Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 19, 1994, 115–132

APPLICATIONS OF GEOMETRIC MEASURE THEORY TO THE STUDY OF GAUSS–WEIERSTRASS AND POISSON INTEGRALS

Neil A. Watson

University of Canterbury, Department of Mathematics
Private Bag 4800, Christchurch, New Zealand; naw@math.canterbury.ac.nz

Abstract. The behaviour as $t \to 0$ of $t^{\delta(n-q)}u(x,t)$ is studied, where u is either the Gauss–Weierstrass or Poisson integral of a signed measure μ on $\mathbf{R}^n$, $q \in [0,n]$, and δ is $\frac{1}{2}$ for a Gauss–Weierstrass integral but 1 for a Poisson integral. Such behaviour is used to characterize rectifiable subsets of $\mathbf{R}^n$, positive sets for μ, and sets to which the restriction of μ is absolutely continuous with respect to q-dimensional Hausdorff measure.

1. Introduction

For any locally finite, signed Borel measure μ on $\mathbf{R}^n$, the convolution

$$u(x,t) = W\mu(x,t) = \int_{\mathbf{R}^n} W(x-y,t)\,d\mu(y),$$

where $W(x,t) = (4\pi t)^{-n/2}\exp(-\|x\|^2/4t)$ for all $(x,t) \in \mathbf{R}^n \times]0,\infty[$, is called the Gauss–Weierstrass integral of μ. If $u(x_0,t_0)$ exists and is finite, then $u(x,t)$ is finite whenever $(x,t) \in \mathbf{R}^n \times]0,t_0[$, and u is a solution of the heat equation there. We shall always assume implicitly that u is finite on $\mathbf{R}^n \times]0,a[$ for some $a > 0$.

Similarly, the convolution

$$w(x,t) = P\mu(x,t) = \int_{\mathbf{R}^n} P(x-y,t)\,d\mu(y),$$

where $P(x,t) = 2s_{n+1}^{-1}t(\|x\|^2+t^2)^{-(n+1)/2}$ for all $(x,t) \in \mathbf{R}^n \times]0,\infty[$, and s_{n+1} is the surface area of the unit sphere in $\mathbf{R}^{n+1}$, is called the (half-space) Poisson integral of μ. If $w(x_0,t_0)$ exists and is finite, then $w(x,t)$ is finite whenever $(x,t) \in \mathbf{R}^n \times]0,\infty[$, and w is harmonic there. We shall always assume implicitly that w is finite on $\mathbf{R}^n \times]0,\infty[$.

It is well-known that these two integrals have much in common, and here we study them together, writing $u = K\mu$ with K either W or P. We are particularly

---

1991 Mathematics Subject Classification: Primary 31B10, 35J05, 35K05; Secondary 28A15, 28A78, 35C15, 35B05.

concerned with the limiting behaviour as $t \to 0$ of $t^{\delta(n-q)}u(x,t)$, where $q \in [0,n]$, and $\delta = \frac{1}{2}$ if $K = W$, $\delta = 1$ if $K = P$. Such behaviour is used to characterize rectifiable subsets of $\mathbf{R}^n$, positive sets for μ, and sets to which the restriction of μ is absolutely continuous with respect to the q-dimensional Hausdorff measure m_q (defined in [7, p. 7]). Our main tool is the link with geometric measure theory in $\mathbf{R}^n$ which is provided by the inequalities

$$\liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)} \le \liminf_{t\to 0} \frac{u(x,t)}{v(x,t)} \le \limsup_{t\to 0} \frac{u(x,t)}{v(x,t)} \le \limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)},$$

where $u = K\mu$, $v = K\nu$ with ν positive, and $B(x,r) = \{\, y \in \mathbf{R}^n : \|x-y\| \le r \,\}$. These inequalities lead to a very precise form of the relative Fatou theorem, whose applications are found throughout this paper; the principal ones we now describe.

In Theorem 4 we show that, if q is an integer and Z is a rectifiable set (as defined in Section 4) with σ-finite m_q-measure then

$$\lim_{t\to 0} t^{\delta(n-q)}u(x,t) = cf(x)$$

for m_q-almost all $x \in Z$, where c is a specific constant, and f is the Radon–Nikodým derivative of the restriction of μ to Z with respect to m_q. Conversely, if μ is positive, $m_q(Z) > 0$, and

$$0 < \lim_{t\to 0} t^{\delta(n-q)}u(x,t) < \infty$$

for m_q-almost all $x \in Z$, then q is an integer and Z is a rectifiable set of σ-finite m_q-measure. Thus the classical Fatou theorem and its converse are generalized. The proof depends essentially on the considerable measure theoretic achievements of Federer [8], Marstrand [9], and Preiss [10].

In Section 5, we consider the problem of determining when a given Borel subset Z of $\mathbf{R}^n$ is a positive set for the signed measure μ. For example, Theorem 8 shows that, if Z has σ-finite m_q-measure, then Z is positive for μ if and only if

$$\liminf_{t\to 0} t^{\delta(n-q)}u(x,t) \ge 0$$

for μ-almost all $x \in Z$. The case $q = n$, $Z = \mathbf{R}^n$, is well-known. This result depends on measure theory due to Wallin [12].

In Section 6, we determine when the restriction of μ to Z is absolutely continuous with respect to m_q, where Z is a Borel set which is σ-finite with respect to m_q. Theorem 9 shows that this is the case if and only if

$$\limsup_{t\to 0} t^{\delta(n-q)}|u(x,t)| < \infty$$

for μ-almost all $x \in Z$. Here the measure theory involved is mainly due to Besicovitch [3], [4].

