,\partial_n\varphi)$ is the gradient of $\varphi$. We define the *Sobolev space* $H^{1,p}(\Omega;w)$ to be the completion of $\{\varphi\in C^\infty(\Omega) : \|\varphi\|_{1,p,w}<\infty\}$ with respect to the norm $\|\cdot\|_{1,p,w}$. That is, $u\in H^{1,p}(\Omega;w)$ if and only if $u\in L^p(\Omega;w)$ and there exists a sequence of functions $\varphi_j\in C^\infty(\Omega)$ such that $\varphi_j$ converges to $u$ in $L^p(\Omega;w)$ and that for all $k=1,2,\dots,n$ the sequence of derivatives $\partial_k\varphi_j$ converges to a function $v_k$ in $L^p(\Omega;w)$. The function $v=(v_1,v_2,\dots,v_n)$ is called the *gradient* of $u$, and we shall write $v=\nabla u$; see Proposition 2.1 below.

It is clear that, equipped with the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u|^p\, dw\right)^{1/p} + \left(\int_\Omega |v|^p\, dw\right)^{1/p},$$

where $v$ is the gradient of $u$, the Sobolev space $H^{1,p}(\Omega;w)$ is a reflexive Banach space.

If $w(x)\equiv 1$, the symbol $w$ may be dropped from the notation $H^{1,p}(\Omega;w)$. However, we write $H^{1,p}(\Omega;dx)$ for the usual Sobolev space.

It follows from the Hölder inequality that the weighted Sobolev space can be embedded into unweighted spaces and that the gradient $v$ of $u$ is the distributional gradient of $u$, and therefore a uniquely defined function; for completeness we include a short proof.

**2.1. Proposition.** *Let* $p_0 = \inf\{q > 1 : w \in A_q\}$ *and* $0 \le \delta < (p - p_0)/p_0$ *. If* $u \in H^{1,p}(\Omega; w)$ *, then* $u \in H^{1,1+\delta}(D; dx)$ *whenever* $D$ *is a bounded open subset of* $\Omega$ *.*

*Moreover, the gradient* $v$ *of* $u$ *in* $H^{1,p}(\Omega; w)$ *is the distributional gradient of* $u$ *, i.e.* $v$ *is locally integrable and*

$$\int_\Omega v\,\varphi\,dx = -\int_\Omega u\,\nabla\varphi\,dx$$

*for all* $\varphi \in C_0^\infty(\Omega)$ *.*

*Proof.* Let $q = p/(1+\delta)$. Then $p_0 < q < p$ and hence $w \in A_q$. In particular, $w^{1/(1-q)}$ is locally integrable in $\mathbf{R}^n$. Therefore, if $D \subset \Omega$ is bounded and if $f \in L^p(D; w)$, we have that

$$\begin{aligned}\Big(\int_D |f|^{1+\delta}\,dx\Big)^{1/(1+\delta)} &= \Big(\int_D |f|^{p/q} w(x)^{1/q} w(x)^{-1/q}\,dx\Big)^{q/p}\\ &\le \Big(\int_D |f|^p\,dw\Big)^{1/p}\Big(\int_D w^{1/(1-q)}\,dx\Big)^{(q-1)/p} \le c\Big(\int_D |f|^p\,dw\Big)^{1/p}.\end{aligned} \tag{2.2}$$

Consequently, if $u \in H^{1,p}(\Omega; w)$ if $\varphi_j \in C^\infty(\Omega)$ which converges to $u$ in $L^p(\Omega; w)$ such that $\nabla\varphi_j$ converges to a vector valued function $v$ in $L^p(\Omega; w)$, then both $u$ and $v$ belong to $L^{1+\delta}_{loc}(\Omega; dx)$. Moreover, for all $\psi \in C_0^\infty(\Omega)$ it holds that

$$\begin{aligned}\Big|\int_\Omega u\nabla\psi - (-)v\,\psi\,dx\Big| &= \Big|\int_\Omega (u - \varphi_j)\nabla\psi - (\nabla\varphi_j - v)\,\psi\,dx\Big|\\ &\le \max|\nabla\psi|\int_{\operatorname{spt}\psi}|u - \varphi_j|\,dx + \max|\psi|\int_{\operatorname{spt}\psi}|v - \nabla\varphi_j|\,dx,\end{aligned}$$

and the last two terms converge to 0 by (2.2). Moreover, it follows that $\varphi_j \to u$ in $H^{1,1+\delta}(\Omega; dx)$, and the theorem follows.

The usual Sobolev embedding theorem [GT, Theorem 7.26] and Proposition 2.1 imply:

**2.3. Corollary.** *Let* $p_0 = \inf\{q > 1 : w \in A_q\}$ *. If* $p_0 < p/n$ *, then each function* $u$ *in* $H^{1,p}(\Omega; w)$ *is continuous (after a redefinition in a set of measure zero). In fact,* $u$ *is locally Hölder continuous in* $\Omega$ *with any exponent* $\alpha$ *such that* $0 < \alpha < 1 - np_0/p$ *.*

**2.4. Remark.** It follows from the proof of Proposition 2.1 that if, for $1 < q \le p$, $w^{1/(1-q)}$ is Lebesgue integrable in $\Omega$, then

$$H^{1,p}(\Omega; w) \subset H^{1,p/q}(\Omega; dx).$$

Hence the number $p_0$ in Corollary 2.3 may be replaced by $\tilde{p}_0 = \min(p_0, q)$. Moreover, if $1/w$ is bounded, then

$$H^{1,p}(\Omega; w) \subset H^{1,p}(\Omega; dx).$$

Both embeddings are continuous.

