Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 19, 1994, 133–146

CONFORMAL INVARIANTS IN THE PUNCTURED UNIT DISK

G.D. Anderson, M. Lehtinen, and M. Vuorinen

Michigan State University, Department of Mathematics
East Lansing, MI 48824 U.S.A.; 21144gda@msu.edu
University of Helsinki, Department of Mathematics
P.O. Box 4 (Hallituskatu 15), FIN-00014 Helsinki, Finland
University of Helsinki, Department of Mathematics
P.O. Box 4 (Hallituskatu 15), FIN-00014 Helsinki, Finland; vuorinen@csc.fi

Abstract. The authors obtain new estimates for some conformal invariants in the punctured unit disk and apply these to derive sharp distortion theorems for plane quasiconformal mappings.

1. Introduction

When one wishes to develop the theory of K-quasiconformal mappings using only conformal invariants, in the spirit of [A1], one needs to find invariants which are practical as computational tools and which also have a natural interpretation in terms of geometric properties of the domains mapped.

In [V2], two conformal invariants given by extremal lengths of curve families were used to study the distortion and other geometric properties of quasiconformal mappings in Euclidean n-space, $n \geq 2$. In the particular case $n = 2$, analogous methods can be used to produce somewhat sharper results, as shown in [LV].

In this paper, we continue to study one of these invariants, namely the invariant λ_G due to J. Ferrand [LF]. Let G be a proper subdomain of the complex plane $\mathbf{C} = \mathbf{R}^2$, and for $z \in G$ let C_z denote any continuum in G joining z to ∂G. For $x, y \in G$, $x \neq y$, let

$$\lambda_G(x, y) = \inf_{C_x, C_y} M\big(\Delta(C_x, C_y; G)\big) \tag{1.1}$$

(for notation, see Section 2).

Let B be the unit disk and let τ denote the capacity of the Teichmüller ring. The following theorem was proved in [LV].

1.2. Theorem. *For $G = B \setminus \{0\}$ and $x, y \in G$ with $|x| \leq |y|$, $x \neq y$, the following inequality holds:*

$$\lambda_G(x, y) \leq C^2 \tau\Big(\frac{|x-y|}{\min\{|x|,\, 1-|y|\}}\Big),$$

where $C < 1.172$.

1991 Mathematics Subject Classification: Primary 30C75; Secondary 33E05.

In general, it is not possible to replace the constant C by 1 in Theorem 1.2. However, in the following result, which is a particular case of our Theorem 3.9, we are able to reduce the coefficient to 1 at the expense of a modification of the argument of τ.

1.3. Theorem. *For distinct $x, y \in B \setminus \{0\} = G$, $|x| \le |y|$, we have*

$$\lambda_G(x, y) \le \tau\Big(\frac{|x-y|\big(1+|x|\big)^2}{2|x|\big(1-\frac{1}{2}|x-y|\big)^2}\Big) \le \tau\Big(\frac{|x-y|\big(1+|x|\big)^2}{2|x|}\Big).$$

As far as we know, there is no explicit formula for $\lambda_G(x,y)$ when $G = B\setminus\{0\}$. Note that Theorem 1.3 is weaker than Theorem 1.2 if $|y|$ is close to 1 (for such y one should use Theorem 3.9 instead of Theorem 1.3).

Results such as this theorem can be interpreted as restrictions on the geometry of the configuration G, x, y. By using the quasi-invariance of λ_G under quasiconformal mappings, one can obtain distortion theorems for these maps. For example, we prove the following theorem, which is the main result in this paper. Recall the distortion function (cf. [LeVi, p. 63]) φ_K given by

$$\varphi_K(r) = \mu^{-1}\Big(\frac{\mu(r)}{K}\Big), \tag{1.4}$$

where μ is the conformal modulus of the Grötzsch ring $B \setminus [0, r]$ (cf. Section 2).

1.5. Theorem. *Let $f\colon B \to f(B) \subset B$ be a K-quasiconformal mapping with $f(0) = 0$ and let $\delta = d\big(0, \partial f(B)\big)$. Then for all $x \in B \setminus \{0\}$, $|x| = r$, $r' = \sqrt{1-r^2}$, we have*

$$\frac{\big|f(x)\big|}{\big(1-\big|f(x)\big|\big)^2} \le \frac{\delta}{(1+\delta)^2}\Big(\frac{\varphi_{2K}(r)}{\varphi_{1/(2K)}(r')}\Big)^2 \le \frac{\varphi_K(r)}{\big(1-\varphi_K(r)\big)^2}. \tag{1.6}$$

There is equality in the second inequality in (1.6) if and only if $\delta = 1$, i.e. $f(B) = B$. Hence Theorem 1.5 improves the quasiconformal Schwarz lemma [LeVi, p. 63].

In the final section of this paper we study the hyperbolic geometry of the unit disk, obtaining a characterization for non-Euclidean ellipses and hyperbolas, then finding a relationship between them.

Acknowledgements. We wish to thank M.K. Vamanamurthy for reading the manuscript carefully and suggesting substantial improvements. The third author also wishes to thank Ch. Pommerenke and J. Becker for their kind hospitality during his visit at the Technical University of Berlin, and the Alexander von Humboldt Foundation for financial support. The paper was completed during the third author's visit at the University of Michigan, on a grant from the Academy of Finland. We are indebted to A.Yu. Solynin for useful discussions.

2. Preliminary results

We shall adopt the relatively standard notation and terminology used in [V2] and [LV]. For $x \in \mathbf{R}^2$ and $r > 0$ we set $B(x,r) = \{z \in \mathbf{R}^2 \mid |x-z| < r\}$, $B(r) = B(0,r)$, $S(x,r) = \partial B(x,r)$, $S(r) = S(0,r)$, $B = B(1)$ and, $S = S(1)$. For $x, y \in \mathbf{R}^2$, we set $[x,y] = \{(1-t)x + ty \mid 0 \le t \le 1\}$, and for $x \neq 0$, $[x,\infty] = \{tx \mid 1 \le t\} \cup \{\infty\}$.

