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QUASISYMMETRIC EMBEDDINGS OF PRODUCTS OF CELLS INTO THE EUCLIDEAN SPACE

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Abstract. We shall mainly consider the following situation. Suppose that A is a p-cell, B is a q-cell and there exists a quasisymmetric embedding $f\colon A\times B\to \mathbf{R}^{p+q}$. What can we then say about the metric properties of A and B? It turns out that, for example, the cell A locally satisfies a $(p-1)$-dimensional bounded turning condition. We also study the packing measures of A and B and show that if B is a quasiconvex arc, this implies certain conditions for the packing measure of A.

Introduction

The motivation for studying quasisymmetric embeddings of products of cells into the Euclidean space of the corresponding dimension derives from the article [Vä$_3$], where the second author considered the product of two curves and showed that the existence of a QS embedding of the product into the plane implies certain regularity properties of the curves. The purpose of this article is to extend these results to cover some higher dimensional cases.

As regards [Vä$_3$], it turns out that similar results can be obtained for the cells in our case, but this requires some new concepts that generalize the usual bounded turning and quasiconvexity properties.

The methods used in this article are similar to those in [Vä$_3$] and [Tu, Lemma 4]. However, some modifications are needed for handling higher dimensional objects; one of these is the use of packing measures. To avoid topological obstructions we restrict our study mainly to products of cells instead of arbitrary manifolds.

1. Metric spaces and manifolds

We start by listing some notation used hereafter.

All spaces considered here will be metric and the distance between points a and b in a space (X,d) is written as $|a-b|$ even if X is not a normed space. If A and B are nonempty subsets of a space X, we let $d(A)$ denote the diameter of A and $d(A,B)$ denote the distance between A and B. Also, we let $B(A,r)$ and $\overline{B}(A,r)$ denote the open and closed neighborhoods of A with radius r.

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The product of two metric spaces A and B is the set $A \times B$ endowed with the metric

$$|(a,b)-(a',b')| = |a-a'| + |b-b'|,$$

whenever $a, a' \in A$ and $b, b' \in B$. If $A_1, \ldots, A_n$ are spaces, we identify the spaces $\prod_{j=1}^{k} A_j \times \prod_{j=k+1}^{n} A_j$ and $\prod_{j=1}^{n} A_j$ for all $1 \le k \le n-1$.

Let $\mathbf{N}$ denote the natural numbers starting from 1, let $\mathbf{R}$ denote the set of reals and let $\mathbf{I} = \{x \in \mathbf{R} \mid 0 \le x \le 1\}$. In the Euclidean n-space $\mathbf{R}^n$ we use the ordinary metric and let $\mathbf{B}^n = B(0,1)$, $\mathbf{S}^{n-1} = \partial \mathbf{B}^n$ and $\Omega_n = m_n(\mathbf{B}^n)$, where m_n is the Lebesgue measure of $\mathbf{R}^n$.

A metric space X is said to be *boundedly precompact* if all bounded subsets of X are precompact. A space X is *boundedly compact* if all closed and bounded subsets of X are compact.

Using elementary topology, we get the following lemma.

1.1. Lemma. *A boundedly compact space is complete and, conversely, a complete and boundedly precompact space is boundedly compact.* □

A boundedly compact space homeomorphic to $\mathbf{R}^p$ is called a *p-string*. It is easy to show that bounded compactness is preserved in products and hence the product of a p-string with a q-string is a $(p+q)$-string.

A manifold may have a boundary unless this is explicitly denied. A metric space X homeomorphic to the space $\mathbf{I}^p$ is called a *p-cell*. The *boundary* of a p-cell X is the image of $\partial \mathbf{I}^p$ under a homeomorphism $\mathbf{I}^p \to X$. A 1-cell is also called an *arc*. For an arc A, we let $s(A, \mathscr{P})$ denote the length of the broken line determined by a finite division $\mathscr{P}$ of A, and let $l(A)$ denote the length of A.

Let X be a p-cell. The *inner radius* of X is the number

$$r(X) = \max\{d(x, \partial X) \mid x \in X\}.$$

Clearly $r(X) \le d(X)$ for all cells.