A similar argument shows that if the weight $w$ belongs to $L^q(\Omega; dx)$ for some $1 < q \leq \infty$, then we have the continuous embedding

$$H^{1,q'p}(\Omega; dx) \subset H^{1,p}(\Omega; w),$$

where $q'$ is the conjugate exponent of $q$, i.e. $1/q + 1/q' = 1$. In particular, we have the trivial result that the weighted space $H^{1,p}(\Omega; w)$ coincides with the unweighted space $H^{1,p}(\Omega; dx)$ if both $w$ and $1/w$ are bounded in $\Omega$.

Another possible way to define the weighted Sobolev spaces is the following: let $W^{1,p}(\Omega; w)$ be the set of all functions $u \in L^p(\Omega; w)$ whose distributional gradient $\nabla u$ belongs to $L^p(\Omega; w)$. A well known theorem of Meyers and Serrin [MS] (see also [DL]) states that in the unweighted case these two definitions result in the same function spaces. We extend this result to the weighted situation.

**2.5. Theorem.** $H^{1,p}(\Omega; w) = W^{1,p}(\Omega; w)$.

*Proof.* Since $w^{1/(1-p)}$ is locally (Lebesgue) integrable in $\mathbf{R}^n$, the space $W^{1,p}(\Omega; w)$ is a Banach space (see the proof of Proposition 2.1). Therefore we have that

$$H^{1,p}(\Omega; w) \subset W^{1,p}(\Omega; w).$$

For the reverse inclusion, let $u \in W^{1,p}(\Omega; w)$. Fix an open set $D \Subset \Omega$. It suffices to show that $u \in H^{1,p}(D; w)$ (cf. [HKM, 1.15]). Thus by multiplying $u$ with a cut-off function we may assume that $u \in W^{1,p}(\mathbf{R}^n; w)$. Let $\eta_j \in C_0^\infty(\mathbf{R}^n)$ be mollifiers as in Lemma 1.5. Then the convolutions $u_j = \eta_j * u$ belong to $C^\infty(\mathbf{R}^n)$ and $\nabla u_j = \eta_j * \nabla u$ (see e.g. [Z, Lemma 2.1.3]). Hence it follows from Lemma 1.5 that $u_j \to u$ in $H^{1,p}(\mathbf{R}^n; w)$ so that $u \in H^{1,p}(D; w)$, as desired.

Next we prove a removability result that is stronger than what is known for general weighted Sobolev spaces [HKM, 2.44].

**2.6. Theorem.** *Suppose that $E$ is a relatively closed subset of $\Omega$. If $E$ is of $(n-1)$ measure zero, then*

$$H^{1,p}(\Omega \setminus E; w) = H^{1,p}(\Omega; w).$$

*Proof.* By Theorem 2.5 it suffices to show that the distributional gradient $\nabla u$ of a function $u \in H^{1,p}(\Omega \setminus E; w)$ satisfies

$$\int_\Omega u \nabla \varphi \, dx = -\int_\Omega \varphi \nabla u \, dx \tag{2.7}$$

for all $\varphi \in C_0^\infty(\Omega)$. To show that this is the case, we first observe that $u \in H^{1,1}(D\setminus E; dx)$ by Theorem 2.1, where $D \subset \Omega$ is a bounded open set that contains the support of $\varphi$. It is easy to see that $u$ belongs to $H^{1,1}(D; dx)$, because the functions in $H^{1,1}(\Omega; dx)$ can be characterized as functions in $L^1(\Omega; dx)$ that are absolutely continuous on almost all line segments in $\Omega$, parallel to the coordinate axes, and whose first partial derivatives are in $L^1(\Omega; dx)$ (see e.g. [Z, Theorem 2.1.4]). Since $u \in H^{1,1}(D; dx)$, (2.7) follows.

**2.8. Weighted capacities.** Next we list basic properties of a concept of capacity that is applicable in the theory of weighted Sobolev spaces. For a more thorough discussion the reader is referred to [HKM, Chapter 2].

We define the variational $(1,p,w)$-*capacity* of a set $E \subset \mathbf{R}^n$ to be the number

$$C_{p,w}(E) = \inf_{u \in \mathscr{A}(E)} \int_{\mathbf{R}^n} \big(|u|^p + |\nabla u|^p\big)\, dw, \tag{2.9}$$

where

$$\mathscr{A}(E) = \big\{ u \in H^{1,p}(\mathbf{R}^n; w) : u \ge 1 \text{ in a neighborhood of } E \big\}.$$

If $\mathscr{A}(E) = \emptyset$, we let $C_{p,w}(E) = \infty$. It is clear that the same number $C_{p,w}(E)$ is achieved if the infimum in (2.9) is taken over all $u \in \mathscr{A}(E)$, $0 \le u \le 1$. If $K$ is a compact set, the infimum may be taken over all smooth functions [HKM, 2.36].