The group of Möbius transformations of $\overline{\mathbf{R}}^2 = \mathbf{R}^2 \cup \{\infty\}$ is denoted by $\mathrm{GM}(\overline{\mathbf{R}}^2)$. For $D \subset \overline{\mathbf{R}}^2$, $D \neq \emptyset$, we let $\mathrm{GM}(D) = \{f \in \mathrm{GM}(\overline{\mathbf{R}}^2) \mid f(D) = D\}$. The subgroup of all sense-preserving Möbius transformations is denoted by $\mathrm{M}(\overline{\mathbf{R}}^2)$ or $\mathrm{M}(D)$. For $x \in B$ we denote by T_x the element of $\mathrm{M}(B)$ satisfying $T_x(x) = 0$, $T_x(0) = -x$. The hyperbolic tangent and its inverse are denoted by th and arth, respectively, and the hyperbolic sine by sh.

The *Poincaré* or *hyperbolic* metric ϱ of B is defined by

$$\mathrm{th}\,\frac{\varrho(x,y)}{2} = \frac{|x-y|}{|1-\overline{x}y|} = \bigl|T_x(y)\bigr|, \tag{2.1}$$

where $\overline{x}$ is the complex conjugate of x (see e.g. [V2, Section 2]). We also call this metric the non-Euclidean metric of B and sometimes use L. Ahlfors' abbreviation n.e. for non-Euclidean [A2]. We denote by $J[x,y]$ the n.e. line segment, that is the arc, with endpoints x and y, of a circle orthogonal to S.

The modulus of a curve family Γ in $\mathbf{R}^2$ is denoted by $M(\Gamma)$ [Vä]. If $E, F, D \subset \overline{\mathbf{R}}^2$, we denote by $\Delta(E,F;D)$ the family of all curves joining E and F in D. If $D = \mathbf{R}^2$ or $D = \overline{\mathbf{R}}^2$, this family is denoted by $\Delta(E,F)$. For $s > 1$ and $t > 0$, the moduli

$$\gamma(s) = M\bigl(\Delta\bigl(B, [s,\infty]\bigr)\bigr),$$
$$\tau(t) = M\bigl(\Delta\bigl([-1,0], [t,\infty]\bigr)\bigr)$$

are the capacities of the Grötzsch and Teichmüller rings, respectively. For convenience, we set $\gamma(1) = \tau(0) = \infty$. The capacities γ and τ satisfy the basic functional identity [V2]

$$\gamma(s) = 2\tau\left(s^2 - 1\right). \tag{2.2}$$

The capacity of the Grötzsch ring can be computed from

$$\gamma(s) = \frac{2\pi}{\mu(1/s)}, \tag{2.3}$$

where

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)},$$

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \qquad \mathcal{K}'(r) = \mathcal{K}(r'),$$

for $0 < r < 1$ and $r' = \sqrt{1-r^2}$, $\mathcal{K}(1-) = \infty$, $\mu(1-) = 0$. The following identities hold [LeVi, p. 64], [V2, p. 68]:

$$\mu(r)\mu(r') = \frac{\pi^2}{4}, \qquad \mu(r)\mu\Big(\frac{1-r}{1+r}\Big) = \frac{\pi^2}{2}, \qquad \mu(r) = 2\mu\Big(\frac{2\sqrt{r}}{1+r}\Big). \tag{2.4}$$

From (2.2) and (2.3) we have, for $s > 0$,

$$\tau(s) = \frac{\pi}{\mu\big(1/\sqrt{1+s}\,\big)}. \tag{2.5}$$

For distinct points x and y on the unit circle S, we let E_{xy} denote the positively oriented arc from x to y.

Let $\alpha \in (0, \pi/2)$, $a = e^{-i\alpha}$, $b = e^{i\alpha}$, and $E = E_{ab}$. Then, using the conformal mapping $z \mapsto \big((1-z)/(1+z)\big)^2$, we obtain from (2.5) and [Vä, Theorem 8.1],

$$M\big(\Delta(E, [-1, 0]; B)\big) = \frac{1}{2}\gamma\Big(\csc\frac{\alpha}{2}\Big) = \frac{4}{\pi}\mu\Big(\cos\frac{\alpha}{2}\Big) = \tau\big(|-\bar{c}, -c, c, \bar{c}|\big), \tag{2.6}$$

where $c = e^{i\alpha/2}$. Here, for an ordered quadruple a, b, c, d of distinct points in $\overline{\mathbf{R}}^2$, $|a, b, c, d|$ denotes the absolute value of the cross ratio $(a, b, c, d) = q(a, c)q(b, d)/(q(a, b)q(c, d))$, where q is the spherical metric in $\overline{\mathbf{R}}^2$. Let G be any Jordan domain and let E be an arc in ∂G. For $z \in G$, define

$$\sigma(z, E; G) = \inf_{C_z} M\big(\Delta(C_z, E; G)\big), \tag{2.7}$$

where the infimum is taken over all arcs C_z in G joining z and $\partial G \setminus E$. When the domain is clear, the G is omitted in (2.7). Then σ is a conformal invariant:

$$\sigma\big(f(z), f(E); f(G)\big) = \sigma(z, E; G) \tag{2.8}$$

whenever f is a conformal mapping of a Jordan domain G onto a Jordan domain $f(G)$. In particular, (2.8) holds if $G = B$ and $f \in \mathrm{M}(B)$.

We next compute $\sigma(z, E)$ in terms of the geometry of the configuration z, E, G. The next result is essentially due to A. Beurling [B].