1.2. Lemma. *If A is an arc, then $r(A) \ge d(A)/4$.*

Proof. Let a and b be the end points of A. Then

$$r(A) = \max_{x \in A}\{|x-a| \wedge |x-b|\},$$

and the assertion follows from [Vä$_3$, 2.7], which gives a point $x \in A$ such that $|x-a| \wedge |x-b| \ge d(A)/4$. □

Let X be a metric space and let $c \ge 1$. The space X is of *c-bounded turning*, abbreviated c-BT, if, given disjoint points a and b in X, there exists an arc $A \subset X$ joining a and b such that $d(A) \le c|a-b|$. The space X is called *locally* c-BT if every point $x \in X$ has a neighborhood U such that all pairs of points in

U can be joined by an arc $A \subset X$ such that $d(A) \le c|a-b|$. If $d(A)$ is replaced with $l(A)$ in the definition of a BT and locally BT space, we get the properties *c-quasiconvex* and *locally c-quasiconvex*. In the literature c-quasiconvex arcs are also said to satisfy the c-chord-arc-condition.

We generalize these concepts as follows.

1.3. Definition. Let $c \ge 1$ and let $p \ge 0$ be an integer. A metric space A is of (c,p)-*bounded turning*, abbreviated as (c,p)-BT, if each continuous mapping $f\colon \mathbf{S}^p \to A$ has a continuous extension $g\colon \overline{\mathbf{B}}^{p+1} \to A$ such that $d\big(g\overline{\mathbf{B}}^{p+1}\big) \le cd(f\mathbf{S}^p)$. We also say that A is *locally* (c,p)-BT if every point has a neighborhood U in A such that every continuous mapping $f\colon \mathbf{S}^p \to A$ with $f\mathbf{S}^p \subset U$ has a continuous extension $g\colon \overline{\mathbf{B}}^{p+1} \to A$ satisfying the above condition.

The space A is said to be (c,p)-CBT if, given a $(p+1)$-cell β in A, there is a $(p+1)$-cell $\gamma \subset X$ such that $\partial\gamma = \partial\beta$ and $d(\gamma) \le cd(\partial\gamma)$. The local CBT-condition is defined by requiring every point to have a neighborhood U such that for every $(p+1)$-cell whose boundary lies in U there is another $(p+1)$-cell in A with the same boundary and satisfying the above turning condition.

Obviously the property $(c,0)$-BT is equivalent to the ordinary concept of c-BT defined earlier. Also, a space is $(c,0)$-CBT if and only if each of its path components is c-BT.

For example, convex sets in normed spaces are $(1,p)$-BT for every $p \ge 0$.

1.4. Lemma. *Let A be a p-cell and let $c \ge 1$. If A is $(c,p-1)$-BT, it is also $(c,p-1)$-CBT.*

Proof. Let $\beta = \partial\gamma$ for some p-cell γ in A. Then γ is the unique p-cell in A with this property, and we must show that $d(\gamma) \le cd(\beta)$. Let $f\colon \mathbf{S}^{p-1} \to A$ be an embedding such that $f\mathbf{S}^{p-1} = \beta$. Then it has an extension $g\colon \overline{\mathbf{B}}^p \to A$ such that $d\big(g\overline{\mathbf{B}}^p\big) \le cd(f\mathbf{S}^{p-1}) = cd(\beta)$, and it suffices to show that $\gamma \subset g\overline{\mathbf{B}}^p$. We may assume that $A = \overline{\mathbf{B}}^p$. Let $h\colon \overline{\mathbf{B}}^p \to A$ be an embedding extending f; thus $h\overline{\mathbf{B}}^p = \gamma$. Then $h \simeq g \operatorname{rel} \mathbf{S}^{p-1}$ so that their degrees agree in $\operatorname{int}\gamma$. The degree of h is clearly ± 1, and therefore $g\overline{\mathbf{B}}^p$ contains γ. □

We do not know whether the converse is true.

The following lemma shows that these properties are preserved in products.

