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# ON THE BEHAVIOR OF MEROMORPHIC FUNCTIONS AROUND SOME NONISOLATED SINGULARITIES

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**Abstract.** Let $G$ be a plane domain, let $E \subset G$ be a Painlevé null set and let $f$ be meromorphic in $G \setminus E$ having at least one essential singularity in $E$. Let $\psi$ be an exhaustion function of $G \setminus E$, i.e., $\psi(z)$ tends to infinity as $z$ tends to $E$. We first show that there is a sequence $(r_n)$ with $r_n \to \infty$ such that the spherical lengths of the images of the level sets $\psi^{-1}(r_n)$ exceed $\pi$. We also derive some estimates for the growth of the spherical derivative of $f$ as $z$ tends to the singularity set. Our result relates the growth rate of the spherical derivative with the Minkowski dimension of the singularity set.

**1.** Let $f$ be a meromorphic function with an isolated singularity at $0$, and let $f^*$ stand for the spherical derivative of $f$, i.e., $f^*(z) = |f'(z)|/(1+|f(z)|^2)$. Then

$$\limsup_{z\to 0} |z| f^*(z) \geq \tfrac{1}{2} \tag{1}$$

and the constant $\frac{1}{2}$ is sharp [8, Theorem 1]. Lehto's proof consists in showing that for some values of $r$, tending to $0$, the circle $|z| = r$ contains two points which are mapped into approximately antipodal points on the Riemann sphere. This implies that $\pi$, the length of the great circles of the Riemann sphere, is an approximate lower bound for the spherical length of the images of these circles. Integration then immediately yields (1). Gauld and Martin [5] have recently clarified the picture by showing that the word "approximately" is actually superfluous in the above description. Their result is general enough to cover the quasimeromorphic mappings in $R^n$.

In this paper we make an attempt to extend these results to meromorphic functions around certain nonisolated singularities. Suppose $G$ is a plane domain, $E \subset G$ is a Painlevé null set, i.e., a set of class $N_B$ in the notation of [1], and $f$ is meromorphic in $G \setminus E$. Let $\psi$ be a reasonably well-behaved exhaustion function of $G \setminus E$. We first show that there is a sequence $(r_n)$, tending to infinity, such that the spherical lengths of the images of the level sets $\psi^{-1}(r_n)$ exceed $\pi$. We then specialize to the case $\psi(z) = d(z,E)^{-1}$, where $d(z,E)$ stands for the distance from $z$ to $E$, and derive some estimates for the growth of the spherical derivative as $z$ tends to $E$. In order to control the growth of the length of the level sets, we have to impose restrictions on the geometry of $E$. The concept of Minkowski dimension appears to be a proper device for this purpose.

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**2.** Let $G \subset \mathbf{C}$ be a bounded domain and let $E \subset G$ be a compact totally disconnected set. Let $\psi$ be a continuous nonnegative function in $\overline{G} \setminus E$ such that $\psi(z) \to \infty$ as $d(z,E) \to 0$. We say that $\psi$ is an exhaustion function of $G \setminus E$ relative to $E$ provided that the following condition holds:

(2) For large $r$, let $G_r$ denote the component of $\{\, z \in G \mid \psi(z) < r \,\}$ with $\partial G \subset \partial G_r$. Then $\partial G_r \setminus \partial G$ consists of a finite number of components $C_r^1, \dots, C_r^{k_r}$, each of which admits a parametrization as a closed rectifiable curve. Furthermore, $l(C_r^i)$, the length of $C_r^i$, equals $H^1(C_r^i)$, the 1-dimensional Hausdorff measure of $C_r^i$, $i = 1, \dots, k_r$.

In what follows, the boundary curves are usually thought of as parametrized in this manner.

**Examples.** (1) Set $\psi(z) = d(z,E)^{-1}$, $z \in \overline{G} \setminus E$. A close examination of Brown's paper [2] (see also Section 3) reveals that $\psi$ fulfils condition (2). Observe that for all but a countable number of $r$ (for large $r$), the components of $\partial G_r \setminus \partial G$ are simple Jordan curves [2, Theorem 3].