Given a signed measure μ and a Borel set Z, we denote by μ_Z the restriction of μ to Z, and by μ^+, μ^-, and $|\mu|$ the positive, negative, and total variations of μ. The term 'positive' is used in the wide sense.

Remark. The referee has pointed out that the results in [3], [4] which are used below, can all be found in [8, Chapters 2.8 and 2.9].

2. Preliminaries

Let $u = K\mu$ and $v = K\nu$, where μ is signed but ν is positive. Consider the inequalities

$$\begin{aligned} \liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)} &\le \liminf_{t\to 0} \frac{u(x,t)}{v(x,t)} \\ &\le \limsup_{t\to 0} \frac{u(x,t)}{v(x,t)} \le \limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)}. \end{aligned} \tag{1}$$

For the case $K = W$, it was proved in [13, Theorem 1] that (1) holds at any point x such that $\nu\big(B(x,r)\big) > 0$ for all $r > 0$. Subsequently, in [1, Theorem 1], Armitage proved that (1) holds when $K = P$, but only at those points x for which

$$\lim_{t\to 0} t^{-1} v(x,t) = \infty. \tag{2}$$

He did not discuss the necessity of this more stringent condition, but it is essential. For example, if $n = 1$, $x = 0$, $l \in]0,\infty[$, and

$$d\nu(s) = l|s|^{-3} \exp(-s^{-2})\, ds,$$

then for every $r > 0$ we have $\nu\big(B(0,r)\big) = l\exp(-r^{-2})$, so that, by a formula in [1, p. 241],

$$\lim_{t\to 0} t^{-1} v(0,t) = 2l\pi^{-1} \int_0^\infty s^{-3} \exp(-s^{-2})\, ds = l\pi^{-1}.$$

Taking μ to be the unit mass at the point 1, it is obvious that

$$\lim_{r\to 0} \frac{\mu\big(B(0,r)\big)}{\nu\big(B(0,r)\big)} = 0,$$

but

$$\frac{u(0,t)}{v(0,t)} \sim \frac{\pi P(-1,t)}{lt} \to \frac{1}{l}$$

as $t \to 0$.

The inequalities (1) are essential for this paper, and the extra condition (2) in the harmonic case is not a real problem because it is satisfied by ν-almost all points x. We introduce a notation which masks this difference in the two cases. Given a positive measure ν such that $W\nu$ is finite at some point, we put

$$X(\nu) = X_W(\nu) = \left\{ x \in \mathbf{R}^n : \nu\bigl(B(x,r)\bigr) > 0 \text{ for all } r > 0 \right\}.$$

Clearly ν is concentrated on $X(\nu)$, and (1) holds for all $x \in X(\nu)$. If ν is a positive measure such that $v = P\nu$ is finite, we put

$$X(\nu) = X_P(\nu) = \left\{ x \in \mathbf{R}^n : \lim_{t\to 0} t^{-1} v(x,t) = \infty \right\}.$$

It was shown in [1, p. 241] that $\lim_{t\to 0} t^{-1} v(x,t)$ exists for every $x \in \mathbf{R}^n$, and therefore $\lim_{t\to 0} v(x,t) = 0$ for every $x \in \mathbf{R}^n \setminus X_P(\nu)$, so that $\nu\bigl(\mathbf{R}^n \setminus X(\nu)\bigr) = 0$ by a result of Brelot [5]. Thus ν is again concentrated on $X(\nu)$, and (1) holds for all $x \in X(\nu)$.

In particular, we have the following result, which is given in [14] for the case $K = W$.

Lemma 1. *Let $u = K\mu$ and $v = K\nu$, where μ is signed and ν is positive. If μ and ν are mutually singular, then*

$$u(x,t) = o\bigl(v(x,t)\bigr) \qquad \text{as } t \to 0$$

for ν-almost all x.

Proof. We can suppose that μ is positive. By [3, Theorem 3],

$$\lim_{r\to 0} \frac{\mu\bigl(B(x,r)\bigr)}{\nu\bigl(B(x,r)\bigr)} = 0$$

for ν-almost all x. Since (1) also holds for ν-almost all x, the result follows.

Another set of inequalities, which can be derived from (1), will also be needed. If $u = K\mu$ and $q \in [0,n]$, then for all $x \in \mathbf{R}^n$

$$\begin{aligned} \liminf_{r\to 0} r^{-q}\mu\bigl(B(x,r)\bigr) &\le c_{n,q} \liminf_{t\to 0} t^{\delta(n-q)} u(x,t) \\ &\le c_{n,q} \limsup_{t\to 0} t^{\delta(n-q)} u(x,t) \le \limsup_{r\to 0} r^{-q}\mu\bigl(B(x,r)\bigr), \end{aligned} \tag{3}$$

where δ is as defined in Section 1, and

$$c_{n,q} = \begin{cases} \pi^{n/2} 2^{n-q} / \Gamma(\frac12 q + 1) & \text{if } K = W, \\ \pi^{(n+1)/2} / \Gamma(\frac12 q + 1)\Gamma\bigl(\frac12 (n+1-q)\bigr) & \text{if } K = P. \end{cases} \tag{4}$$

These inequalities were proved for $K = W$ in [13], and for $K = P$ in [1]. Their proofs are valid for all $q \in [0,\infty[$ and all $q \in [0, n+1[$ respectively, but the extended ranges are not useful here. In particular, it follows from (3) that if $\lim_{r\to 0} r^{-q}\mu\big(B(x,r)\big) = \lambda$ then $\lim_{t\to 0} t^{\delta(n-q)}u(x,t) = c_{n,q}^{-1}\lambda$. For Theorem 4 below, we require the reverse implication under the Tauberian condition that μ is positive. For $K = W$ this was proved in [13], for all $\lambda, q \in [0,\infty[$ (despite the statement of [13, Theorem 4]). For $K = P$ a different Tauberian condition was used in [1], while that of positivity was used by Rudin in [11] for the case $\lambda \in [0,\infty[$, $q = n$ only. We now extend the result of [11, Theorem A] to an arbitrary $q \in [0, n+1[$.

