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ZETA FUNCTIONS OF RATIONAL FUNCTIONS ARE RATIONAL

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Abstract. We prove that if f is a rational function of degree at least 2 in the Riemann sphere then its zeta function $\exp\{\sum_{n=1}^{\infty} N_n t^n/n\}$ is a rational function. Here N_n denotes the number of distinct fixed points of the n^{th} iterate of f. Furthermore we show that all the zeros of the zeta function, if any, have modulus 1.

1. Introduction and results

1.1. Zeta functions have been defined and studied for a great variety of algebraic structures and dynamical systems. Suppose that N_n is a sequence of complex numbers, usually integers, defined for $n \geq 1$ and associated with an algebraic or geometric object. The corresponding *zeta function* is defined first as a formal power series

$$Z(t) = \exp\left\{\sum_{n=1}^{\infty} \frac{N_n t^n}{n}\right\}. \tag{1.1}$$

Then the question arises if this series has a positive radius of convergence, if it can be continued to a meromorphic function in the plane, if it perhaps represents a rational function, and what can be said about the location of the zeros and poles of Z. One also describes the case when Z is a rational function by saying that the numbers N_n can be determined from a finite amount of information, since a rational function is determined by finitely many parameters.

To give an example of a zeta function arising from an algebraic structure, let q be a power of a prime and let $\mathbf{F}_q$ be the finite field with q elements. Let $f(x_1, \ldots, x_n)$ be a homogeneous polynomial of degree d with coefficients in $\mathbf{F}_q$, such that the partial derivatives of f with respect to the x_i have no common projective zeros in any algebraic extension of $\mathbf{F}_q$. Then $Z(t)$ is defined by taking

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N_s to be the number of solutions to $f = 0$ in the projective space $P^n(\mathbf{F}_{q^s})$. Dwork [3] proved that then $Z(t)$ is a rational function of t. In fact,

$$Z(t) = \frac{P(t)^{(-1)^n}}{(1-t)(1-qt)\cdots(1-q^{n-1}t)}$$

where P is a polynomial whose degree can be given explicitly in terms of d and n. Deligne [2] proved that all the zeros of P have modulus $q^{-(n-1)/2}$, which was conjectured by A. Weil [12] and which is the statement corresponding to Riemann's hypothesis in this connection. One may consider the function $Z(q^{-s})$ for a complex variable s to get a closer analogy with Riemann's zeta function, and we see that all the zeros of $P(q^{-s})$ have real part $(n-1)/2$. For a more detailed survey of the above results, see e.g. [6, pp. 151–163]. Deligne [2] has obtained similar results, conjectured by Weil [12], for more general varieties over finite fields.

In the theory of dynamical systems, zeta functions have been considered for at least individual functions and flows. We shall only consider individual functions. Let X be a compact manifold, and let $f\colon X \to X$ be a continuous function. A point $x \in X$ is called a fixed point of f if $f(x) = x$. We denote the iterates of f by $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n \ge 2$. The (geometric) zeta function $Z_f(t)$ of f is defined by the right hand side of (1.1), taking N_n to be the number of *distinct* isolated fixed points of f^n (cf. [11, p. 764], [13, p. 335]). The algebraic zeta function, obtained by taking N_n to be the sum of the Lefschetz indices of the fixed points of f^n (when X and f are such that these indices are defined), is sometimes simpler. Properties of such zeta functions have been considered particularly for diffeomorphisms and other homeomorphisms, and it has been proved in many cases that the zeta function has a positive radius of convergence or is a rational function or a root of a rational function. We cannot provide a complete list of references to such results here, and merely refer to the survey [11] and to [8]. Noninvertible functions satisfying suitable side-conditions have been considered by some authors. Zhang [14] has proved that the zeta functions of expanding maps are rational.

It seems to me that zeta functions for rational functions mapping the Riemann sphere onto itself have not been previously studied, and I have not been able to find in the literature a more general result that would cover *all* rational functions as a special case. Smale's survey [11] on diffeomorphisms makes passing references to Julia's work [7] on the iteration of rational functions [11, pp. 792, 807], but not in connection with zeta functions.

It is the purpose of this paper to note that zeta functions of rational functions are rational, with all the zeros, if any, on the unit circle. In fact, we shall find a formula for such zeta functions. Furthermore, we raise the question of how to define zeta functions for transcendental entire or meromorphic functions.