The variational capacity is a monotone subadditive set function, i.e. it enjoys the following properties:

(i) $C_{p,w}(\emptyset) = 0$.

(ii) If $E_1 \subset E_2$, then $C_{p,w}(E_1) \le C_{p,w}(E_2)$.

(iii) If $E_i \subset \mathbf{R}^n$, $i = 1, 2, \ldots$, then

$$C_{p,w}\Big(\bigcup_i E_i\Big) \le \sum_i C_{p,w}(E_i).$$

(iv) If $E_1 \subset E_2 \subset \ldots \subset \mathbf{R}^n$, then

$$C_{p,w}\Big(\bigcup_i E_i\Big) = \lim_{i\to\infty} C_{p,w}(E_i).$$

(v) If $K_1 \supset K_2 \supset \cdots$ are compact sets, then

$$C_{p,w}\Big(\bigcap_i K_i\Big) = \lim_{i\to\infty} C_{p,w}(K_i).$$

For the proof of these statements see [HKM, Chapter 2]; see also the proofs of [Z, Lemma 2.6.3 and Theorem 2.6.7]. Observe that the assertion (v) above is not true in general for noncompact sets. To display an example, let $w = 1$ and

$p > n$, and let $E_i$ be a decreasing sequence of nonempty sets whose intersection is the empty set. It is easy to see that $C_{p,w}(E_i) \geq c$, where $c$ is a positive constant depending only on $n$ and $p$ (see (2.10) below).

We have the following capacity estimate (see [HKM, 2.18, 2.19, 2.40]). If $0 < r \leq 1$, then

$$\begin{aligned} c_1 C_{p,w}\big(B(x_0,r)\big) &\leq \bigg(\int_{A(x_0,r)} |x-x_0|^{p(1-n)/(p-1)} w(x)^{1/(1-p)}\,dx\bigg)^{1-p} \\ &\leq c_2 C_{p,w}\big(B(x_0,r)\big), \end{aligned} \tag{2.10}$$

where $A(x_0,r)$ is the annulus $B(x_0,2)\setminus B(x_0,r)$ and $c_1 = c_1(n,p,c_{p,w})$ and $c_2 = c_2(n,p)$ are positive constants.

**Example.** Let $w(x) = |x|^\gamma$, where $-n < \gamma < n(p-1)$. Then it follows from the estimate in Theorem 2.10 that

$$C_{p,w}(\{0\}) > 0 \qquad \text{if and only if } -n < \gamma < p-n$$

and for $x \in \mathbf{R}^n$, $x \neq 0$,

$$C_{p,w}(\{x\}) > 0 \qquad \text{if and only if } p > n.$$

In particular we have that for $p > n$ and $-n < \gamma < p-n$ the only set of $(1,p,w)$-capacity zero is the empty set.

Finally we record estimates for the Hausdorff dimension of sets of zero $(1,p,w)$-capacity; see [HKM, 2.32. and 2.33].

**2.11. Theorem.** *Let $q_0 = \inf\{q > 1 : w^{1/(1-q)} \in L^1_{\mathrm{loc}}(\mathbf{R}^n;dx)\}$ and let $E$ be a nonempty set with $C_{p,w}(E) = 0$. Then $\mathrm{C_{p/q}}(E) = 0$ whenever $q > q_0$. In particular, $p \leq q_0\, n$ and the Hausdorff dimension of $E$ does not exceed $n - p/q_0 < n-1$.*

**2.12. Corollary.** *Let $p_0 = \inf\{q > 1 : w \in A_q\}$. If $E \neq \emptyset$ and $C_{p,w}(E) = 0$, then $p \leq p_0\, n$, and the Hausdorff dimension of $E$ does not exceed $n - p/p_0$.*

Uniform Hausdorff measure estimates do not provide a very accurate description of the smallness of sets of capacity zero because the weighted capacity is not distributed in a uniform manner. For example, let again $w(x) = |x|^\gamma$, where $-n < \gamma < n\,(p-1)$. Then $C_{p,w}(\{0\}) = 0$ if and only if $\gamma \geq n-p$. Moreover, if the origin $0$ does not belong to the closure of $E$, then $C_{p,w}(E) = 0$ if and only if $C_{p,1}(E) = 0$, and sharp estimates for the Hausdorff dimension of $E$ can be derived from known results for unweighted capacities (cf. [HKM, 2.26, 2.27]). But the sets $E$ with $0 \in \overline{E}$ cause troubles. A way out of this difficulty is proposed in [N], where the author considers "weighted" Hausdorff measures and their connection with capacities. However, it may be very difficult to find estimates for "weighted" Hausdorff measures.