(2) Suppose $E$ is of logarithmic capacity zero. Then there is a positive harmonic function $\psi$ in $G$ such that $\psi(z) \to \infty$ as $z \to E$; $\psi$ is the so-called Evans–Selberg potential. Again (2) holds true.

Let $F$ be a closed (parametrized) curve or a finite union of disjoint simple arcs in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then $s(F)$ stands for the spherical length of $F$.

**Theorem 1.** *Let $G \subset \mathbf{C}$ be a bounded domain, let $E \subset G$ be a compact set of class $N_B$, and let $f$ be meromorphic in $G \setminus E$ having at least one essential singularity in $E$. Let $\psi$ be an exhaustion function of $G \setminus E$ relative to $E$, and set $\partial G_r \setminus \partial G = C_r^1 \cup \dots \cup C_r^{k_r}$* (cf. (2)). *Then there is a sequence $\bigl(r(n)\bigr)$, tending to infinity, such that*

$$\sum_{i=1}^{k_{r(n)}} s\bigl(f(C_{r(n)}^i)\bigr) \geq \pi \qquad \text{for all } n.$$

*Proof.* Suppose there is $r_0$ such that $\sum_{i=1}^{k_r} s\bigl(f(C_r^i)\bigr) < \pi$ for all $r \geq r_0$. In particular, $S = \sum_{i=1}^{k_{r_0}} s\bigl(f(C_{r_0}^i)\bigr) < \pi$. By Crofton's theorem (see e.g. [3, p. 33]), the measure of the (oriented) great circles of the Riemann sphere, which meet $F = \bigcup_{i=1}^{k_{r_0}} f(C_{r_0}^i)$, each counted a number of times equal to the number of its common points with $F$ (taken as a union of parametrized curves), is equal to two times the total length of $F$ $(= 2S)$ (observe the difference in scaling!). The measure referred to above is the area measure of the Riemann sphere arising via identification of an oriented great circle with the intersection point of the sphere and the positive normal to the plane of the circle. Since $F$ consists of closed curves, almost all great circles, which meet $F$, intersect $F$ in at least two points. It follows that there is a great circle, say $K$, such that $K \cap F = \emptyset$.

Since $f$ has at least one essential singularity in $E$ and $E$ is of class $N_B$, $f$ assumes in any neighborhood of $E$ all values of $\hat{\mathbf{C}}$ except for those of a Painlevé null set. On the other hand, it is known that a linear set is a Painlevé null set if and only if it is of linear measure zero [1, Theorem 11]. Hence it makes sense to define

$$r' = \inf\{\, r \mid r > r_0 \text{ and } s\big(f(G_r \setminus \overline{G}_{r_0}) \cap K\big) \geq \tfrac{1}{2}\pi = \tfrac{1}{2}s(K)\,\}.$$

Let $K_i$ be the shortest spherical segment in $K$ which contains $f(C^i_{r'}) \cap K$, $i = 1, \dots, k_{r'}$. Obviously $\sum_{i=1}^{k_{r'}} s(K_i) < \frac{1}{2}\pi$. Since $s\big(K \setminus \bigcup_{i=1}^{k_{r'}} K_i\big) > \frac{1}{2}\pi$, there is, by the choice of $r'$, at least one point, say $p$, in $K \setminus \bigcup_{i=1}^{k_{r'}} K_i$, which is not assumed by $f$ in $G_{r'} \setminus \overline{G}_{r_0}$. By rotating the Riemann sphere, we may assume that $p = \infty$; in other words, we may regard $f$ as holomorphic in $\overline{G}_{r'} \setminus G_{r_0}$. We now apply the argument principle to $f$ in $\overline{G}_{r'} \setminus G_{r_0}$. Clearly the winding number of each curve $f(C^i_{r_0})$, $i = 1, \dots, k_{r_0}$, with respect to any point of $K$ is zero. The same is true of each curve $f(C^i_{r'})$, $i = 1, \dots, k_{r'}$, with respect to any point of $K \setminus \bigcup_{i=1}^{k_{r'}} K_i$. It follows that $f$ omits all points of $K \setminus \bigcup_{i=1}^{k_{r'}} K_i$ in $\overline{G}_{r'} \setminus G_{r_0}$. But since $s\big(K \setminus \bigcup_{i=1}^{k_{r'}} K_i\big) > \frac{1}{2}\pi$, this state of affairs is in disagreement with the choice of $r'$. The proof is complete. □