1.2. Let f be a rational function of degree $d \ge 2$ defined in the extended complex plane or Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ where $\mathbf{C}$ denotes the complex plane. Thus f^n is a rational function of degree d^n, and so f^n has d^n+1 fixed points in $\overline{\mathbf{C}}$, with due count of multiplicity. We say that $z_0 \in \mathbf{C}$ is a fixed point of multiplicity m of a rational function g if g has the expansion

$$g(z) - z = a_m(z-z_0)^m + O\big((z-z_0)^{m+1}\big)$$

as $z \to z_0$, where $m \geq 1$ and $a_m \neq 0$. We say that g has the multiplier $\lambda = g'(z_0)$ at the fixed point z_0. It follows that we have $m \geq 2$ if and only if $\lambda = 1$. If $g(\infty) = \infty$, we consider $1/g(1/z)$ at the origin to determine the multiplicity and the multiplier at infinity. A fixed point of f^n whose multiplier is equal to 1 is called a *parabolic* fixed point. Thus the multiplicity of a fixed point is greater than 1 exactly for parabolic fixed points. A cycle of distinct points $z, f(z), \ldots, f^{p-1}(z)$ that are parabolic fixed points of some f^n, with $f^p(z) = z$, is called a *parabolic cycle* of f of *length* p. Then $(f^p)'(z)$ is a root of unity.

Let N_n be the number of distinct fixed points of f^n in $\overline{\mathbf{C}}$, so that $1 \leq N_n \leq d^n + 1$. As in the general theory of dynamical systems, the (geometric) *zeta function* Z_f of f is first defined as the formal power series

$$Z_f(t) = \exp\left\{\sum_{n=1}^{\infty} \frac{N_n t^n}{n}\right\}.$$

Since $N_n \leq d^n+1$, it follows that the radius of convergence of $\sum_{n=1}^{\infty} N_n t^n/n$ and, hence, that of Z_f, is at least $1/d$. If $N_n = d^n + 1$ for all $n \geq 1$ then

$$Z_f(t) = (1-dt)^{-1}(1-t)^{-1}. \tag{1.2}$$

One could define an algebraic zeta function of f by taking N_n to be the number of fixed points of f^n with due count of multiplicity. However, then $N_n = d^n + 1$ for all $n \geq 1$ so that this results in a rather trivial zeta function that is equal to $(1-dt)^{-1}(1-t)^{-1}$ for every rational function f of degree $d \geq 2$.

Theorem 1. *Let f be a rational function of degree d where $d \geq 2$. Then the zeta function of f is a rational function given by*

$$Z_f(t) \equiv \exp\left\{\sum_{n=1}^{\infty} \frac{N_n t^n}{n}\right\} = (1-dt)^{-1}(1-t)^{-1} \prod_{\text{parabolic cycles}} (1-t^{pq})^{\ell} \tag{1.3}$$

where the product is taken over all the (finitely many) parabolic cycles of f, and where p, q, and ℓ denote certain positive integers associated with and depending on these cycles. The integer p is the length of the cycle, and the multiplier of f^p at each point of the cycle is a primitive q^{th} root of unity. An empty product is taken to be equal to 1.

Consequently, all the zeros of Z_f, if any, occur at certain roots of unity and hence lie on the unit circle. The zeta function of f has a simple pole at the point $1/d$ and, possibly, at the point 1, and no other poles. There is a pole at the point 1 if, and only if, f has no parabolic cycles.

Thus the numbers N_n for a rational f can be deduced from a finite amount of information. Of course, this is to be expected, not only because f itself depends on finitely many parameters, but because N_n can differ from d^n+1 only if there are parabolic cycles, and because there can only be finitely many parabolic cycles. One may also note that an expanding rational function has no parabolic cycles so that its zeta function is given by (1.2).

2. Proof of Theorem 1

2.1. Let the assumptions of Theorem 1 be satisfied. We make use of the Fatou–Julia theory of iteration of rational functions. We define the set of normality $N(f)$ of f as the set of all $z \in \overline{\mathbf{C}}$ that have a neighbourhood U such that the family $\{\, f^n \mid U : n \geq 1 \,\}$ of the iterates of f restricted to U is a normal family. The Julia set $J(f)$ of f is defined by $J(f) = \overline{\mathbf{C}} \setminus N(f)$. Clearly $N(f)$ is open, while $J(f)$ is known to be a nonempty perfect set which coincides with $\overline{\mathbf{C}}$ or is nowhere dense in $\overline{\mathbf{C}}$. For the basic results in the iteration theory of rational functions based on these concepts we refer to the papers of Fatou [4, 5] and of Julia [7], and to the book of Beardon [1]. First we formulate two lemmas.