**2.13. Bessel potentials and capacity.** By Calderón's well known theorem [C] (see also [S]) the Sobolev space $H^{1,p}(\mathbf{R}^n; dx)$ is equivalent to the space of Bessel potentials $G_1 * f$, $f \in L^p(\mathbf{R}^n; dx)$. Recall that the Bessel kernel $G_1$ is the function whose Fourier transform is $\hat{G}_1(x) = (1+|x|^2)^{-1/2}$. This result can also be extended for weighted spaces (see [M, Theorem 3.3] or [N, Section 5]):

**2.14. Theorem.** *A function $u$ belongs to $H^{1,p}(\mathbf{R}^n; w)$ if and only if there is $f \in L^p(\mathbf{R}^n; w)$ such that $u = G_1 * f$. Moreover, there is a positive constant $c = c(n,p,c_{p,w})$ such that*

$$c^{-1}\|f\|_{L^p(\mathbf{R}^n;w)} \le \|u\|_{1,p,w} \le c\|f\|_{L^p(\mathbf{R}^n;w)}.$$

The $(1,p,w)$-*Bessel capacity* of a set $E \subset \mathbf{R}^n$ is the number

$$B_{p,w}(E) = \inf \int_{\mathbf{R}^n} f^p \, dw,$$

where the infimum is taken over all nonnegative functions $f \in L^p(\mathbf{R}^n; w)$ such that $G_1 * f \ge 1$ on $E$.

Because the Bessel potential $G_1 * f$ is lower semicontinuous if $f \ge 0$, the Bessel capacity and the variational capacity are equivalent: for every $E \subset \mathbf{R}^n$

$$c^{-1}\, C_{p,w}(E) \le B_{p,w}(E) \le c\, C_{p,w}(E), \tag{2.15}$$

where $c = c(n,p,c_{p,w}) > 0$.

## 3. Pointwise behavior of functions in Sobolev spaces

In this section we show that a function $u \in H^{1,p}(\Omega; w)$ can be redefined in a set of measure zero so that it is quasicontinuous, i.e. its restriction to the complement of a set of arbitrary small $(1,p,w)$-capacity is continuous. Moreover, we show that the quasicontinuous representative possesses quasieverywhere Lebesgue points with respect to either Lebesgue measure or the weighted measure $w$.

We start with an extension result, which we need in establishing a capacitary weak type inequality for a Sobolev function.

**3.1. Lemma.** *Suppose that $u \in H^{1,p}(B; w)$, where $B$ is an open ball in $\mathbf{R}^n$. Then there exists a function $v \in H^{1,p}(2B; w)$ such that $v = u - u_B$ in $B$, $\operatorname{spt} v \subset 2B$, and*

$$\int_{2B} |\nabla v|^p \, dw \le c \int_B |\nabla u|^p \, dw,$$

*where $c = c(n,p,c_{p,w}) > 0$ and $u_B = \fint_B u \, dw$ is the weighted average of $u$ in $B$.*

*Proof.* Let $B = B(x_0, r)$. By [Ch, Theorem D] there is $f \in H^{1,p}_{\mathrm{loc}}(\mathbf{R}^n; w)$ such that $f = u$ in $B$ and

$$\int_{2B} |\nabla f|^p \, dw \le c \int_B |\nabla u|^p \, dw,$$

where $c = c(n, p, c_{p,w}) > 0$. Then applying the Poincaré inequality [HKM, 15.30] twice we obtain

$$\begin{aligned}
\int_{2B} |f - u_B|^p \, dw &\le 2^p \int_{2B} |f - f_{2B}|^p \, dw + 2^p \int_{2B} |f_{2B} - f_B|^p \, dw \\
&\le c\, r^p \int_{2B} |\nabla f|^p \, dw + 2^p \, \frac{w(2B)}{w(B)} \int_B |f - f_{2B}|^p \, dw \\
&\le c\, r^p \int_{2B} |\nabla f|^p \, dw \le c\, r^p \int_B |\nabla u|^p \, dw.
\end{aligned} \tag{3.2}$$

Let $v = \eta(f - u_B)$, where $\eta \in C_0^\infty(2B)$ is a cut-off function such that $0 \le \eta \le 1$, $\eta = 1 \in B$, and $|\nabla \eta| \le 202/r$. Then $v$ is the desired function because

$$\begin{aligned}
\int_{2B} |\nabla v|^p \, dw &\le 2^p \int_{2B} \eta^p |\nabla f|^p \, dw + 2^p \int_{2B} |\nabla \eta|^p |f - u_B|^p \, dw \\
&\le c \int_B |\nabla u|^p \, dw + \frac{c}{r^p} \int_{2B} |f - u_B|^p \, dw \le c \int_B |\nabla u|^p \, dw,
\end{aligned}$$

where in the last step we employed (3.2).

We write

$$M_w f(x) = \sup_{r>0} \fint_{B(x,r)} |f| \, dw$$

for the weighted maximal function of $f$.