2.9. Lemma. *Let* $a, b \in S$ *and* $c \in B$. *Then*

$$\sigma(c, E_{ab}) = \frac{4}{\pi}\mu\Big(\cos\frac{\alpha}{2}\Big),$$

where $\alpha \in (0, \pi/2)$ *is given by*

$$2 \sin \alpha = \frac{|a-b|\big(1-|c|^2\big)}{|1-a\overline{c}||1-b\overline{c}|}.$$

Proof. Let $f \in \mathrm{M}(B)$ be such that $f(c) = 0$, $f(a) = e^{-i\alpha} = a'$, $f(b) = e^{i\alpha} = b'$. By (2.6) and (2.8),

$$\sigma(c, E_{ab}) = \sigma(0, E_{a'b'}) = \frac{4}{\pi}\mu\Big(\cos\frac{\alpha}{2}\Big).$$

Further,

$$|c, a, 1/\overline{c}, b| = |0, e^{-i\alpha}, \infty, e^{i\alpha}|$$

gives

$$2 \sin \alpha = \frac{\big(1-|c|^2\big)|a-b|}{|c-a||1-b\overline{c}|} = \frac{\big(1-|c|^2\big)|a-b|}{|1-a\overline{c}||1-b\overline{c}|}. \square$$

2.10. Corollary. *Let* f *be a* K*-quasiconformal automorphism of* $\overline{B} = B \cup S$ *with* $f(0) = 0$ *and* $a = e^{i\alpha}$, $b = e^{i\beta}$, $\alpha, \beta \in (0, \pi/2)$. *If* $f(E_{1a}) = E_{1b}$, *then*

$$\varphi_{1/K}\Big(\sin\frac{\alpha}{2}\Big) \le \sin\frac{\beta}{2} \le \varphi_K\Big(\sin\frac{\alpha}{2}\Big).$$

Proof. Use 2.9 and the inequalities $\sigma(0, E_{1a})/K \le \sigma(0, E_{1b}) \le K\sigma(0, E_{1a})$. $\square$

2.11. Remark. Corollary 2.10 is essentially due to J. Hersch [H, p. 5].

2.12. Theorem. *Let* $f\colon B \to \mathbf{R}^2$ *be a* K*-quasiconformal mapping such that* $f(B)$ *is a Jordan domain, let* $a, b \in S$, *and suppose that* $\big|f(x)\big| \le \varepsilon < 1$ *for all* $x \in E_{ab}$. *Then*

$$(2.13) \qquad \big|f(c)\big| \le \frac{\varepsilon}{\varphi_{1/(2K)}\big(\sin(\alpha/2)\big)}; \qquad 2\sin\alpha = \frac{|a-b|\big(1-|c|^2\big)}{|1-a\overline{c}||1-b\overline{c}|},$$

for all $c \in B$. *Further, if* $f(B) \subset B$, *then*

$$(2.14) \qquad \big|f(c)\big| \le \frac{A+\varepsilon}{A\varepsilon+1}\varepsilon, \qquad A = \frac{1}{\varphi_{1/(2K)}\big(\sin(\alpha/2)\big)},$$

for all $c \in B$.

Proof. To prove (2.13), fix $c \in B$. Since the inequality is trivial for $\big|f(c)\big| \le \varepsilon$, we may assume that $\big|f(c)\big| > \varepsilon$. Let $F' = [f(c), \infty]$, $F = f^{-1}(F')$, $\Gamma' = \Delta\big(f(E_{ab}), F'; fB\big)$, and $\Gamma = f^{-1}(\Gamma') = \Delta(E_{ab}, F; B)$. By Lemma 2.9 we obtain

$$\frac{1}{2}\gamma\Big(\csc\frac{\alpha}{2}\Big) = \sigma(c, E_{ab}) \le M(\Gamma),$$

while $M(\Gamma') \le \gamma\big(|f(c)|/\varepsilon\big)$ by an extremal property of the Grötzsch ring [LeVi, p. 54]. These two inequalities together with $M(\Gamma) \le KM(\Gamma')$ yield

$$\big|f(c)\big| \le \varepsilon\gamma^{-1}\Big(\frac{1}{2K}\gamma\Big(\csc\frac{\alpha}{2}\Big)\Big) = \frac{\varepsilon}{\varphi_{1/(2K)}\big(\sin(\alpha/2)\big)},$$

as desired.

The proof of (2.14) is similar to that of (2.13) except that we take $F' = \big[f(c), f(c)/|f(c)|\big]$ and use the sharper majorant

$$M(\Gamma') \le \gamma\Big(\frac{\big|f(c)\big| - \varepsilon^2}{\varepsilon\big(1 - |f(c)|\big)}\Big)$$

(see e.g. [V2, 5.54 (2)]). □

2.15. Remark. Theorem 2.12 is closely related to the so-called two-constants theorem for quasiconformal mappings. For further results see [R], [V1], [GLM], [M].

3. Majorants for conformal invariants

From (2.1) it follows that

$$\text{(3.1)} \qquad \operatorname{sh}^2\frac{\varrho(x,y)}{2} = \frac{|x-y|^2}{\big(1-|x|^2\big)\big(1-|y|^2\big)}$$

for x, $y \in B$. Given x, $y \in B$ choose $z \in B$ such that $T_z x = -T_z y$. Then it is easy to show (cf. e.g. [V2, (2.27)]) that

$$\text{(3.2)} \qquad |T_z x| = |T_z y| = \operatorname{th}\frac{\varrho(x,y)}{4} \ge \frac{|x-y|}{2}.$$

For $z \in B$ let $G = B \setminus \{z\}$. For x, $y \in G$, define

$$\text{(3.3)} \qquad p_z(x,y) = \inf_{C_x, C_y} M\big(\Delta(C_x, C_y; G)\big),$$

where C_x is any curve in G joining x and z and C_y is any curve in G joining y and S. We abbreviate $p_0(x,y)$ as $p(x,y)$.