Theorem 1. *Let $u = P\mu$ with μ positive, let $q \in [0, n+1[$, and suppose that*

$$\lim_{t\to 0} t^{n-q}u(x,t) = \lambda < \infty \tag{5}$$

for some x. Then

$$\lim_{r\to 0} r^{-q}\mu\big(B(x,r)\big) = c_{n,q}\lambda,$$

where $c_{n,q}$ is given by (4).

Proof. The case $q = 0$ is simple, because it is always true that $\lim_{r\to 0}\mu\big(B(x,r)\big) = \mu(\{x\})$, and therefore that $\lim_{t\to 0} t^n u(x,t) = \mu(\{x\})c_{n,0}^{-1}$ by (3).

The proof when $q > 0$ is similar to that given for $q = n$ in [11], so we give only an outline, in which all unexplained notations have the same meanings as in [11], and all omitted calculations are similar to the corresponding ones there. Note that our Poisson integral differs from Rudin's by a factor of $\pi^{n/2}\Gamma(\frac{1}{2}n+1)$.

If $v(t) = t^{n-q}u(x,t)$, then (5) implies that

$$\lim_{r\to 0}(H_q * v)(r) = \lambda.$$

Furthermore, it follows from (3) that, if $\mu\big(B(x,\varrho)\big) = 0$ for some $\varrho > 0$, then $\lim_{t\to 0} t^{n-q}u(x,t) = 0$; therefore our hypotheses are unaffected if we replace μ by $\mu_{B(x,1)}$, and since our conclusion is also unaffected by such a change, we may assume that $\mu(\mathbf{R}^n) < \infty$. We shall also take $x = 0$.

The definitions of H_q and v give

$$(H_q * v)(r) = qr^{-q}\int_{\mathbf{R}^n} d\mu(y)\int_0^r P(y,s)s^{n-1}\,ds. \tag{6}$$

If $M_q(r) = r^{-q}\mu\big(B(0,r)\big)$ for all $r \in]0,\infty[$, then M_q is bounded. The function k_q, defined for all $t \in]0,\infty[$ by

$$k_q(t) = 2s_{n+1}^{-1}qt^{n-q+1}(1+t^2)^{-(n+1)/2},$$

satisfies

$$P\big(y, \|y\|t\big) = q^{-1}t^{q-n}\|y\|^{-n}k_q(t)$$

whenever $y \neq 0$. Since M_q is bounded and $q > 0$, we have $\mu(\{0\}) = 0$. Therefore, if we replace s by $\|y\|t$ in (6), we obtain

$$(H_q * v)(r) = (M_q * k_q)(r),$$

so that the bounded function M_q satisfies $\lim_{r\to 0}(M_q * k_q)(r) = \lambda$. Furthermore, the Fourier transform of k_q is given by

$$\begin{aligned}\hat{k}_q(s) &= 2s_{n+1}^{-1}q\int_0^\infty (1+t^2)^{-(n+1)/2}t^{n-q-is}\,dt \\ &= s_{n+1}^{-1}q\Gamma\Big(\frac{q+is}{2}\Big)\Gamma\Big(\frac{n+1-q-is}{2}\Big)\Big/\Gamma\Big(\frac{n+1}{2}\Big),\end{aligned}$$

since $q \in]0, n+1[$. Hence $\hat{k}_q$ has no zeros, and $\hat{k}_q(0) = c_{n,q}^{-1}$. It now follows from Wiener's tauberian theorem that

$$\lim_{r\to 0}(M_q * f)(r) = c_{n,q}\lambda$$

for every f such that

$$\int_0^\infty f(r)r^{-1}\,dr = 1.$$

The desired conclusion, that $\lim_{r\to 0} M_q(r) = c_{n,q}\lambda$, now follows.

3. A relative Fatou theorem

Here we give a precise form of the relative Fatou theorem for approach along lines normal to the boundary. This is needed for the proofs of most of the subsequent results. The existence of parabolic limits of quotients of Gauss–Weierstrass integrals, and of nontangential limits of quotients of Poisson integrals, is proved in [6, p. 292 and p. 31, respectively]. However, we give a simpler argument for the present situation. The case $K = W$, $\nu = m_n$, is given in [15].

Theorem 2. *Let $u = K\mu$ and $v = K\nu$, where μ is signed and ν is positive, let*

$$f(x) = \lim_{t\to 0}\frac{u(x,t)}{v(x,t)}$$

whenever the limit exists, let $Z^+ = \{x \in \mathbf{R}^n : f(x) = \infty\}$, and let $Z^- = \{x \in \mathbf{R}^n : f(x) = -\infty\}$. Then f is defined and finite ν-a.e. on $\mathbf{R}^n$, and there are positive ν-singular measures σ^+ and σ^-, concentrated on Z^+ and Z^- respectively, such that

$$d\mu = f\,d\nu + d\sigma^+ - d\sigma^-. \tag{7}$$

Proof. The inequalities (1) hold for all $x \in X(\nu)$, and hence for ν-almost all x, so that $f(x)$ exists and is equal to

$$\lim_{r \to 0} \frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)} \tag{8}$$

for ν-almost all x at which the limit (8) exists. Therefore f is defined and finite ν-a.e., by [3, Theorem 2]. Furthermore, [4, Theorem 6] shows that the limit (8) is the Radon–Nikodým derivative of μ with respect to ν, so that (7) holds with σ the ν-singular part of μ.