**3.3. Lemma.** *Let $u \in H^{1,p}(\mathbf{R}^n; w)$ such that $\operatorname{spt} u \subset B(0, R)$. Then there is a positive constant $c = c(n, p, c_{p,w}, R)$ such that for $t > 0$ it holds that*

$$C_{p,w}\bigl(\{x : M_w u(x) > t\}\bigr) \le \frac{c}{t^p} \|u\|^p_{1,p,w}.$$

*Proof.* Fix $t > 0$ and write $E_t = \{x : M_w u(x) > t\}$; note that $E_t$ is open. For each $x \in E_t$ choose $r_x \le 2R$ such that

$$\fint_{B(x,r_x)} u \, dw > t.$$

By the Besicovitch covering theorem [Z, Theorem 1.3.5] we may select $N = N(n)$ disjoint sequences $B(x_{i,j}, r_{x_{i,j}})$, $i = 1, 2, \ldots, N$, of the balls $B(x, r_x)$, $x \in E_t$, such that

$$E_t \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{\infty} B(x_{i,j}, r_{x_{i,j}}).$$

Write $B_{i,j} = B(x_{i,j}, r_{x_{i,j}})$ and

$$u_{i,j} = \fint_{B_{i,j}} u \, dw.$$

Let $v_{i,j} \in H^{1,p}(\mathbf{R}^n; w)$ such that $\operatorname{spt} v_{i,j} \subset 2B_{i,j}$, $v_{i,j} = |u - u_{i,j}|$ in $B_{i,j}$ and

$$\int_{\mathbf{R}^n} |\nabla v_{i,j}|^p \, dw \le c \int_{B_{i,j}} |\nabla u|^p \, dw,$$

where $c = c(n, p, c_{p,w}) > 0$ (Lemma 3.1). Fix $i = 1, 2, \ldots, N$ and let $v_i = \sup_j v_{i,j}$. Then $v_i \in H^{1,p}(\mathbf{R}^n; w)$ because the balls $B_{i,j}$ are disjoint and by the Poincaré inequality [HKM, 15.30] we have that

$$\begin{aligned} \|v_i\|_{1,p,w}^p &\le \sum_{j=1}^{\infty} \int_{B_{i,j}} |v_{i,j}|^p + |\nabla v_{i,j}|^p \, dw \\ &\le c\,(R^p + 1) \sum_{j=1}^{\infty} \int_{2B_{i,j}} |\nabla v_{i,j}|^p \, dw \\ &\le c \sum_{j=1}^{\infty} \int_{B_{i,j}} |\nabla u|^p \, dw \le c \int_{\mathbf{R}^n} |\nabla u|^p \, dw. \end{aligned}$$

Put $v = \sum_{i=1}^{N} v_i$. Then $v + u > t$ in $E_t$ and $v \in H^{1,p}(\mathbf{R}^n; w)$ with

$$\|v\|_{1,p,w} \le c \left( \int_{\mathbf{R}^n} |\nabla u|^p \, dw \right)^{1/p}$$

where $c = c(n, p, c_{p,w}, R) > 0$. In conclusion,

$$C_{p,w}(E_t) \le \left\| \frac{u+v}{t} \right\|_{1,p,w}^p \le \frac{c}{t^p} \|u\|_{1,p,w}^p.$$

**Remark.** The proof of Lemma 3.3 is due to W.P. Ziemer.

We say that a property holds $(1, p, w)$-*quasieverywhere on* $E$, if it holds on $E \setminus F$, where $C_{p,w}(F) = 0$. For short, we usually write q.e. or quasieverywhere instead of $(1, p, w)$-quasieverywhere.

**3.4. Theorem.** *If* $u \in H^{1,p}(\mathbf{R}^n; w)$, *then the limit*

$$\lim_{r\to 0} \fint_{B(x,r)} u\,dw$$

*exists* $(1,p,w)$*-quasieverywhere in* $\mathbf{R}^n$.

*Proof.* Because the capacity is subadditive we may by multiplying $u$ with a cut-off function assume that the support of $u$ is compactly contained in a ball $B(0,R)$. Write

$$\Phi(u,x) = \limsup_{r\to 0} \fint_{B(x,r)} u\,dw - \liminf_{r\to 0} \fint_{B(x,r)} u\,dw \geq 0.$$

We show that $\Phi(u,x) = 0$ q.e. To this end fix $\varepsilon > 0$ and $t > 0$. Let $\varphi \in C_0^\infty(B(0,R))$ be such that

$$\|u-\varphi\|_{1,p,w} \leq \frac{t^p}{2^p c}\,\varepsilon,$$

where $c$ is the constant in Lemma 3.3. Now

$$\Phi(u,x) = \Phi(u-\varphi,x) \leq 2\,M_w(u-\varphi)(x),$$

whence Lemma 3.3 implies

$$\begin{aligned} C_{p,w}\big(\{x : \Phi(u,x) > t\}\big) &\leq C_{p,w}\big(\{x : M_w(u-\varphi)(x) > \tfrac{1}{2}t\}\big) \\ &\leq \frac{2^p c}{t^p}\|u-\varphi\|_{1,p,w} < \varepsilon. \end{aligned}$$

Since $t > 0$ and $\varepsilon > 0$ were arbitrary, $\Phi(u,x) = 0$ $(1,p,w)$-quasieverywhere, and the theorem follows.