**Remarks.** (1) In [3], Crofton's theorem is proved only for smooth curves, but, as observed in [3, p. 34], the theorem is true also for (unions of) rectifiable curves.

(2) Suppose $E$ fails to be of class $N_B$. Then $\hat{\mathbf{C}} \setminus E$ tolerates nonconstant bounded holomorphic functions. By scaling, we can then shorten the length of the image curves as much as we please. Hence it is highly plausible that one cannot extend the above theorem beyond the class $N_B$.

**3.** We are going to employ Theorem 1 for deriving estimates for the growth of the spherical derivative of meromorphic functions in terms of $d(z, E)$. For this reason, we henceforth specialize to the case $\psi(z) = d(z, E)^{-1}$. Unfortunately, the level sets of $\psi$ may be extremely intricate even if $E$ is only countable. Hence we will measure the size of sets in terms of the Minkowski content or dimension, which are more sensitive to irregularities in the distribution of points than, say, the Hausdorff measure and dimension.

Let $E \subset \mathbf{C}$ be a compact nonempty set and set $E(r) = \{\, z \in \mathbf{C} \mid d(z, E) < r\,\}$ for $r > 0$. Obviously, $\partial E(r) = \psi^{-1}(1/r)$. Further, let $m$ stand for the area measure and set $B(z_0, r) = \{\, z \in \mathbf{C} \mid |z - z_0| < r\,\}$. Let $\alpha \geq 0$. The $\alpha$-dimensional (upper) Minkowski content and the Minkowski dimension of $E$ are defined to be

$$M_\alpha(E) = \limsup_{r \to 0} \frac{m\big(E + B(0, r)\big)}{r^{2-\alpha}} \qquad \text{and}$$

$$\dim_M(E) = \sup\{\, \alpha \geq 0 \mid M_\alpha(E) = \infty\,\},$$

respectively; see e.g. [9, Section 3]. It is known that $\dim_M(E) \ge \dim_H(E)$ (= the Hausdorff dimension of $E$) for any compact set $E \subset \mathbf{C}$, but for quite a large class of sets, including the self-similar sets, the two numbers are equal [9, Theorem 4.19].

Martio and Vuorinen have exhibited a connection between the Minkowski dimension of $E$ and the distribution of squares in the Whitney decomposition of $\mathbf{C} \setminus E$. Recall that the Whitney decomposition of $\mathbf{C} \setminus E$ is a representation

$$\mathbf{C} \setminus E = \bigcup_{k \in Z} \bigcup_{j=1}^{n_k} Q_j^k, \tag{3}$$

where each $Q_j^k$ is a closed dyadic square with sides parallel to the coordinate axes, and the side length of $Q_j^k$ is $2^{-k}$. Furthermore, the interiors of the squares are pairwise disjoint, and $d(Q_j^k, E)$, the distance of $Q_j^k$ and $E$, satisfies

$$2^{-k}\sqrt{2} \le d(Q_j^k, E) \le 4 \cdot 2^{-k}\sqrt{2} \tag{4}$$

[12, pp. 167–168], [9, Section 2].

We require an upper bound for the length of the part of $\partial E(r)$ lying in a single Whitney square.