It remains to prove that σ^+ and σ^- are concentrated on Z^+ and Z^- respectively. Since ν and $\mu - \sigma^+$ are both σ^+-singular, it follows from Lemma 1 that $v(x,t) = o\big(K\sigma^+(x,t)\big)$ and $K(\mu - \sigma^+)(x,t) = o\big(K\sigma^+(x,t)\big)$ as $t \to 0$, for σ^+-almost all x. Hence

$$\frac{u(x,t)}{v(x,t)} = \frac{K(\mu - \sigma^+)(x,t) + K\sigma^+(x,t)}{v(x,t)} \to \infty$$

as $t \to 0$ for σ^+-almost all x, so that σ^+ is concentrated on Z^+. Similarly σ^- is concentrated on Z^-.

4. Boundary singularities and rectifiable sets

The main result in this section is Theorem 4, which establishes a relationship between the sets of points x where $\lim_{t \to 0} t^{\delta(n-q)} K\mu(x,t)$ exists and the rectifiable subsets of $\mathbf{R}^n$. In its proof, we require a result that sharpens parts of both [13, Theorem 6] and the unstated version of [1, Theorem 4] which relates to normal limits.

Theorem 3. *Let $u = K\mu$ for some positive measure μ, and let $q \in [0,n]$. Then the set*

$$Y = \Big\{x \in \mathbf{R}^n : \limsup_{t \to 0} t^{\delta(n-q)} u(x,t) > 0\Big\}$$

is a Borel set which is σ-finite with respect to m_q.

Proof. It follows from (3) that

$$Y \subseteq \Big\{x \in \mathbf{R}^n : \limsup_{r \to 0} r^{-q} \mu\big(B(x,r)\big) > 0\Big\}. \tag{9}$$

Furthermore, for any $x \in \mathbf{R}^n$,

$$\begin{aligned} t^{-q\delta} \mu\big(B(x,t^\delta)\big) &\le \kappa t^{\delta(n-q)} \int_{B(x,t^\delta)} K(x-y,t)\, d\mu(y) \\ &\le \kappa t^{\delta(n-q)} u(x,t), \end{aligned}$$

where $\kappa = 2^n \pi^{n/2} e^{1/4}$ if $K = W$, and $\kappa = 2^{(n-1)/2} s_{n+1}$ if $K = P$. Therefore

$$\limsup_{r\to 0} r^{-q} \mu\big(B(x,r)\big) \le \kappa \limsup_{t\to 0} t^{\delta(n-q)} u(x,t), \tag{10}$$

so that equality holds in (9). Hence Y is a Borel set. For each $j \in \mathbf{N}$, let $\mu_j = \mu_{B(0,j+1)}$. Then $Y = \bigcup_{k,j=1}^{\infty} Y_{k,j}$, where

$$Y_{k,j} = \big\{x \in B(0,j) : \limsup_{r\to 0} r^{-q} \mu_j\big(B(x,r)\big) > k^{-1}\big\}.$$

If $q > 0$, then [8, p. 181 (3)] shows that $m_q(Y_{k,j}) \le k\mu_j(\mathbf{R}^n) < \infty$; and if $q = 0$ then obviously $m_q(Y_{k,j}) < \infty$. Hence Y is σ-finite with respect to m_q.

Following Federer [8, p. 251], we say that a Borel set Z is *countably* (m_q, q) *rectifiable* if it can be written in the form

$$\bigcup_{j=1}^{\infty} f_j(B_j) \cup Y,$$

where $m_q(Y) = 0$ and each f_j is a Lipschitz function from a bounded subset B_j of $\mathbf{R}^q$ into $\mathbf{R}^n$.

Theorem 4. *Let $u = K\mu$, let $q \in [0,n]$ and let Z be a Borel subset of $\mathbf{R}^n$ such that $m_q(Z) > 0$.*

(i) *If q is an integer, μ is signed, and Z is a countably (m_q, q) rectifiable set with σ-finite m_q-measure, then*

$$\lim_{t\to 0} t^{\delta(n-q)} u(x,t) = 2^q c_{n,q}^{-1} f(x)$$

for m_q-almost all $x \in Z$, where $c_{n,q}$ is given by (4) and f is the Radon–Nikodým derivative of μ_Z with respect to m_{qZ}.

(ii) *Conversely, if μ is positive and*

$$0 < \lim_{t\to 0} t^{\delta(n-q)} u(x,t) < \infty \tag{11}$$

for m_q-almost all $x \in Z$, then q is an integer and Z is a countably (m_q, q) rectifiable set with σ-finite m_q-measure.

Proof. (i) If $q = 0$ then Z is countable, and for every $x \in Z$ it follows from (3) that

$$f(x) = \mu\big(\{x\}\big) = \lim_{r\to 0} \mu\big(B(x,r)\big) = c_{n,0} \lim_{r\to 0} t^{\delta n} u(x,t).$$

Now suppose that $q \in]0,n]$. (The case $q = n$ is well-known, but is included for completeness.) We can assume that $m_q(Z) < \infty$. If $\nu = m_{qZ}$, then

$\lim_{r\to 0} r^{-q}\nu\big(B(x,r)\big) = 2^q$ for ν-almost all x, by a theorem of Federer [8, p. 256] if $q < n$. It follows from (3) that

$$\lim_{t\to 0} t^{\delta(n-q)} K\nu(x,t) = 2^q c_{n,q}^{-1}$$

for the same values of x. By Theorem 2, the function

$$f(x) = \lim_{t\to 0} \frac{u(x,t)}{K\nu(x,t)}$$

is defined for ν-almost all x, and is the Radon–Nikodým derivative of μ_Z with respect to ν. The result follows.