A sequence of real valued functions $\varphi_j$ *converges* $(1,p,w)$*-quasiuniformly* on a set $E$ if for each $\varepsilon > 0$ there is an open set $G$ with $C_{p,w}(G) < \varepsilon$ such that $\varphi_j$ converges uniformly on $E \setminus G$. The sequence $\varphi_j$ is said to *converge locally* $(1,p,w)$*-quasiuniformly on* $E$ if it converges $(1,p,w)$-quasiuniformly on each compact subset of $E$.

Clearly, a locally quasiuniformly convergent sequence converges q.e. and hence a.e.

Let $u \in H^{1,p}(\mathbf{R}^n; w)$. We say that $v$ is a $(1,p,w)$*-refined representative* of $u$ if there exists a sequence $\varphi_j \in C_0^\infty(\mathbf{R}^n)$ such that $\varphi_j$ converges to $u$ both $(1,p,w)$-quasiuniformly in $\mathbf{R}^n$ and in $H^{1,p}(\mathbf{R}^n; w)$. Moreover, a function $f$ is $(1,p,w)$*-quasicontinuous* in $E$ if for each $\varepsilon > 0$ there is an open set $G$ with $C_{p,w}(G) < \varepsilon$ such that the restriction $f|_{E\setminus G}$ is continuous.

It follows from [HKM, 4.6] that each function in $H^{1,p}(\mathbf{R}^n; w)$ has a $(1,p,w)$-refined representative and that representative is $(1,p,w)$-quasicontinuous. (In [HKM] the class of $(1,p,w)$-refined representatives is denoted by $Q^{1,p}$.)

**3.5. Theorem.** *Let $u$ be a $(1,p,w)$-refined representative in $H^{1,p}(\mathbf{R}^n;w)$. Then*

$$\lim_{r\to 0} ⨍_{B(x,r)} u\,dw = u(x)$$

*$(1,p,w)$-quasieverywhere in $\mathbf{R}^n$.*

*Proof.* Again there is no loss of generality in assuming that the support of $u$ is contained in an open ball $B$. For $t>0$ write

$$E_t = \Big\{x : \Big|\lim_{r\to 0} ⨍_{B(x,r)} u\,dw - u(x)\Big| > t\Big\};$$

note that the above limit exists $(1,p,w)$-q.e. by Theorem 3.4. Choose a sequence $\varphi_j \in C_0^\infty(B)$ such that $\varphi_j \to u$ both $(1,p,w)$-quasiuniformly and in $H^{1,p}(\mathbf{R}^n;w)$. Then

$$\begin{aligned}\limsup_{r\to 0}\Big| ⨍_{B(x,r)} u\,dw - u(x)\Big| &\le \limsup_{r\to 0} ⨍_{B(x,r)} |u-\varphi_j|\,dw + |\varphi_j(x)-u(x)| \\ &\le M_w\big(u-\varphi_j\big)(x) + |\varphi_j(x)-u(x)|,\end{aligned}$$

and hence

$$E_t \subset \big\{x : M_w\big(u-\varphi_j\big)(x) > t/2\big\} \cup \big\{x : |\varphi_j(x)-u(x)| > t/2\big\}.$$

Now

$$C_{p,w}\big(\big\{x : M_w\big(u-\varphi_j\big)(x) > t/2\big\}\big) \le \frac{c}{t^p}\|u-\varphi_j\|_{1,p,w} \to 0$$

by Lemma 3.3 and

$$C_{p,w}\big(\big\{x : |\varphi_j(x)-u(x)| > t/2\big\}\big) \to 0$$

because $\varphi_j \to u$ $(1,p,w)$-quasiuniformly. Therefore $C_{p,w}(E_t) = 0$, and consequently,

$$\begin{aligned}C_{p,w}\Big(\Big\{x : \Big|\lim_{r\to 0} ⨍_{B(x,r)} u\,dw - u(x)\Big| > 0\Big\}\Big) &= C_{p,w}\Big(\bigcup_{k=1}^{\infty} E_{1/k}\Big) \\ &\le \sum_{k=1}^{\infty} C_{p,w}(E_{1/k}) = 0,\end{aligned}$$

as desired.

**3.6. Corollary.** *Let $u$ be a $(1,p,w)$-refined representative in $H^{1,p}(\mathbf{R}^n;w)$. Then*

$$\lim_{r\to 0} \fint_{B(x,r)} |u-u(x)|\,dw = 0 \tag{3.7}$$

*$(1,p,w)$-quasieverywhere in $\mathbf{R}^n$.*

*Proof.* Because the $(1,p,w)$-capacity is subadditive, it suffices to verify (3.7) q.e. in a ball $B$. To this end, let $\eta \in C_0^\infty(\mathbf{R}^n)$ be a cut-off function such that $\eta = 1$ in $B$. If $q$ is a rational number, then $|\eta(u-q)|$ is clearly a $(1,p,w)$-refined representative in $H^{1,p}(\mathbf{R}^n;w)$. Hence by Theorem 3.5

$$\lim_{r\to 0} \fint_{B(x,r)} |\eta(u-q)|\,dw = \big|\eta(x)\big(u(x)-q\big)\big| \tag{3.8}$$

for all $x \in \mathbf{R}^n \setminus E_q$, where $C_{p,w}(E_q) = 0$. If $E = \bigcup_{q\in\mathbf{Q}} E_q$, then $C_{p,w}(E) = 0$. Moreover, for $x \in \mathbf{R}^n \setminus E$ (3.8) holds whenever $q$ is rational, and hence by continuity it holds for every real $q$. In particular,

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y)-u(x)|\,dw(y) = 0$$

for all $x \in B \setminus E$, as desired.