Lemma 1. *If α is a fixed point of f^p of multiplicity ν where $p \geq 1$ and $\nu \geq 2$, then for each $n \geq 1$, the point α is a fixed point of f^{np} of the same multiplicity ν. The same applies even if $\nu = 1$ provided that the multiplier $\lambda = (f^p)'(\alpha)$ is not a root of unity.*

If $\nu = 1$ then $\lambda \neq 1$. Now if λ is a primitive q^{th} root of unity, where $q \geq 2$, and if q does not divide n, then α is a fixed point of f^{np} of multiplicity 1.

Proof of Lemma 1. We provide the simple proof only for completeness. Let α be a fixed point of f^p of multiplicity ν. We may assume without loss of generality that $p = 1$ and that $\alpha = 0$. If $\nu \geq 2$ then $f(z) = z + Az^\nu + O(z^{\nu+1})$ as $z \to 0$, where $A \neq 0$, and so $f^n(z) = z + nAz^\nu + O(z^{\nu+1})$ as $z \to 0$. Thus $\alpha = 0$ is a fixed point of f^n of multiplicity ν.

If $\nu = 1$ and $f'(0) = \lambda$ then $f(z) = \lambda z + O(z^2)$. Therefore we have $f(z) - z = (\lambda - 1)z + O(z^2)$ while $f^n(z) - z = (\lambda^n - 1)z + O(z^2)$ as $z \to 0$. If λ is not a root of unity then $\lambda^n - 1 \neq 0$ for all $n \geq 1$ and therefore $\alpha = 0$ is a fixed point of f^n of multiplicity 1, for each $n \geq 1$. Next if $\nu = 1$ (for f), we have $\lambda \neq 1$. Now if $\lambda^q = 1$ while $\lambda^j \neq 1$ for $1 \leq j \leq q-1$ then $\lambda^n = 1$ if, and only if, q divides n. So if q does not divide n then $\alpha = 0$ is a fixed point of f^n of multiplicity 1. This proves Lemma 1.

Lemma 2. *Suppose that $f^p(\alpha) = \alpha$ and $(f^p)'(\alpha) = \lambda$ where $p \geq 1$ and λ is a primitive q^{th} root of unity for some $q \geq 1$. Define $\alpha_j = f^j(\alpha)$ and suppose that the α_j are distinct and that $\alpha_j \in \mathbf{C}$ for $0 \leq j \leq p-1$. Then*

$$f^{pq}(z) = \alpha_j + (z - \alpha_j) + b_{m,j}(z - \alpha_j)^m + O\big((z - \alpha_j)^{m+1}\big) \tag{2.1}$$

for z in a neighbourhood of α_j where $b_{m,j} \neq 0$, $m \geq 2$, the number m is independent of j, and $m = 1 + \ell q$ for some integer $\ell \geq 1$. There are pairwise disjoint components $U_{i,j}$ of $N(f)$ for $1 \leq i \leq \ell q$ and $0 \leq j \leq p-1$ such that $\alpha_j \in \partial U_{i,j}$ and

(i) $f^{pq}(U_{i,j}) \subset U_{i,j}$;
(ii) $f^{np}(z) \to \alpha_j$ *locally uniformly for* $z \in U_{i,j}$ *as* $n \to \infty$.

For any z in any $U_{i,j}$, the values $f^{np}(z)$ visit q of the $U_{r,j}$ cyclically, namely those with $r \equiv i \pmod{\ell}$ provided that the $U_{i,j}$ are properly labelled. Thus for each j, the domains $U_{i,j}$ are divided into ℓ cycles, each consisting of q domains. The function f maps such a cycle of domains $U_{i,j}$ into another cycle of domains $U_{i,j+1}$ where $U_{i,p} = U_{i,0}$. In each cycle of domains there is a component of $N(f)$ that contains a critical point of f, that is, a point at which f is not locally homeomorphic.

Furthermore, all these domains $U_{i,j}$ are disjoint, that is, $U_{i,j} \cap U_{r,s} = \emptyset$ unless $(i,j) = (r,s)$ where $1 \leq i$, $r \leq \ell q$ and $0 \leq j$, $s \leq p-1$.