(ii) There is nothing to prove if $q = n$. If $q = 0$, then (11) holds for all $x \in Z$, and it follows from (3) that

$$0 < \lim_{r\to 0} \mu\big(B(x,r)\big) = \mu\big(\{x\}\big)$$

for all $x \in Z$, so that Z is countable.

Now suppose that $q \in]0,n[$, and put $\nu = m_{qZ}$ again. By Theorem 3, the set Z is σ-finite with respect to m_q, and so we may suppose that $0 < m_q(Z) < \infty$. By (11) and either [13, Theorem 4] or Theorem 1 above,

$$0 < \lim_{r\to 0} r^{-q}\mu\big(B(x,r)\big) < \infty$$

for ν-almost all x. By [3, Theorem 2],

$$\lim_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)}$$

exists and is finite for ν-almost all x, and so it follows that

$$\lim_{r\to 0} r^{-q}\nu\big(B(x,r)\big) \tag{12}$$

exists and is strictly positive for ν-almost all x. By [7, Corollary 2.5],

$$\limsup_{r\to 0} r^{-q}\nu\big(B(x,r)\big) \le 2^q,$$

so that the limit (12) is finite, for ν-almost all x. Therefore, by a theorem of Marstrand [9], q is an integer. It follows that Z is countably (m_q, q) rectifiable, by a theorem of Preiss [10, p. 613].

5. Positivity of sets for signed measures

We begin by establishing two conditions which ensure that a set is null for a positive measure.

Theorem 5. *Let* $u = K\mu$ *for some positive measure* μ, *let* $q \in [0, n]$, *and let* Z *be a Borel subset of* $\mathbf{R}^n$.

(i) *If* $m_q(Z) = 0$ *and*

$$\limsup_{t \to 0} t^{\delta(n-q)} u(x,t) < \infty$$

for μ*-almost all* $x \in Z$, *then* $\mu(Z) = 0$.

(ii) *If* Z *is* σ*-finite with respect to* m_q *and*

$$\lim_{t \to 0} t^{\delta(n-q)} u(x,t) = 0$$

for μ*-almost all* $x \in Z$, *then* $\mu(Z) = 0$.

Proof. If $q = 0$ then (i) is trivial, and in case (ii) Z is countable and (10) shows that $\mu(\{x\}) = 0$ for all $x \in Z$, so that $\mu(Z) = 0$.

Suppose that $q \in]0, n]$. In case (i), it follows from (10) that μ_Z is concentrated on $\bigcup_{k=1}^{\infty} S_k$, where

$$S_k = \bigl\{x \in Z : \limsup_{r \to 0} r^{-q} \mu_Z\bigl(B(x,r)\bigr) < k\bigr\}.$$

Furthermore, by [8, p. 181 (1)], we have $\mu_Z(S_k) \le k m_q(S_k) \le k m_q(Z) = 0$, and therefore μ_Z is null. In case (ii), we may suppose that $m_q(Z) < \infty$. By (10), for every $\gamma > 0$ the measure μ_Z is concentrated on

$$S = \bigl\{x \in Z : \limsup_{r \to 0} r^{-q} \mu_Z\bigl(B(x,r)\bigr) < \gamma\bigr\},$$

and as before $\mu_Z(S) \le \gamma m_q(Z) < \infty$. Hence μ_Z is again null.

Remark. In the case $q = n$, Theorem 5(ii) was proved in [14] for $K = W$ and in [5] for $K = P$.

We are now in a position to prove three theorems on the positivity of sets for signed measures. In the case $K = W$, the first generalizes [15, Theorem 7] in several directions, and also [14, Theorem 6]. In the case $K = P$, the analogue of the latter result is [2, Theorem 2], which is complicated by the fact that behaviour at infinity has to be taken into account; that aside, the conditions therein which ensure the positivity of the appropriate boundary measure are relaxed and made applicable to an arbitrary Borel set.

Theorem 6. *Let $u = K\mu$ and $v = K\nu$, where μ is signed and ν is positive, let $q \in [0, n]$ and let Z be a Borel subset of $\mathbf{R}^n$. Let Y be a ν-null Borel subset of Z, and suppose that*

$$\limsup_{t \to 0} \frac{u(x,t)}{v(x,t)} > -\infty$$

for all $x \in Z \setminus Y$, and that

$$\limsup_{t \to 0} \frac{u(x,t)}{v(x,t)} \geq 0 \tag{13}$$

for ν-almost all $x \in Z \setminus Y$.

(i) *If $m_q(Y) = 0$ and*

$$\liminf_{t \to 0} t^{\delta(n-q)} u(x,t) > -\infty \tag{14}$$

for μ-almost all $x \in Y$, then μ_Z is positive.

(ii) *If Y is σ-finite with respect to m_q and*

$$\liminf_{t \to 0} t^{\delta(n-q)} u(x,t) \geq 0 \tag{15}$$

for μ-almost all $x \in Y$, then μ_Z is positive.