**Lemma.** *Let $Q \subset \mathbf{C}$ be the closed square with vertices at the points $\pm\frac{1}{2} \pm \frac{1}{2}i$ and let $E \subset \mathbf{C}$ be a nonempty compact set with $d(0, E) \ge \frac{1}{2} + \sqrt{2}$. Then $H^1\bigl(\partial E(r) \cap Q\bigr) < 10$ for all $r > 0$.*

*Proof.* Fix $r > 0$ and suppose that $F = \partial E(r) \cap Q \ne \emptyset$. Let $S_j$ stand for the cone $\{\, z \in \mathbf{C} \mid \frac{1}{4}(2j-1)\pi \le \arg z \le \frac{1}{4}(2j+1)\pi \,\}$ and set $F_j = \{\, z \in F \mid |z - w| = r \text{ for some } w \in E \cap S_j \,\}$, $j = 0, 1, 2, 3$. Then each $F_j$ is closed and $\bigcup_{j=0}^{3} F_j = F$.

Fix $z_0 \in F_0$ and pick out a point $w_0 \in E \cap S_0$ such that $|z_0 - w_0| = r$. Clearly $F \cap B(w_0, r) = \emptyset$. Let $w_1$ and $w_2$ be the points on the rays $\arg z = -\frac{1}{4}\pi$ and $\arg z = \frac{1}{4}\pi$, respectively, such that $|w_j - z_0| = r$, $j = 1, 2$. Set $r_j = \max\{\frac{1}{2} + \sqrt{2}, |w_j|\}$, $j = 1, 2$, and $w_1' = r_1 e^{-i\pi/4}$, $w_2' = r_2 e^{i\pi/4}$. Obviously $(E \cap S_0) \cap \bigl(B(0, \frac{1}{2} + \sqrt{2}\,) \cup B(z_0, r)\bigr) = \emptyset$.

A moment's reflection reveals that $F_0 \subset \bigl(\overline{B}(w_1', |w_1' - z_0|) \cup \overline{B}(w_2', |w_2' - z_0|)\bigr) \setminus B(w_0, r)$. It follows that $F_0$ may be regarded as the graph of a function, defined in a closed part of a closed interval of length 1, which satisfies a Lipschitz condition. By considering the obvious extremal positions, one realizes that the Lipschitz constant may be chosen to be $5 - 2\sqrt{2}$. Hence $H^1(F_0) \le \sqrt{1 + (5 - 2\sqrt{2}\,)^2} < 2\frac{1}{2}$. Of course, the same is true of any $F_j$, $j = 0, 1, 2, 3$, so that $H^1(F) \le \sum_{j=0}^{3} H^1(F_j) < 10$ as was asserted. ▫

**Corollary 1.** *Let $E \subset \mathbf{C}$ be a nonempty compact set and let $Q$ be a square of side length $2^{-k}$ in the Whitney decomposition of $\mathbf{C} \setminus E$. Then $H^1\bigl(\partial E(r) \cap Q\bigr) < 10 \cdot 2^{-k}$ for all $r > 0$.*

Let $E \subset R^n$ $(n > 1)$ be a nonempty bounded set. Then there is a constant $A$, depending only on $n$ and the diameter of $E$, such that $H^{n-1}\big(\partial E(r)\big) < Ar^{-1}$ for sufficiently small $r$ [11, p. 422]. Owing to the previous lemma, we are able to improve this result (at least in dimension 2) as follows.

**Corollary 2.** *Let $E \subset \mathbf{C}$ be a nonempty compact set. Then* $\lim_{r\to 0} rH^1\big(\partial E(r)\big) = 0$.

*Proof.* Fix $r > 0$ and let $k = k(r)$ denote the unique integer such that $2^{-k+1} \le r < 2^{-k+2}$. Suppose $Q_j^m \cap \partial E(r) \ne \emptyset$ (cf. (3)). Then by (4) $r - \sqrt{2}\,2^{-m} \le 4\sqrt{2}\,2^{-m}$, whence $2^{-k+1} \le r \le 5\sqrt{2}\,2^{-m}$. Also by (4) $\sqrt{2}\,2^{-m} \le r < 2^{-k+2}$. It follows that $k-1 \le m \le k+1$. In other words, there are only three "generations" of squares, which may have elements intersecting $\partial E(r)$.