We are going to show that each $(1,p,w)$-quasicontinuous representative of a function $u \in H^{1,p}(\Omega;w)$ is also a $(1,p,w)$-refined representative. This indeed follows from [HKM, 4.12] but we give another proof that avoids the use of the theory of variational inequalities. The proof for the following lemma is much simpler than that of [HKM, 4.9].

**3.9. Lemma.** *Let $G_j$ be a sequence of open sets with $C_{p,w}(G_j) \to 0$. Then for $(1,p,w)$-quasievery $x$ there is an index $j = j(x)$ such that*

$$w\big(G_j \cap B(x,r)\big) \le \tfrac{1}{2} w\big(B(x,r)\big)$$

*for $r > 0$ small enough.*

*Proof.* For each $j$ choose a $(1,p,w)$-refined function $u_j \in H^{1,p}(\mathbf{R}^n;w)$ such that $u_j = 1$ a.e. in $G_j$ and

$$\int_{\mathbf{R}^n} (|u_j|^p + |\nabla u_j|^p)\,dw \le C_{p,w}(G_j) + \frac{1}{j}.$$

Since $u_j \to 0$ in $H^{1,p}(\mathbf{R}^n;w)$, we may choose a subsequence, denoted again by $u_j$, that converges $(1,p,w)$-quasiuniformly to $0$ (see [HKM, 4.8]). In particular,

$u_j \to 0$ $(1,p,w)$-quasieverywhere. Therefore, for q.e. $x$ there is an index $j$ such that

$$\lim_{r\to 0} \fint_{B(x,r)} u_j \, dw = u_j(x) < \tfrac{1}{2}$$

by Theorem 3.5. Consequently,

$$\limsup_{r\to 0} \frac{w\big(G_j \cap B(x,r)\big)}{w\big(B(x,r)\big)} = \limsup_{r\to 0} \frac{1}{w\big(B(x,r)\big)} \int_{G_j \cap B(x,r)} u_j \, dw < \tfrac{1}{2},$$

as desired.

The next theorem follows from Lemma 3.9 in the same way as [HKM, 4.12]; for completeness we include a proof.

**3.10. Theorem.** *Suppose that $u$ and $v$ are $(1,p,w)$-quasicontinuous in an open set $\Omega$. If $u = v$ a.e., then $u = v$ $(1,p,w)$-quasieverywhere in $\Omega$.*

*Proof.* There is no loss of generality in assuming that $v = 0$ and $\Omega = \mathbf{R}^n$. Choose open sets $G_j$ with $C_{p,w}(G_j) \to 0$ such that the restriction $u|_{\complement G_j}$ is continuous. For $(1,p,w)$-quasievery $x \in \Omega$ we may choose an index $j$ such that

$$w\big(G_j \cap B(x,r)\big) \le \tfrac{1}{2} w\big(B(x,r)\big)$$

for $r > 0$ small enough (Lemma 3.9). Such a point $x$ does not belong to $G_j$, and since $u = 0$ a.e., we conclude that for each $r > 0$ there is $y \in B(x,r) \cap \complement G_j$ such that $u(y) = 0$. But because $u|_{\complement G_j}$ is continuous, it follows that $u(x) = 0 = v(x)$, as required.

We collect the above results in the following theorem.

**3.11. Theorem.** *Suppose that $u \in H^{1,p}(\mathbf{R}^n; w)$. Then the following are equivalent:*

(i) *$u$ is $(1,p,w)$-refined.*

(ii) *For $(1,p,w)$-quasievery $x$*

$$\lim_{r\to 0} \fint_{B(x,r)} u \, dw = u(x).$$

(iii) *$u$ is $(1,p,w)$-quasicontinuous.*

*Moreover, for each $v \in H^{1,p}(\mathbf{R}^n; w)$ there exists a function $u$ such that $u = v$ a.e. and that $u$ satisfies* (i)–(iii) *above.*

Also we have the following local version.