Proof. By Theorem 2, there are positive ν-singular and mutually singular measures σ^+ and σ^- on Z such that

$$d\mu_Z = f\, d\nu_Z + d\sigma^+ - d\sigma^-,$$

where $f(x)$ is the (upper) limit in (13) and σ^- is concentrated on Y. By (13), f is positive. The measures $\mu + \sigma^-$ and σ^- are mutually singular, so that by Lemma 1 $K(\mu + \sigma^-)(x,t) = o\big(K\sigma^-(x,t)\big)$ as $t \to 0$ for σ^--almost all x. Therefore

$$\lim_{t \to 0} \frac{u(x,t)}{K\sigma^-(x,t)} = -1 \tag{16}$$

for σ^--almost all x.

(i) It follows from (14) and (16) that

$$\limsup_{t \to 0} t^{\delta(n-q)} K\sigma^-(x,t) < \infty$$

for σ^--almost all x, so that $\sigma^-(Y) = 0$ by Theorem 5(i). Hence σ^- is null and μ_Z is positive.

(ii) It follows from (15) and (16) that

$$\lim_{t \to 0} t^{\delta(n-q)} K\sigma^-(x,t) = 0$$

for σ^--almost all x, so that Theorem 5(ii) can be used to show that μ_Z is positive.

If Z is σ-finite with respect to m_q, for some $q \in [0,n]$, we can derive from Theorem 6 two criteria for μ_Z to be positive, neither of which explicitly involves an auxiliary function v. The necessity of the conditions is shown in our next result.

Lemma 2. *Let $u = K\mu$ for some signed measure μ, let $q \in [0,n]$, and let Z be a Borel subset of $\mathbf{R}^n$ which is σ-finite with respect to m_q. If μ_Z is positive, then*

$$\liminf_{t\to 0} t^{\delta(n-q)} u(x,t) \ge 0 \tag{17}$$

for μ-almost all and m_q-almost all $x \in Z$.

Proof. If $v = K\mu_Z$, then Lemma 1 implies that

$$K(\mu - \mu_Z)(x,t) = o\big(v(x,t)\big)$$

as $t \to 0$ for μ_Z-almost all x. Therefore $u(x,t) \sim v(x,t)$ for the same values of x, so that (17) holds for μ_Z-almost all x because the corresponding inequality for v is obvious.

For the second part, we may suppose that $m_q(Z) < \infty$. Let $\omega = m_{qZ}$ and $w = K\omega$. By Theorem 2,

$$f(x) = \lim_{t\to 0} \frac{u(x,t)}{w(x,t)}$$

is defined and finite for ω-almost all x, and there is an ω-singular measure σ on Z such that $d\mu_Z = f\,d\omega + d\sigma$. The function f is positive because μ_Z is, and since

$$\liminf_{t\to 0} t^{\delta(n-q)} w(x,t) \le c_{n,q}^{-1} \limsup_{r\to 0} r^{-q}\omega\big(B(x,r)\big) < \infty$$

for ω-almost all x by (3) and [7, Corollary 2.5], it follows that

$$\liminf_{t\to 0} t^{\delta(n-q)} u(x,t) = f(x) \liminf_{t\to 0} t^{\delta(n-q)} w(x,t) \ge 0$$

for ω-almost all x.

We shall prove in Theorem 8 that, if (17) holds for μ-almost all $x \in Z$ then μ_Z is positive. The alternative hypothesis that (17) holds m_q-a.e. is not generally sufficient. If $q = 0$, Z is countable, and (17) holds for (m_0-almost) all $x \in Z$, then (3) shows that

$$\mu\big(\{x\}\big) = \lim_{r\to 0} \mu\big(B(x,r)\big) = c_{n,0} \lim_{t\to 0} t^{\delta n} u(x,t) \ge 0$$

for all $x \in Z$, so that μ_Z is positive. However, if $q \in]0,n]$ then $-t^{\delta(n-q)}K(0,t) = -c_{n,0}^{-1} t^{-\delta q} \to -\infty$ and $-t^{\delta(n-q)}K(x,t) \to 0$ as $t \to 0$ if $x \ne 0$, so that (17) can hold m_q-a.e. on Z without μ_Z being positive. Hence an extra condition is required in this case, which is labelled (19) in the theorem below. Observe that if $q = 0$ then condition (18) below is assumed to hold for all $x \in Z$, so that (19) is vacuous.

Theorem 7. *Let $u = K\mu$ for some signed measure μ, let $q \in [0, n]$ and let Z be a Borel subset of $\mathbf{R}^n$ which is σ-finite with respect to m_q. Then μ_Z is positive if and only if both*

$$\liminf_{t\to 0} t^{\delta(n-q)} u(x,t) \geq 0 \tag{18}$$

for m_q-almost all $x \in Z$, and

$$\liminf_{t\to 0} t^{\delta(n-q)} u(x,t) > -\infty \tag{19}$$

for μ-almost all $x \in Z$.

Proof. Suppose that (18) and (19) hold as described. We may suppose that $m_q(Z) < \infty$. If $\nu = m_{qZ}$ then

$$\limsup_{r\to 0} r^{-q} \nu\big(B(x,r)\big) \geq 1$$

for ν-almost all x, by [7, Corollary 2.5] if $q \in]0, n[$. It therefore follows from (10) that

$$\limsup_{t\to 0} t^{\delta(n-q)} K\nu(x,t) \geq \kappa^{-1}$$

for the same values of x. This combines with (18) to yield

$$\limsup_{t\to 0} \frac{u(x,t)}{K\nu(x,t)} \geq 0$$

for ν-almost all x, so that it follows from (19) and Theorem 6(i) (with Y the set of $x \in Z$ for which the last inequality fails) that μ_Z is positive.

The converse is given in Lemma 2.

In Theorem 7, if μ is negative and $m_q(Z) = 0$, then the result reduces to Theorem 5(i) and its converse. However, if we suppose only that μ is negative, then we do not obtain Theorem 5(ii), but the following variant thereof.