Proof of Lemma 2. The statements in Lemma 2 other than the one involving critical points follow from the discussion in the paper of Fatou [4, pp. 217–220] (cf. [1, Theorems 6.5.4 and 6.5.8, pp. 116, 124]). In particular, the fact that $m-1$ is divisible by q is obtained by considering the identity $f^{pq} \circ f^p = f^p \circ f^{pq}$ for power series at α_j. The fact that m is independent of j, can be seen by considering the set $N(f)$ close to the points α_j, as Fatou does. We mention that alternatively, the independence of m can be proved by purely formal calculations based on the chain rule, without referring to $N(f)$, but we omit the details. As was proved by Fatou and by Julia (cf. [1, Theorem 9.3.2, p. 194]), in each cycle of domains there is a component of $N(f)$ that contains a critical point of f. This proves Lemma 2.

2.2. We continue with the proof of Theorem 1. Suppose that w is a fixed point of f^p where $p \geq 1$. We may assume that w is finite and that z is finite whenever $z = f^n(w)$ or $f^n(z) = w$ for some positive integer n, as the analysis is the same in the general case. Choosing p to be as small as possible, we see that then w is a fixed point of f^n if, and only if, $n = kp$ for some positive integer k. The point w has multiplicity 1 as a fixed point of f^{kp} unless the multiplier of f^{kp} at w equals 1. By Lemma 1, w has multiplicity 1 as a fixed point of f^{kp} for all $k \geq 1$ unless the multiplier λ of f^p at w is a root of unity, say $\lambda = e^{2\pi i K/q}$ where the positive integers K and q are relatively prime. Then w has multiplicity at least 2 as a fixed point of f^{kp} exactly when $k = mq$ for some positive integer m.

We have

$$f^{pq}(z) - w = (z-w) + a_s(z-w)^s + O\big((z-w)^{s+1}\big)$$

as $z \to w$, where $s \geq 2$ and $a_s \neq 0$. Here s is the multiplicity of w as a fixed point of f^{pq}. Furthermore, by Lemma 2, $s-1$ is divisible by q, so $s = 1 + \ell q$

for some positive integer ℓ. By Lemma 1, w is a fixed point of multiplicity s of f^{mpq}, for each $m \geq 1$.

The function f^n has d^n+1 fixed points, with due count of multiplicity. The number of distinct fixed points of f^n is therefore d^n+1-N where N is the sum of the numbers $\nu-1$ for the multiple fixed points of f^n where ν is the multiplicity of such a point as a fixed point of f^n. Thus, corresponding to the fixed point w of f^{mpq}, we have to subtract $s-1$, that is, ℓq, from $d^{mpq}+1$.

Consider the points $w_j = f^j(w)$ for $j \geq 0$. The points $w = w_0, w_1, \ldots, w_{p-1}$ are distinct while $w_p = w$. Suppose that $0 \leq j \leq p-1$. Then w_j is a fixed point of f^p with the same multiplier as w since by the chain rule,

$$(f^p)'(w) = \prod_{i=0}^{p-1} f'(w_i).$$

Hence each w_j is also a fixed point of f^{kp} for all $k \geq 1$, and is a fixed point of multiplicity at least 2 if, and only if, k is an integral multiple of q. By Lemma 2, each w_j has the same multiplicity s as w as a fixed point of f^{mpq}, for all $m \geq 1$. We deduce that corresponding to the cycle $w = w_0, w_1, \ldots, w_{p-1}$ of fixed points of f^p, we need to subtract altogether $pq\ell$ from $d^{mpq}+1$ when dealing with the fixed points of f^{mpq}.

2.3. The cycle $w = w_0, w_1, \ldots, w_{p-1}$ of fixed points of f^p determines a parabolic cycle since each of these points is a parabolic fixed point of f^{pq}. By Lemma 2, for each j with $0 \leq j \leq p-1$, there are ℓq components $U_{i,j}$ of $N(f)$ with $w_j \in \partial U_{i,j}$ where $1 \leq i \leq \ell q$, such that $f^{mpq}(z) \to w_j$ as $m \to \infty$, locally uniformly for $z \in U_{i,j}$. The action of f divides these components into ℓ cycles, each consisting of pq components permuted by f. Further by Lemma 2, at least one component in each cycle contains a critical point of f. Since f has at most $2d-2$ distinct critical points, it follows that there are only finitely many (at most $2d-2$) parabolic cycles.