3.4. Definitions. Let $a > 0$. (1) For b, $c \in B(a)$ and $t \geq \varrho_{B(a)}(b,c)$, the set

$$E(b,c,t) = \{z \in B(a) \mid \varrho_{B(a)}(b,z) + \varrho_{B(a)}(z,c) = t\}$$

is called a *non-Euclidean* or *n.e. ellipse* with foci b and c.

(2) For b, $c \in B(a)$ and $t \in (0,\infty)$ the set

$$H(b,c,t) = \{z \in B(a) \mid |\varrho_{B(a)}(b,z) - \varrho_{B(a)}(z,c)| = t\}$$

is called a *non-Euclidean* or *n.e. hyperbola* with foci b and c.

The next result, which is implied by a result due to R. Kühnau [K, p. 24], provides a useful characterization of an n.e. ellipse. An independent proof of this result, as well as an analog for n.e. hyperbolas, will be given in Section 5.

3.5. Theorem. *Given $a > 0$ and $b, c \in B(a)$ let f_{bc} be a conformal mapping of a plane annulus $B \setminus \overline{B(t)}$ onto the ring $B(a) \setminus J[b,c]$. A set $E \subset B(a)$ is a non-Euclidean ellipse with foci b, c if and only if $f_{bc}(S(u)) = E$ for some $u \in (t,1)$.*

3.6. Lemma. *Let x, y, $z \in B$. Then*

$$p_z(x,y) \leq M\big(\Delta\big([0,s],[t,1];B\big)\big) = \tau\Big(\frac{(t-s)(1-ts)}{s(1-t)^2}\Big),$$

where $s = \operatorname{th}\frac{1}{2}\varrho(x,z)$ and $t = \operatorname{th}\frac{1}{4}(\varrho(y,z) + \varrho(z,x) + \varrho(x,y))$. The bound is attained if z, x, and y lie on an n.e. line in this order.

Proof. Let $c = \varrho(y,x) + \varrho(y,z)$ and let $f_{xy}\colon B \setminus \overline{B(r)} \to B \setminus J[x,y]$ be the mapping as in Theorem 3.5. For $y \in E(x,z,c)$ let y' be the point where $J[z,x]$, produced, meets $E(x,z,c)$. Let L' be the hyperbolic ray $J[y',w]$, where w is the intersection of S and the hyperbolic ray from z through x and y'. Let $C' = f_{zx}^{-1}(L')$ and let C be the rotation of C' so as to pass through $f_{zx}^{-1}(y)$. Take $F_y = f_{zx}(C)$. Then, by conformal invariance,

$$M\big(\Delta\big(J[x,z],F_y;B\big)\big) = M\big(\Delta\big([0,s],[t,1];B\big)\big),$$

where $s = \operatorname{th}\frac{1}{2}\varrho(x,z)$ and $t = \operatorname{th}\frac{1}{4}\big(\varrho(y,z)+\varrho(x,y)+\varrho(z,x)\big)$. This fact, together with [LV, 2.8], yields the desired estimate for $p_z(x,y)$. □

It follows immediately from the definition of λ_G that for $G = B \setminus \{z\}$

$$\lambda_G(x,y) = \min\{p_z(x,y), p_z(y,x), \lambda_B(x,y)\} \tag{3.7}$$

for all distinct x, $y \in G$. Since by [V2, 8.6] and (3.1)

$$\lambda_B(x,y) = \frac{1}{2}\tau\Big(\frac{|x-y|^2}{\big(1-|x|^2\big)\big(1-|y|^2\big)}\Big) = \frac{1}{2}\tau\Big(\operatorname{sh}^2\frac{\varrho(x,y)}{2}\Big) \tag{3.8}$$

for distinct x, $y \in B$, we obtain a majorant for $\lambda_G(x,y)$ by combining Lemma 3.6 with (3.7) and (3.8) as follows.

3.9. Theorem. *For distinct* x, y *in* $G = B \setminus \{0\}$ *we have*

$$\lambda_G(x,y) \le \min\Big\{\tau\Big(\frac{(t-s)(1-ts)}{s(1-t)^2}\Big),\ \frac{1}{2}\tau\Big(\frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\Big)\Big\},$$

where $t = \operatorname{th}\frac{1}{4}\big(\varrho(y,0)+\varrho(x,y)+\varrho(x,0)\big)$ *and* $s = \operatorname{th}\frac{1}{2}\min\{\varrho(x,0),\varrho(y,0)\} = \min\{|x|,|y|\}$.

3.10. Remark. Let $G = B \setminus \{0\}$, and $0 < x < y < 1$. We claim that

$$\lambda_G(x,y) = p_0(x,y) = \tau\Big(\frac{(y-x)(1-xy)}{x(1-y)^2}\Big). \tag{3.11}$$

First, by Lemma 3.6,

$$p_0(x,y) = \tau\Big(\frac{(y-x)(1-xy)}{x(1-y)^2}\Big) = \tau\Big(\frac{\operatorname{th}(c/2)}{(1-\operatorname{th}(c/2))^2}\ \frac{(1+x)^2}{x}\Big)$$

if $\varrho(x,y) = c$. Then $g \equiv T_y^{-1}\circ\sigma\circ T_y$ maps the segment $[0,1]$ onto itself with $x \to y$, $y \to x'$, and $\varrho(x,y) = \varrho(y,x')$, where σ is a circular symmetrization about 0. Hence

$$p_0(x,y) \le p_0(y,x') \le p_0(y,x).$$

Next, by (3.8), Lemma 3.6, (2.2), and (2.4),

$$p_0(x,y) < \frac{1}{2}\tau\Big(\frac{(y-x)^2}{(1-x^2)(1-y^2)}\Big) = \lambda_B(x,y),$$

and (3.11) follows.