Corollary. *Let $u = K\mu$ for some positive measure μ and let Z be a Borel subset of $\mathbf{R}^n$ which is σ-finite with respect to m_q. Then $\mu(Z) = 0$ if and only if both*

$$\lim_{t\to 0} t^{\delta(n-q)} u(x,t) = 0$$

for m_q-almost all $x \in Z$, and

$$\limsup_{t\to 0} t^{\delta(n-q)} u(x,t) < \infty$$

for μ-almost all $x \in Z$.

Theorem 8 below reduces to Theorem 5(ii) and its converse when μ is negative. In its proof, we use the following terminology from [12]. For any $q \in]0,n]$ and $x \in \mathbf{R}^n$ the *convex upper q-derivate* of a positive measure ν at x is the supremum of the set of extended-real numbers l such that there exists a sequence of convex sets $\{C_j\}$, with strictly positive diameters $d(C_j)$, such that $x \in C_j$ for all j, $d(C_j) \to 0$ as $j \to \infty$, and

$$\lim_{j\to\infty} \frac{\nu(C_j)}{d(C_j)^q} = l.$$

It is denoted by $\overline{D}_q\nu(x)$.

Theorem 8. *Let $u = K\mu$ for some signed measure μ, let $q \in [0,n]$, and let Z be a Borel subset of $\mathbf{R}^n$ which is σ-finite with respect to m_q. Then μ_Z is positive if and only if*

$$\liminf_{t\to 0} t^{\delta(n-q)} u(x,t) \geq 0 \tag{20}$$

for μ-almost all $x \in Z$.

Proof. If μ_Z is positive, then Lemma 2 shows that (20) holds for μ-almost all $z \in Z$.

For the converse, we may suppose that $m_q(Z) < \infty$ and that (20) holds for all $x \in Z$. If $q \in]0,n]$ then, by [12, Theorem 3], there is a finite, positive measure ν such that $\overline{D}_q\nu(x) \geq 1$ for *all* $x \in Z$. If $x \in Z$ and C is a convex superset of $\{x\}$ with diameter $d(C) > 0$, then

$$\frac{\nu(C)}{d(C)^q} \leq \frac{\nu\big(B\big(x, d(C)\big)\big)}{d(C)^q}$$

and hence

$$1 \leq \overline{D}_q\nu(x) \leq \limsup_{r\to 0} r^{-q}\nu\big(B(x,r)\big). \tag{21}$$

If $q = 0$ we take $\nu = m_{0Z}$, and the corresponding inequality is then trivial. It now follows from (10) that

$$\kappa^{-1} \leq \limsup_{t\to 0} t^{\delta(n-q)} K\nu(x,t) \tag{22}$$

for all $x \in Z$. Furthermore, (20) and (22) together imply that

$$\limsup_{t\to 0} \frac{u(x,t)}{K\nu(x,t)} \geq 0$$

for all $x \in Z$, and so it follows from Theorem 6 that μ_Z is positive.

6. Absolute continuity with respect to Hausdorff measures

Here we establish the following characterization of those Borel sets Z of σ-finite m_q-measure for which μ_Z is absolutely continuous with respect to m_q.

Theorem 9. *Let $u = K\mu$ for some signed measure μ, let $q \in [0,n]$, and let Z be a Borel subset of $\mathbf{R}^n$ which is σ-finite with respect to m_q. If*

$$\limsup_{t\to 0} t^{\delta(n-q)}|u(x,t)| < \infty \tag{23}$$

for μ-almost all $x \in Z$, then μ_Z is absolutely continuous with respect to m_q. Conversely, if μ_Z is absolutely continuous with respect to m_q, then (23) *holds for m_q-almost all $x \in Z$.*

Proof. Suppose that (23) holds μ-a.e. on Z. Since it suffices to prove the result locally, we may suppose that $|\mu|(\mathbf{R}^n) < \infty$. We may also assume that $m_q(Z) < \infty$.

Let $\nu = m_{qZ}$. By Theorem 2, if

$$f(x) = \lim_{t\to 0} \frac{u(x,t)}{K\nu(x,t)},$$

then f is defined and finite ν-a.e., and there is a ν-singular measure τ such that $d\mu_Z = f\,d\nu + d\tau$. The measures $\mu + \tau^-$ and τ^- are mutually singular, so that

$$K(\mu + \tau^-)(x,t) = o\big(K\tau^-(x,t)\big)$$

as $t \to 0$ for τ^- almost all x, by Lemma 1. Therefore

$$\lim_{t\to 0} \frac{u(x,t)}{K\tau^-(x,t)} = -1$$

for the same values of x, and so it follows from (23) that

$$\limsup_{t\to 0} t^{\delta(n-q)} K\tau^-(x,t) < \infty$$

for τ^--almost all x. Since τ^- is ν-singular, it is concentrated on a subset Y of Z such that $m_q(Y) = 0$, and it therefore follows from Theorem 5(i) that $\tau^-(Y) = 0$, so that τ^- is null. Similarly τ^+ is null, so that $d\mu_Z = f\,d\nu$.

For the converse, we need consider only the case where μ is positive and $0 < m_q(Z) < \infty$.

Let $\nu = m_{qZ}$ again, and f be the Radon–Nikodým derivative of μ with respect to ν. Then

$$f(x) = \lim_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)} < \infty$$

for ν-almost all x, by [4, Theorem 6] and [3, Theorem 2]. Furthermore,

$$\limsup_{r\to 0} r^{-q}\nu\big(B(x,r)\big) \le 2^q$$

for ν-almost all x, by [7, Corollary 2.5] if $q \in]0,n[$. In view of (3), it follows that

$$\begin{aligned} c_{n,q}\limsup_{t\to 0} t^{\delta(n-q)}u(x,t) &\le \limsup_{r\to 0} r^{-q}\mu\big(B(x,r)\big) \\ &= f(x)\limsup_{r\to 0} r^{-q}\nu\big(B(x,r)\big) < \infty \end{aligned}$$

for ν-almost all x.