Let $n_m$ be the number of the squares in the $m^{\text{th}}$ generation. As an expression for the area of a bounded plane set the series $\sum_{m=1}^{\infty} n_m 2^{-2m}$ is convergent. Hence

$$c_m = n_m 2^{-2m} \to 0 \qquad \text{as } m \to \infty. \tag{5}$$

Invoking the preceding lemma we get

$$H^1\big(\partial E(r)\big) < 10n_{k-1}2^{-k+1} + 10n_k 2^{-k} + 10n_{k+1}2^{-k-1}$$

and

$$\begin{aligned} rH^1\big(\partial E(r)\big) &< 10n_{k-1}2^{-k+2}\cdot 2^{-k+1} + 10n_k 2^{-k+2}\cdot 2^{-k} + 10n_{k+1}2^{-k+2}\cdot 2^{-k-1} \\ &= 20n_{k-1}2^{-2(k-1)} + 40n_k 2^{-2k} + 80n_{k+1}2^{-2(k+1)}. \end{aligned}$$

Since $r \to 0$ implies $k \to \infty$, we infer from (5) that $\lim_{r\to 0} rH^1\big(\partial E(r)\big) = 0$. □

**Remark.** It should be clear that the corresponding results are valid in any $n$-space, $n \ge 2$.

Let $h$: $(0,\infty) \to (0,\infty)$ be an increasing function. Following Martio and Vuorinen, we say that $E$ satisfies the Whitney square #-condition with the function $h$ if there exists $k_0 \ge 0$ such that $n_k \le h(k)$ for $k \ge k_0$; here $n_k$ is as in (3). This definition gives rise to the following rather implicit consequence of Theorem 1.

**Theorem 2.** *Let $G \subset \mathbf{C}$ be a domain, let $h$: $(0,\infty) \to (0,\infty)$ be increasing and let $E \subset G$ be a compact set of class $N_B$, which satisfies the Whitney square #-condition with $h$. Let $f$ be meromorphic in $G \setminus E$ having at least one essential singularity in $E$. Then*

$$\limsup_{d(z,E)\to 0} f^*(z)d(z,E)h\left({}^2\log\frac{8}{d(z,E)}\right) \ge \frac{2\pi}{35}.$$

*Proof.* Suppose there is $r_0 > 0$ such that

$$f^*(z) < \frac{2\pi}{35d(z,E)h\left({}^2\log\dfrac{8}{d(z,E)}\right)} \qquad \text{for } d(z,E) \le r_0.$$

Fix $z \in G\backslash E$ such that $r = d(z,E) \le r_0$ and pick $k \in Z$ with $2^{-k+1} \le r < 2^{-k+2}$. By Corollary 1 and by the proof of Corollary 2,

$$\begin{aligned} H^1\big(\partial E(r)\big) &< 10\cdot 2^{-k+1}h(k-1) + 10\cdot 2^{-k}h(k) + 10\cdot 2^{-k-1}h(k+1) \\ &= 20\cdot 2^{-k}h(k-1) + 10\cdot 2^{-k}h(k) + 5\cdot 2^{-k}h(k+1) \\ &\le 35\cdot 2^{-k}h(k+1) \le \frac{35}{2}rh\left({}^2\log\frac{8}{r}\right). \end{aligned}$$

It follows that $s\big(f(\partial E(r)\big) < \pi$ (cf. (2)). But this contradicts Theorem 1. □