Hence in this case one of the extremal continua always joins 0 and x, while the other joins y and S. In particular, the inequality in Theorem 3.9 is sharp.

3.12. Lemma. *Let* $0 < x < y < 1$ *and* $\Gamma_{xy} = \Delta\big([0,x],[y,1]\cup S; B\big)$. *Then*

$$M(\Gamma_{xy}) = \tau\Big(\frac{(y-x)(1-xy)}{x(1+y)^2}\Big).$$

Proof. Let $f(z) = (1-z)/(1+z)$, $g(z) = z^2$, and $h = g\circ f$. Then by conformal invariance,

$$M(\Gamma_{xy}) = M\big(h(\Gamma_{xy})\big) = \tau\Big(\frac{h(x)-h(y)}{1-h(x)}\Big) = \tau\Big(\frac{(y-x)(1-xy)}{x(1+y)^2}\Big). \ \square$$

3.13. Lemma. *Let* x, y, z *be distinct points in* B. *Then there exists an arc* F *in* $\overline{B}$ *joining* y *and* S *such that*

$$M\big(\Delta\big(J[z,x], F\cup S; B\big)\big) = \tau\Big(\frac{(t-s)(1-ts)}{s(1+t)^2}\Big),$$

where t *and* s *are as in Lemma 3.6.*

Proof. Let notation be as in the proof of Lemma 3.6. Let $y \in E(x,z,c)$ and let y_o, w_o be the points where $J[z,x]$, produced, meets $E(x,z,c)$ and S, respectively. Let L be the hyperbolic ray $J[y_o,w_o]$ and $L' = f_{zx}^{-1}(L)$. Let F' be the rotation of L' so as to pass through $y' = f_{zx}^{-1}(y)$, and take $F = f_{zx}(F')$.

Next, there is a Möbius transformation that maps B onto itself and carries z, x, y_o, w_o to $0, s, t, 1$, respectively, where s and t are as in Lemma 3.6. A second Möbius transformation takes B onto the right half plane and $0, s, t, 1$ onto 1, $(1-s)/(1+s), (1-t)/(1+t), 0$, respectively. Finally, the square mapping carries this right half plane configuration onto the Teichmüller ring whose complementary components are $\bigl(-\infty, \bigl((1-t)/(1+t)\bigr)^2\bigr] \cup \{\infty\}$ and $\bigl[\bigl((1-s)/(1+s)\bigr)^2, 1\bigr]$. Thus if $\Gamma = \Delta\bigl(J[z,x], F \cup S]; B\bigr)$ we have

$$M(\Gamma) = \tau\Bigl(\frac{\bigl((1-s)/(1+s)\bigr)^2 - \bigl((1-t)/(1+t)\bigr)^2}{1 - \bigl((1-s)/(1+s)\bigr)^2}\Bigr) = \tau\Bigl(\frac{(t-s)(1-st)}{s(1+t)^2}\Bigr). \ \square$$

3.14. Lemma. *Let x, y, z be three distinct points in B. Let $E \subset \overline{B}$ be a continuum joining x to z and $F \subset \overline{B}$ a continuum joining y to S. Then*

$$M\bigl(\Delta(E, F \cup S; B)\bigr) \ge M\bigl(\Delta\bigl([-s,0],[t,1] \cup S; B\bigr)\bigr) = \tau\Bigl(\frac{t(1-s)^2}{s(1+t)^2}\Bigr),$$

where $s = \operatorname{th}\frac{1}{2}\varrho(x,z)$ and $t = \operatorname{th}\frac{1}{2}\varrho(x,y)$. Equality holds if $x \in J[z,y]$ and F is a subarc of the hyperbolic line through x and y.

Proof. The proof is a standard symmetrization argument consisting of two steps (cf. also [LV, 3.7]). First apply T_x and then perform a circular symmetrization with center 0. Application of Lemma 3.12 completes the proof. $\square$

4. Applications

We now apply the majorants for λ_G derived in the previous section to quasiconformal mappings. Additional results of this type can be obtained by combining Lemma 3.6 or Theorem 3.9 with results from [LV]. We begin by proving a technical lemma.

4.1. Lemma. *Let x, y, $z \in B$ with $\varrho(z,x) \le \varrho(z,y)$. Let $u = \operatorname{th}\bigl(\varrho(x,y)/4\bigr)$, $s = \operatorname{th}\bigl(\varrho(z,x)/2\bigr), t = \operatorname{th}\bigl(\frac{1}{4}\bigl(\varrho(x,y) + \varrho(y,z) + \varrho(z,x)\bigr)\bigr)$. Then*

$$\frac{u}{s} \le \frac{(t-s)(1-st)}{s(1-t)^2}.$$

Proof. For each $s \in (0,1)$, $f(t) \equiv (t-s)(1-st)/\bigl(s(1-t)^2\bigr)$ is increasing on $(s,1)$. Now

$$t \ge \operatorname{th}\Bigl(\frac{1}{4}\bigl(\varrho(x,y) + 2\varrho(z,x)\bigr)\Bigr) = \frac{u+s}{1+us}.$$

Hence

$$f(t) \geq f\Big(\frac{u+s}{1+us}\Big) = \frac{u(1+s)^2}{s(1-u)^2} \geq \frac{u}{s}. \ \square$$

4.2. Theorem. *Let* $: B \to f(B) \subset B$ *be a* K*-quasiconformal mapping with* $f(0)=0$*. Then, for* $x,\, y \in B \setminus \{0\}$*,* $0 < |x| \leq |y|$*,*

$$\frac{\big|f(x)-f(y)\big|}{2} \leq \operatorname{th}\frac{\varrho(f(x),f(y))}{4} \leq \big|f(x)\big|\tau^{-1}\bigg(\frac{1}{K}\tau\Big(\frac{|y|\big(1+|x|\big)^2}{|x|\big(1-|y|\big)^2}\Big)\bigg).$$