Theorem 9 gives rise to the following apparently weaker test for absolute continuity.

Corollary. *Let $q \in]0,n]$, let $\{q_i\}$ be a sequence in $[0,q[$, let $u = K\mu$ for some signed measure μ, let Z be a Borel subset of $\mathbf{R}^n$ which is σ-finite with respect to m_q, and let $\{Z_i\}$ be a sequence of disjoint Borel subsets of Z such that $m_{q_i}(Z_i) = 0$ for all i. If*

$$\limsup_{t\to 0} t^{\delta(n-q)}|u(x,t)| < \infty$$

for μ-almost all $x \in Z \setminus \bigcup_{i=1}^{\infty} Z_i$, and for every i

$$\limsup_{t\to 0} t^{\delta(n-q_i)}|u(x,t)| < \infty$$

for μ-almost all $x \in Z_i$, then μ_Z is absolutely continuous with respect to m_q.

Proof. Let $Y = Z \setminus \bigcup_{i=1}^{\infty} Z_i$. By Theorem 9, μ_Y is absolutely continuous with respect to m_q; and each μ_{Z_i} is absolutely continuous with respect to the corresponding m_{q_i}, and hence is null.

Remarks. If q is an integer and Z is a countably (m_q, q) rectifiable set, we know from Theorem 4 that the Radon–Nikodým derivative f of μ_Z with respect to m_{qZ} is given by

$$f(x) = 2^{-q}c_{n,q}\lim_{t\to 0} t^{\delta(n-q)}u(x,t).$$

In general, such precise information is not available. However, if μ is positive we can obtain estimates for f in terms of the upper limits in (23), as follows. We know from [4, Theorem 6] and [3, Theorem 2] that if $\nu = m_{qZ}$ then

$$f(x) = \lim_{r\to 0}\frac{\mu\big(B(x,r)\big)}{\nu\big(B(x,r)\big)} < \infty,$$

and from [7, Corollary 2.5] that

$$1 \le \limsup_{r\to 0} r^{-q}\nu\big(B(x,r)\big) \le 2^q,$$

for ν-almost all x. It therefore follows from (3) and (10) that

$$\begin{aligned} 2^{-q}c_{n,q}\limsup_{t\to 0} t^{\delta(n-q)}u(x,t) &\le 2^{-q}\limsup_{r\to 0} r^{-q}\mu\big(B(x,r)\big) \\ &= 2^{-q}f(x)\limsup_{r\to 0} r^{-q}\nu\big(B(x,r)\big) \le f(x) \end{aligned}$$

and

$$\begin{aligned} \kappa\limsup_{t\to 0} t^{\delta(n-q)}u(x,t) &\ge \limsup_{r\to 0} r^{-q}\mu\big(B(x,r)\big) \\ &= f(x)\limsup_{r\to 0} r^{-q}\nu\big(B(x,r)\big) \ge f(x) \end{aligned}$$

for ν-almost all x.

References

[1] Armitage, D.H.: Normal limits, half-spherical means and boundary measures of half-space Poisson integrals. - Hiroshima Math. J. 11, 1981, 235–246.

[2] Armitage, D.H.: Domination, uniqueness and representation theorems for harmonic functions in half-spaces. - Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1981, 161–172.

[3] Besicovitch, A.S.: A general form of the covering principle and relative differentiation of additive functions. - Proc. Cambridge Phil. Soc. 41, 1945, 103–110.

[4] Besicovitch, A.S.: A general form of the covering principle and relative differentiation of additive functions II. - Proc. Cambridge Phil. Soc. 42, 1946, 1–10.

[5] Brelot, M.: Remarques sur les zéros à la frontières des fonctions harmoniques positives. - Boll. Un. Mat. Ital. 12, 1975, 314–319.

[6] Doob, J.L.: Classical potential theory and its probabilistic counterpart. - Springer-Verlag, 1984.

[7] Falconer, K.J.: The geometry of fractal sets. - Cambridge University Press, 1985.

[8] Federer, H.: Geometric measure theory. - Springer-Verlag, 1969.

[9] Marstrand, J.M.: The (φ, s)-regular subsets of n-space. - Trans. Amer. Math. Soc. 113, 1964, 369–392.

[10] Preiss, D.: Geometry of measures in R^n: Distribution, rectifiability, and densities. - Ann. Math. 125, 1987, 537–643.

[11] Rudin, W.: Tauberian theorems for positive harmonic functions. - Nederl. Akad. Wetensch. Indag. Math. 40, 1978, 376–384.

[12] Wallin, H.: Hausdorff measures and generalized differentiation. - Math. Ann. 183, 1969, 275–286.

[13] Watson, N.A.: Initial singularities of Gauss–Weierstrass integrals and their relations to Laplace transforms and Hausdorff measures. - J. London Math. Soc. (2) 21, 1980, 336–350.

[14] Watson, N.A.: Initial and relative limiting behaviour of temperatures on a strip. - J. Austral. Math. Soc. (Ser. A) 33, 1982, 213–228.

[15] Watson, N.A.: On the representation of solutions of the heat equation and weakly coupled parabolic systems. - J. London Math. Soc. (2) 34, 1986, 457–472.

Received 22 December 1992