Consider the case $M_\alpha(E) < \infty$ for some $0 \le \alpha < 1$. Since $\dim_H(E) \le \dim_M(E) < 1$, $E$ is of class $N_B$. Furthermore, by [9, Theorem 3.11] $E$ satisfies the Whitney square #-condition with $h(t) = A2^{\alpha t}$ for some $A < \infty$. It follows from Theorem 2 that

$$\limsup_{d(z,E)\to 0} f^*(z)d(z,E)^{1-\alpha} > 0.$$

However, it turns out that in case $\alpha > 0$ much better estimates can be derived making use of the fact that sets of class $N_D$ are removable singularities for meromorphic functions with a finite spherical Dirichlet integral [6, Theorem 2]. Recall that by definition $E \in N_D$ provided that $E$ is a null set for holomorphic functions with a finite Dirichlet integral. Hence Theorem 2 is of interest only in case $\dim_M(E) = 0$. We return to this case at the end of the paper.

**Theorem 3.** *Let $G \subset \mathbf{C}$ be a domain and let $E \subset G$ be a compact set of class $N_D$ with $M_\alpha(E) < \infty$ for some $\alpha \in [0,2)$. Let $f$ be meromorphic in $G \setminus E$ with at least one essential singularity in $E$. Then*

$$\limsup_{d(z,E)\to 0} f^*(z)d(z,E)^{\beta} = \infty \qquad \text{for all } \beta < 1 - \tfrac{1}{2}\alpha.$$

*Proof.* Suppose, on the contrary, that there are positive constants $C$ and $r_0$ such that $f^*(z)d(z,E)^\beta \le C$ for $z \in G \setminus E$ with $r = d(z,E) \le r_0$ and for some $\beta < 1 - \frac{1}{2}\alpha$. Using [9, Theorem 3.11] as above we see that $H^1\big(\partial E(r)\big) \le Ar^{1-\alpha}$ for some $A < \infty$. Hence by [4, Lemma 3.2.34]

$$\begin{aligned} \iint_{E(r_0)\backslash E} \big(f^*(z)\big)^2\,dx\,dy &= \int_0^{r_0}\left(\int_0^{H^1(\partial E(r))}\big(f^*(z)\big)^2\,ds\right)dr \\ &\le \int_0^{r_0}\frac{AC^2r^{1-\alpha}}{r^{2\beta}}\,dr = AC^2\int_0^{r_0} r^{1-\alpha-2\beta}\,dr < \infty. \end{aligned}$$

It follows from [6, Theorem 2] that $f$ admits a meromorphic extension to the whole of $G$. This contradiction completes the proof. □

**Remarks.** (1) The above result is sharp in the sense that $1-\frac{1}{2}\alpha$ cannot be replaced by a larger constant. Indeed, given $\alpha$ with $0 \le \alpha < 2$ one can construct a countable set $E$, clustering only at the origin, such that $M_\alpha(E) < \infty$ and $d(z,E) \le A|z|^{2/(2-\alpha)}$ for $z \in \mathbf{C}$ with $A < \infty$; see the proof of [7, Theorem B]. On the other hand, there is a meromorphic function $f$ in $\mathbf{C} \setminus \{0\}$ having an essential singularity at 0 such that $\limsup_{z\to 0} |z| f^*(z) = \frac{1}{2}$ [8, Theorem 1]. It follows that

$$\limsup_{d(z,E)\to 0} f^*(z) d(z,E)^{1-(\alpha/2)} \le \tfrac{1}{2} A^{1-(\alpha/2)} < \infty.$$

(2) In the cited work, Koskela considers a similar problem from the point of view of holomorphic functions. His result ([7, Theorem A]) may be rephrased as follows. Let $G \subset \mathbf{C}$ be a domain and let $E \subset G$ be a compact set with $M_\alpha(E) < \infty$ for some $\alpha \in [0,2)$ and $H^1(\{\operatorname{Re} z \mid z \in E\}) = H^1(\{\operatorname{Im} z \mid z \in E\}) = 0$ (observe that the latter condition implies $E \in N_D$ by [1, Theorem 10]). Let $f$ be holomorphic in $G \setminus E$ having at least one singularity in $E$. Then

$$\limsup_{d(z,E)\to 0} |f'(z)| d(z,E)^\beta = \infty \qquad \text{for all } \beta < 2-\alpha.$$

This result is sharp in the same sense as Theorem 3.