Proof. It follows from [LV, (3.8)] and Lemma 3.6 that

$$\tau\Big(\frac{|y|\big(1+|x|\big)^2}{|x|\big(1-|y|\big)^2}\Big) \leq p(x,y) \leq Kp\big(f(x),f(y)\big) \leq K\tau\Big(\frac{(t-s)(1-ts)}{s(1-t)^2}\Big),$$

where $t = \operatorname{th}\frac{1}{4}\big(\varrho\big(f(y),0\big) + \varrho\big(f(x),f(y)\big) + \varrho\big(f(x),0\big)\big)$ and $s = \big|f(x)\big|$. We consider two cases. If $\big|f(x)\big| \leq \big|f(y)\big|$, then the asserted inequality follows from Lemma 4.1. If $\big|f(x)\big| > \big|f(y)\big|$, then by the previous case the inequality holds with $\big|f(x)\big|$ in place of $\big|f(y)\big|$. Since now $\big|f(y)\big| < \big|f(x)\big|$, the result follows. $\square$

We observe that by [V2, 7.53]

$$\tau^{-1}\Big(\frac{1}{K}\tau(t)\Big) = \frac{\varphi_K^2(r)}{\varphi_{1/K}^2(r')} \leq 16^{K-(1/K)}(1+t)^{K-(1/K)}\,t^{1/K}$$

for $K>1$ and $t>0$, where $r=\sqrt{t/(1+t)}$, $r'=\sqrt{1/(1+t)}$.

4.3. Proof of Theorem 1.3. With notation as in Theorem 4.1, we have

$$\begin{aligned}\lambda_G(x,y) \leq p(x,y) &\leq \tau\Big(\frac{(t-s)(1-ts)}{s(1-t)^2}\Big) \leq \tau\Big(\frac{\operatorname{th}\big(\varrho(x,y)/2\big)}{\big(1-\operatorname{th}\big(\varrho(x,y)/2\big)\big)^2}\,\frac{\big(1+|x|\big)^2}{|x|}\Big)\\ &\leq \tau\Big(\frac{|x-y|}{2\big(1-\frac{1}{2}|x-y|\big)^2}\,\frac{\big(1+|x|\big)^2}{|x|}\Big) \leq \tau\Big(\frac{|x-y|\big(1+|x|\big)^2}{2|x|}\Big). \ \square\end{aligned}$$

4.4. Remark. By conformal invariance, we can replace the expression

$$|y|\big(1+|x|\big)^2/\big(|x|(1-|y|)\big)^2$$

in Theorem 4.2 by $|T_x y|\big(1+|x|\big)^2/\big(|x|(1-|T_x y|\big)^2\big)$.

4.5. Proof of Theorem 1.5. Let $\Gamma' = \Delta\big(f[0,x], \partial f(B); f(B)\big)$ and let $\Gamma = f^{-1}(\Gamma')$. Then by Lemma 3.14,

$$M(\Gamma') \geq \tau\Big(\frac{\delta(1-s)^2}{s(1+\delta)^2}\Big); \qquad s = \big|f(x)\big|.$$

The inequality $M(\Gamma) \leq \gamma\big(1/|x|\big) = 2\tau\big((1/|x|^2)-1\big)$ is obvious. Since $M(\Gamma') \leq KM(\Gamma)$ (cf. [V2, 10.14]) we obtain

$$\frac{(1-s)^2}{s} \geq \frac{(1+\delta)^2}{\delta}\tau^{-1}\Big(2K\tau\Big(\frac{1}{|x|^2}-1\Big)\Big).$$

Using (2.2) we obtain

$$\frac{\big|f(x)\big|}{\big(1-\big|f(x)\big|\big)^2} \leq \frac{\delta}{(1+\delta)^2}\frac{\varphi_{2K}\big(|x|\big)^2}{1-\varphi_{2K}\big(|x|\big)^2}, \tag{4.6}$$

which proves the first inequality. The second inequality follows from [LV, Section 3.2, p. 64] and the fact that $g(t) \equiv t/(1+t)^2$ is strictly increasing on $[0,1]$.

4.7. Remark. Theorem 1.5 holds for quasiregular mappings as well if we make the additional assumption that $B \setminus f(B)$ contains a connected set E with $S(t) \cap E \neq \emptyset \neq E \cap S$.

5. Hyperbolic geometry

In this section we first give a direct proof of Theorem 3.5, then obtain an analog for non-Euclidean hyperbolas, and finally derive a relationship between n.e. ellipses and n.e. hyperbolas.

5.1. Proof of Theorem 3.5. By conformal invariance we may take $a = 1/\sqrt{r}$, $0 < r < 1$, and $b = -1$, $c = 1$ as foci. Thus we consider the conformal mapping [BF, 129.51], [N, p. 295, (49)]

$$w = f(z) = \mathrm{sn}(\zeta, r), \qquad \zeta = \frac{2i\mathcal{K}}{\pi}\log\frac{z}{t} + \mathcal{K}, \qquad z = x+iy, \qquad w = u+iv,$$

of the plane annulus $B \setminus \overline{B(t)}$ onto the ring $B(1/\sqrt{r}\,) \setminus [-1,1]$, $t, r \in (0,1)$, $t = \exp\big(-\pi\mathcal{K}'/(4\mathcal{K})\big)$, where $\mathcal{K}, \mathcal{K}'$ are elliptic integrals as in Section 2, and sn is the Jacobian elliptic sine function. Here there is no ambiguity in the logarithm, since we may first define the mapping f in the portion of the annulus in the first quadrant, using the principal branch of the logarithm, and then complete the mapping by reflecting in both axes. Let D be the quarter disk $\big\{w : |w| < 1/\sqrt{r}, u > 0, v > 0\big\}$. Under the mapping $\zeta = \zeta(z)$ the quarter circles $|z| = \tau$,

$x > 0,\ y > 0,\ t < \tau < 1$, correspond to the horizontal line segments joining the vertical sides of the rectangle

$$\mathcal{R} = \{\zeta = (\alpha, \beta) : 0 < \alpha < \mathcal{K}, 0 < \beta < \mathcal{K}'/2\}.$$

By symmetry it will be sufficient to prove that an arc in D is an arc of an n.e. ellipse with foci ± 1 if and only if it is the image of a horizontal line segment joining the vertical sides of $\mathcal{R}$.