So we see that

$$\begin{aligned}
\sum_{n=1}^{\infty} N_n \frac{t^n}{n} &= \sum_{n=1}^{\infty} (d^n+1)\frac{t^n}{n} \; - \sum_{\text{par. cycles}} \sum_{m=1}^{\infty} pq\ell \frac{t^{mpq}}{mpq} \\
&= \sum_{n=1}^{\infty} \frac{(dt)^n}{n} + \sum_{n=1}^{\infty} \frac{t^n}{n} \; - \sum_{\text{par. cycles}} \ell \sum_{m=1}^{\infty} \frac{(t^{pq})^m}{m} \\
&= -\log(1-dt) - \log(1-t) \; + \sum_{\text{par. cycles}} \ell \log(1-t^{pq}).
\end{aligned}$$

If there are no parabolic cycles, we consider the resulting empty sum to be equal to zero. Similarly, we shall consider an empty product to be equal to 1. It follows that

$$Z_f(t) = \exp\Biggl\{\sum_{n=1}^{\infty} \frac{N_n t^n}{n}\Biggr\} = (1-dt)^{-1}(1-t)^{-1} \prod_{\text{par. cycles}} (1-t^{pq})^{\ell}.$$

This gives (1.3), and this expression obviously proves all the remaining statements of Theorem 1. Thus the proof of Theorem 1 is complete.

Remark 1. Let f be a continuous, open, discrete mapping of $\overline{\mathbf{C}}$ onto itself. By Stoïlow's theorem, we may write $f = \varphi \circ h$ where φ is a rational function and h is a homeomorphism. Every iterate of f is clearly also of this type. We define N_n as before, counting only isolated fixed points of f^n. Note that already the set of fixed points of f could be quite complicated since h is an arbitrary homeomorphism. One may ask to what extent the conclusions of Theorem 1 remain valid. If f is topologically conjugate to a rational function g, that is, if $f = H \circ g \circ H^{-1}$ where H is a homeomorphism, then clearly N_n is the same for f and g so that $Z_f \equiv Z_g$ and the conclusions of Theorem 1 hold for f. More generally, one may ask, in particular, what the situation is if it is assumed that for every $n \geq 1$, the fixed points of f^n are isolated.

Remark 2. One can ask if there is any reasonable way to define a zeta function for a transcendental entire or meromorphic function in the plane. The problem is to define a formal power series whose properties could then be studied. Of course, now f^n has infinitely many fixed points except possibly for $n = 1$. If one tries to approach the question by approximation, one might first consider finitely many f^n in a disk centred at the origin of radius r and then let the number of iterates as well as r tend to infinity. Or one could consider a sequence f_n of rational functions such that for $1 \leq k \leq N = N(n)$, the function f_n^k has the same fixed points as f^k in a disk centred at the origin of radius r_n, with due count of multiplicity. We assume that $N(n)$ and r_n tend to infinity with n. One can ask if there is any way to choose the f_n and then normalize the zeta functions $Z_{f_n}(t)$, possibly after substituting for t a suitable expression depending on n, so that the sequence of normalized functions tends to a limit that one could define to be the zeta function of f.

Another possibility is to set, for example,

$$Z_f(t) = \exp\Biggl\{\sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{f^n(z)=z} \frac{1}{\max\{1, |(f^n)'(z)|\}}\Biggr\}$$

or possibly

$$Z_f(t) = \exp\Biggl\{\sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{f^n(z)=z} \frac{1}{\bigl((f^n)'(z)\bigr)^{+}}\Biggr\}$$

where for a complex number c, we write $c^+ = c$ if $|c| \geq 1$ and $c^+ = 1$ otherwise, provided that the sum over the distinct fixed points of f^n converges for each $n \geq 1$. More generally, if ψ is a suitable function defined in the complex plane, such as $\psi(z) = \max\{1, |z|^\alpha\}$ for some $\alpha > 0$, we could set

$$Z_f(t) = \exp\biggl\{\sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{f^n(z)=z} \frac{1}{\psi\bigl((f^n)'(z)\bigr)}\biggr\} \tag{2.2}$$

or perhaps

$$Z_f(t) = \exp\biggl\{\sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{f^n(z)=z} \prod_{j=0}^{n-1} \frac{1}{\psi\bigl(f'\bigl(f^j(z)\bigr)\bigr)}\biggr\}.$$

It might be of some interest to determine if such sums converge for all f in a reasonable class of functions. In particular, one may ask if (2.2) makes sense for $f(z) = e^z$ whenever $\psi(z) = \max\{1, |z|^\alpha\}$ and $\alpha > 1$. In general, one might expect that some choice for ψ depending on (the growth of) f might yield a well defined zeta function for that f. These formulas are somewhat analogous to Ruelle's definition of a zeta function for expanding maps on compact manifolds [9, 10].

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