**Examples.** (1) Given $t \in (0,1)$, let $E_t \subset [0,1]$ be the self-similar Cantor set obtained by the construction in which one removes from the interval $[0,1]$ centrally a segment of length $1-t$ and continues with the remaining segments in the same manner etc. As is well known, $\dim_H(E_t) = \log 2/\log(2/t)$. Furthermore, it follows from [9, Theorem 4.19 and Remark 4.20] that $\dim_M(E_t) = \dim_H(E_t)$ and $M_\alpha(E_t) < \infty$ for $\alpha = \log 2/\log(2/t)$. Suppose now that $f$ is meromorphic in some neighborhood of $E_t$ having at least one essential singularity in $E_t$. Then by Theorem 3

$$\limsup_{d(z,E_t)\to 0} f^*(z) d(z,E_t)^\beta = \infty \qquad \text{for all } \beta < 1 - \log 2/2\log(2/t).$$

If one replaces $E_t$ with $E_t \times E_t$ (note that $E_t \times E_t \in N_D$ by [1, Theorem 10]), Theorem 3 yields

$$\limsup_{d(z,E_t\times E_t)\to 0} f^*(z) d(z,E_t\times E_t)^\beta = \infty \qquad \text{for all } \beta < 1 - \log 2/\log(2/t),$$

because $\dim_M(E_t \times E_t) = 2\log 2/\log(2/t)$.

(2) We now construct a symmetric Cantor set as follows. Let $I_0$ stand for the interval $[0,1]$. We first remove from $I_0$ centrally a segment of length $1-(1/2)^2$. The remaining set $I_1$ consists of equal segments $I_{1,1}$ and $I_{1,2}$ of total length $(1/2)^2$. Inductively we remove centrally from each segment $I_{nj}$ of $I_n$ with $n = 1,2,\ldots$ and $j = 1,\ldots,2^n$ a segment of length $(1/2)^n\bigl(1-(1/2)^{n+1}\bigr)(1/2)^{2^{n+1}-2}$ so as to obtain a set $I_{n+1}$ of total length $(1/2)^{2^{n+2}-2}$. The resulting set $E = \bigcap_{n=0}^\infty I_n$ is of logarithmic capacity zero [10, p. 153]. Obviously $\dim_M(E) = 0$.

It would be possible to obtain upper bounds for the number of Whitney squares in various generations in the decomposition of $\mathbf{C}\setminus E$ and then employ Theorem 2. However, it is a simple matter to estimate $H^1\big(\partial E(r)\big)$ directly and then apply Theorem 1. Set $l_n = l(I_{nj}) = (1/2)^n(1/2)^{2^{n+1}-2}$ and $r_n = l_n/2$, $n \in N$. Assume $z \in \mathbf{C}\setminus E$ is such that $r = d(z,E) < 1/20$ and pick out $k \in N$ such that $r \in [r_{k+1}, r_k)$. By elementary geometry $H^1\big(\partial E(r)\big) \le 2^{k+3}\pi r = 2^{k+1}4\pi r$. On the other hand, $r_k = l_k/2 = 2(1/2)^{2^{k+1}+k}$, i.e., ${}^2\log(2/r_k) = 2^{k+1}+k$, whence $2^{k+1} < {}^2\log(2/r_k) < {}^2\log(2/r)$. It follows that $H^1\big(\partial E(r)\big) < 4\pi r \cdot {}^2\log(2/r)$. Given a function $f$, meromorphic in some neighborhood of $E$ and with at least one essential singularity in $E$, we have by Theorem 1 that

$$\limsup_{d(z,E)\to 0} f^*(z)d(z,E)\log\big(2/d(z,E)\big) \ge \tfrac{1}{4}\log 2.$$

It is to be noted that this result cannot be obtained by the method of Theorem 3.

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