Let $z \in B \setminus \overline{B(t)}$ be in the first quadrant, and let $\Omega = B(1/\sqrt{r}\,)$. If we denote

$$(5.2)\quad a = |\operatorname{sn}\zeta - 1|, \qquad A = |\operatorname{sn}\zeta - \frac{1}{r}|, \qquad \hat{a} = |\operatorname{sn}\zeta + 1|, \qquad \hat{A} = |\operatorname{sn}\zeta + \frac{1}{r}|,$$

then by [Bo, p. 46]

$$(5.3)\quad \begin{aligned} a &= \frac{1 - sd_1}{\sqrt{1 - d^2 s_1^2}}, & \hat{a} &= \frac{1 + sd_1}{\sqrt{1 - d^2 s_1^2}}, \\ A &= \frac{d_1 - rs}{r\sqrt{1 - d^2 s_1^2}}, & \hat{A} &= \frac{d_1 + rs}{r\sqrt{1 - d^2 s_1^2}}, \end{aligned}$$

where

$$\begin{aligned} s &= \operatorname{sn}(\alpha, r), & c &= \operatorname{cn}(\alpha, r), & d &= \operatorname{dn}(\alpha, r), \\ s_1 &= \operatorname{sn}(\beta, r'), & c_1 &= \operatorname{cn}(\beta, r'), & d_1 &= \operatorname{dn}(\beta, r'). \end{aligned}$$

Hence

$$\hat{a} + a = \frac{\hat{a} - a}{sd_1} = \frac{r(\hat{A} + A)}{d_1} = \frac{\hat{A} - A}{s} = \frac{2}{\sqrt{1 - d^2 s_1^2}}.$$

We then have by (5.3),

$$\begin{aligned} \varrho_1 &\equiv \varrho_\Omega(w, 1) = 2\operatorname{arth}\Big(\sqrt{r}\Big|\frac{1 - w}{1 - rw}\Big|\Big) = 2\operatorname{arth}\Big(\frac{a}{A\sqrt{r}}\Big), \\ \varrho_2 &\equiv \varrho_\Omega(w, -1) = 2\operatorname{arth}\Big(\sqrt{r}\Big|\frac{1 + w}{1 + rw}\Big|\Big) = 2\operatorname{arth}\Big(\frac{\hat{a}}{\hat{A}\sqrt{r}}\Big), \end{aligned}$$

so that

$$\frac{1 + \dfrac{a}{\sqrt{r}A}}{1 - \dfrac{a}{\sqrt{r}A}}\ \frac{1 + \dfrac{\hat{a}}{\sqrt{r}A}}{1 - \dfrac{\hat{a}}{\sqrt{r}A}} = \frac{(d_1 + \sqrt{r}\,)(1 - \sqrt{r}s)(d_1 + \sqrt{r}\,)(1 + \sqrt{r}s)}{(d_1 - \sqrt{r}\,)(1 + \sqrt{r}s)(d_1 - \sqrt{r}\,)(1 - \sqrt{r}s)} = \Big(\frac{d_1 + \sqrt{r}}{d_1 - \sqrt{r}}\Big)^2$$

by (5.3) and algebraic simplification. Thus

$$\varrho_1 + \varrho_2 = 2\log\frac{\operatorname{dn}(\beta, r') + \sqrt{r}}{\operatorname{dn}(\beta, r') - \sqrt{r}}$$

is constant if and only if β is constant. □

5.4. Theorem. *Let* f_{bc} *be a conformal mapping of a plane annulus* $B\setminus\overline{B(t)}$ *onto the ring* $B\setminus J[b,c]$. *A set* $E\subset B$ *is a non-Euclidean hyperbola with foci* b,c *if and only if* $f_{bc}(E)=L_\vartheta$ *for some* $\vartheta\in(0,\pi/2)$, *where*

$$L_\vartheta=\bigl\{z\in B\setminus\overline{B(t)}:|\operatorname{Arg}z|=\vartheta \text{ or } |\pi-\operatorname{Arg}z|=\vartheta\bigr\}.$$

Proof. By conformal invariance it is sufficient, as in the proof of Theorem 3.5, to take $a=1/\sqrt{r}$, $0<r<1$, and $b=-1$, $c=1$ as foci. Thus we consider the conformal mapping f of the plane annulus $B\setminus\overline{B(t)}$ onto the ring $B(1/\sqrt{r}\,)\setminus[-1,1]$, where $t,r\in(0,1)$ and $t=\exp\bigl(-\pi\mathcal{K}'/(4\mathcal{K})\bigr)$. With the same notation as in the previous proof, we have

$$\frac{1+\dfrac{a}{\sqrt{r}A}}{1-\dfrac{a}{\sqrt{r}A}}\,\frac{1-\dfrac{\hat a}{\sqrt{r}\hat A}}{1+\dfrac{\hat a}{\sqrt{r}\hat A}}=\frac{(d_1+\sqrt{r})(1-\sqrt{r}s)(d_1-\sqrt{r})(1-\sqrt{r}s)}{(d_1-\sqrt{r})(1+\sqrt{r}s)(d_1+\sqrt{r})(1+\sqrt{r}s)}=\Bigl(\frac{1-\sqrt{r}s}{1+\sqrt{r}s}\Bigr)^2.$$

Thus

$$\varrho_1-\varrho_2=2\log\frac{1-\sqrt{r}\,\operatorname{sn}(\alpha,r)}{1+\sqrt{r}\,\operatorname{sn}(\alpha,r)}$$

is constant if and only if α is constant. □

5.5. Theorem. *The right* (*left*) *branch of a non-Euclidean hyperbola in the unit disk* $|w|<1$ *is a subarc of a non-Euclidean ellipse in the right* (*left*) *half plane* H, $\operatorname{Re}w>0$ $(\operatorname{Re}w<0)$.