**3.12. Theorem.** *Suppose that $u \in H^{1,p}(\Omega; w)$. Then the following are equivalent:*

(i) *There is a sequence $\varphi_j \in C^\infty(\Omega) \cap H^{1,p}(\Omega; w)$ such that $\varphi_j \to u$ both in $H^{1,p}(\Omega; w)$ and locally $(1, p, w)$-quasiuniformly in $\Omega$.*

(ii) *For $(1, p, w)$-quasievery $x \in \Omega$*

$$\lim_{r\to 0} \fint_{B(x,r)} u\, dw = u(x).$$

(iii) *$u$ is $(1, p, w)$-quasicontinuous in $\Omega$.*

*Moreover, for each $v \in H^{1,p}(\Omega; w)$ there exists a function $u$ such that $u = v$ a.e. and that $u$ satisfies* (i)–(iii) *above.*

One may also ask if functions in weighted Sobolev spaces possess Lebesgue points with respect to the Lebesgue measure. Of course the embedding of $H^{1,p}(\Omega; w)$ into unweighted Sobolev space (Proposition 2.1) and known results in the unweighted case (see e.g. Ziemer's book [Z]) result in the following: the limit

$$\lim_{r\to 0} \fint_{B(x,r)} u(y)\, dy$$

exists $(1, q, 1)$-quasieverywhere whenever $1 \le q < p/p_0$, where $p_0 = \inf\{s > 1 : w \in A_s\}$. More accurate results in this direction are obtained by using the capacitary weak type inequality of the following lemma. Recall that

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f|\, dx$$

is the Hardy–Littlewood maximal function of $f$.

**3.13. Lemma.** *Let $u \in H^{1,p}(\mathbf{R}^n; w)$. Then there is a positive constant $c = c(n, p, c_{p,w})$ such that*

$$B_{p,w}\big(\{x : Mu(x) \ge t\}\big) \le \frac{c}{t^p} \|u\|_{1,p,w}^p$$

*for $t > 0$.*

*Proof.* For fixed $r > 0$ let

$$g = |B(0, r)|^{-1} \chi_{B(0,r)},$$

where $\chi_{B(0,r)}$ is the characteristic function of the ball $B(0, r)$ (i.e. $\chi_{B(0,r)}$ is 1 in $B(0, r)$ and 0 on $\complement B(0, r)$). Choose $f \in L^p(\mathbf{R}^n; w)$ such that $u = G_1 * f$ a.e. and

$$\|f\|_{L^p(\mathbf{R}^n;w)} \approx \|u\|_{1,p,w}$$

(Theorem 2.14). Then

$$\begin{aligned}\fint_{B(x,r)} u(y)\,dy &= \frac{1}{B(0,r)}\int_{\mathbf{R}^n}\chi_{B(0,r)}(x-y)u(y)\,dy\\ &= \big(g*|u|\big)(x) \le g*\big(G_1*|f|\big)(x)\\ &= G_1*\big(g*|f|\big)(x) \le \big(G_1*Mf\big)(x),\end{aligned}$$

and hence by Theorem 1.2

$$\begin{aligned}B_{p,w}\big(\{x: Mu(x)\ge t\}\big) &\le B_{p,w}\big(\{x: \big(G_1*Mf\big)(x)\ge t\}\big)\\ &\le t^{-p}\int_{\mathbf{R}^n}|Mf|^p\,dw \le c\,t^{-p}\int_{\mathbf{R}^n}|f|^p\,dw \le \frac{c}{t^p}\|u\|_{1,p,w}^p,\end{aligned}$$

as desired.

Now we have

**3.14. Theorem.** *If* $u\in H^{1,p}(\mathbf{R}^n;w)$ *is* $(1,p,w)$*-refined, then the limit*

$$\lim_{r\to 0}\fint_{B(x,r)} u\,dy$$

*exists and equals* $u(x)$ $(1,p,w)$*-quasieverywhere in* $\mathbf{R}^n$.

*Proof.* Mimic the proofs of Theorems 3.4 and 3.5. Just replace weighted averages and maximal functions by unweigted ones and use Lemma 3.13 instead of Lemma 3.3.

**3.15. Corollary.** *For* $u\in H^{1,p}(\mathbf{R}^n;w)$ *the following are equivalent:*

(i) $u$ *is* $(1,p,w)$*-refined.*

(ii) $u$ *is* $(1,p,w)$*-quasicontinuous.*

(iii) *For* $(1,p,w)$*-quasievery* $x$

$$\lim_{r\to 0}\fint_{B(x,r)} u\,dw = u(x).$$

(iv) *For* $(1,p,w)$*-quasievery* $x$

$$\lim_{r\to 0}\fint_{B(x,r)} u(y)\,dy = u(x).$$

*Moreover, for each* $v\in H^{1,p}(\mathbf{R}^n;w)$ *there exists a function* $u$ *such that* $u=v$ *a.e. and that* $u$ *satisfies* (i)–(iv) *above.*

**3.16. Remark.** For problems in partial differential equations it is often desirable to determine when a function can be approximated in $H^{1,p}(\Omega; w)$ by functions from $C_0^\infty(\Omega)$. A very useful characterization is the following: *Let* $u \in H^{1,p}(\Omega; w)$. *Then there is a sequence of functions* $\varphi_j \in C_0^\infty(\Omega)$ *converging to* $u$ *in* $H^{1,p}(\Omega; w)$ *if and only if there is a* $(1,p,w)$*-quasicontinuous function* $v$ *in* $\mathbf{R}^n$ *such that* $v = u$ *a.e. in* $\Omega$ *and* $v = 0$ $(1,p,w)$*-quasieverywhere on* $\complement\Omega$. This theorem was first proved by Bagby [B]. For a proof in the weighted case see [HKM, Theorem 4.5].

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