Proof. By symmetry it will be sufficient to prove the result for the right branch. In the right half plane the hyperbolic density is $|dw|/(\operatorname{Re}w)$, and the hyperbolic distance between two points w_1,w_2 in the right half plane H is

$$\varrho_H(w_1,w_2)=2\operatorname{arth}\Bigl|\frac{w_1-w_2}{w_1+\overline{w_2}}\Bigr|.$$

Let $\varrho_1=\varrho_H(w,1)$, $\varrho_2=\varrho_H(w,1/r)$. Then, using (5.3), we have

$$\frac{1+\dfrac{|w-1|}{|w+1|}}{1-\dfrac{|w-1|}{|w+1|}}\,\frac{1+\dfrac{|w-(1/r)|}{|w+(1/r)|}}{1-\dfrac{|w-(1/r)|}{|w+(1/r)|}}=\frac{\hat a+a}{\hat a-a}\,\frac{\hat A+A}{\hat A-A}=\frac{1}{r\bigl(\operatorname{sn}(\alpha,r)\bigr)^2}.$$

Thus $\varrho_1+\varrho_2=\varrho_H(w,1)+\varrho_H(w,1/r)$ is constant if and only if α is constant, that is, if and only if w is on the right branch of a fixed n.e. hyperbola in the unit disk. □

5.6. Theorem. *The right* (*left*) *branch of an n.e. ellipse in the unit disk* $|w|<1$ *is a subarc of an n.e. hyperbola in the right* (*left*) *half plane* H, $\operatorname{Re} w > 0$ $(\operatorname{Re} w < 0)$.

Proof. By symmetry it will be sufficient to prove the result for the right half of an n.e. ellipse. Using the notation of the previous theorem we have

$$\frac{1+\dfrac{|w-1|}{|w+1|}}{1-\dfrac{|w-1|}{|w+1|}}\,\frac{1-\dfrac{|w-(1/r)|}{|w+(1/r)|}}{1+\dfrac{|w-(1/r)|}{|w+(1/r)|}}=\frac{\hat{a}+a}{\hat{a}-a}\,\frac{\hat{A}-A}{\hat{A}+A}=\frac{r}{\big(\mathrm{dn}(\beta,r')\big)^2}.$$

Thus $\varrho_1-\varrho_2=\varrho_H(w,1)-\varrho_H(w,1/r)$ is constant if and only if β is constant, that is, if and only if w is on the right half of a fixed n.e. ellipse in the unit disk. □

References

[A1] Ahlfors, L.V.: Conformal invariants. - McGraw-Hill, New York, 1973.

[A2] Ahlfors, L.V.: Möbius transformations in several dimensions. - School of Mathematics, University of Minnesota, 1981.

[B] Beurling, A.: Collected works, Volume 1. - Birkhäuser, Boston, 1989.

[Bo] Bowman, F.: Introduction to elliptic functions with applications. - Dover, New York, 1961.

[BF] Byrd, P.F., and M.D. Friedman: Handbook of elliptic integrals for engineers and physicists. - Grundlehren Math. Wiss. 57. Springer-Verlag, Berlin–Göttingen–Heidelberg, 1954.

[GLM] Granlund, S., P. Lindqvist, and O. Martio: F-harmonic measure in space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 7, 1982, 233–247.

[H] Hersch, J.: Contribution à la theorie des fonctions pseudo-analytiques. - Comm. Math. Helv. 30, 1956, 1–19.

[K] Kühnau, R.: Geometrie der konformen Abbildung auf der hyperbolischen und elliptischen Ebene. - VEB Deutscher Verlag der Wissenschaften, Berlin, 1974.

[LV] Lehtinen, M., and M. Vuorinen: On Teichmüller's modulus problem in the plane. - Rev. Roumaine Math. Pures Appl. 33, 1988, 97–106.

[LeVi] Lehto, O., and K.I. Virtanen: Quasiconformal mappings in the plane. - Springer-Verlag, Berlin–Heidelberg–New York, 1973.

[LF] Lelong-Ferrand, J.: Invariants conformes globaux sur les varietes riemanniennes. - J. Differential Geom. 8, 1973, 487–510.

[M] Martio, O.: F-harmonic measures, quasihyperbolic distance and Milloux's problem. - Ann. Acad. Sci. Fenn. Ser. A I Math. 12, 1987, 151–162.

[N] Nehari, Z.: Conformal mapping. - McGraw Hill, New York, 1952.

[R] Rickman, S.: Asymptotic values and angular limits of quasiregular mappings of a ball. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 185–196.

[Vä] Väisälä, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Math. 229. Springer-Verlag, Berlin–Heidelberg–New York, 1971.

[V1] Vuorinen, M.: Capacity densities and angular limits of quasiregular mappings. - Trans. Amer. Math. Soc. 263, 1981, 343–354.

[V2] Vuorinen, M.: Conformal geometry and quasiregular mappings. - Lecture Notes in Math. 1319, Springer-Verlag, Berlin–Heidelberg–New York–London–Paris–Tokyo, 1988.

Received 22 January 1993