1.5. Lemma. *Let A and B be metric spaces, let $p \ge 0$ and let $c \ge 1$. If A and B are (locally) (c,p)-BT, then $A \times B$ is (locally) $(2c,p)$-BT and, conversely, if $A \times B$ is (locally) (c,p)-BT, then both A and B are (locally) (c,p)-BT. For (local) quasiconvexity the same is true even without the factor 2.*

Proof. We start with the global case. Let A and B be (c,p)-BT and let $f\colon \mathbf{S}^p \to A \times B$ be continuous. Then $\mathrm{pr}_1 \circ f\colon \mathbf{S}^p \to A$ and $\mathrm{pr}_2 \circ f\colon \mathbf{S}^p \to B$ are continuous and thus have extensions $g_1\colon \overline{\mathbf{B}}^{p+1} \to A$ and $g_2\colon \overline{\mathbf{B}}^{p+1} \to B$ such that

$d(g_i\overline{\mathbf{B}}^{p+1}) \le cd(\mathrm{pr}_i f\mathbf{S}^p) \le cd(f\mathbf{S}^p)$, $i=1,2$. Then $g=g_1\times g_2\colon \overline{\mathbf{B}}^{p+1}\to A\times B$ is an extension of f and

$$d(g\overline{\mathbf{B}}^{p+1}) \le d(g_1\overline{\mathbf{B}}^{p+1}) + d(g_2\overline{\mathbf{B}}^{p+1}) \le 2cd(f\mathbf{S}^p).$$

Thus $A\times B$ is $(2c,p)$-BT.

Conversely, suppose that $A\times B$ is (c,p)-BT. To show that A is then (c,p)-BT let $f\colon \mathbf{S}^p\to A$. Choose an arbitrary point $b\in B$ and let $f_1\colon \mathbf{S}^p\to A\times B$ be defined by $f_1(x)=(f(x),b)$. Then f_1 has an extension $g_1\colon \overline{\mathbf{B}}^{p+1}\to A\times B$ such that $d(g_1\overline{\mathbf{B}}^{p+1})\le cd(f_1\mathbf{S}^p)=cd(f\mathbf{S}^p)$. Thus $g=\mathrm{pr}_1\circ g_1$ is the required extension of f. Hence A is c-BT.

Turning to the local case, let A and B be locally (c,p)-BT and let $x=(a,b)\in A\times B$. Choose neighborhoods U and V of a and b, respectively, such that the BT-conditions are satisfied. Then $U\times V$ is a neighborhood of x, and the same kind of reasoning as above shows it to satisfy the turning condition.

Finally, assume that $A\times B$ is locally (c,p)-BT and let $a\in A$. Choose a point $b\in B$ and a neighborhood $U\times V$ of (a,b), which is contained in a neighborhood satisfying the turning condition. We show that U is the required neighborhood of a. Let $f\colon \mathbf{S}^p\to A$ be continuous with $f\mathbf{S}^p\subset U$. Define $f_1\colon \mathbf{S}^p\to A\times B$ by setting $f_1(x)=(f(x),b)$. Then $f_1\mathbf{S}^p\subset U\times V$, and therefore f_1 has an extension $g_1\colon \overline{\mathbf{B}}^{p+1}\to A\times B$ such that $d(g_1\overline{\mathbf{B}}^{p+1})\le cd(f_1\mathbf{S}^p)$. Then $g=\mathrm{pr}_1\circ g_1$ is an extension of f. As before, it is easily shown that the turning condition is also satisfied for g.

The corresponding proofs for quasiconvexity are similar, if one considers arcs instead of extensions of maps. □

We recall the definition of a quasisymmetric mapping from [TV, p. 97]. Let X and Y be metric spaces and let $\eta\colon [0,\infty[\to[0,\infty[$ be a homeomorphism. An embedding $f\colon X\to Y$ is *η-quasisymmetric*, abbreviated η-QS, if $|f(a)-f(x)|\le \eta(t)|f(b)-f(x)|$ for all points $a,b,x\in X$ such that $|a-x|\le t|b-x|$. The inverse of an η-QS embedding $f\colon X\to Y$ is η'-QS in fX with $\eta'(t)=\eta^{-1}(t^{-1})^{-1}$ for $t>0$. Hereafter, the expression "f is η-QS" contains the assumption that η is a self-homeomorphism of $[0,\infty[$. For convenience we shall assume that $\eta(1)\ge 1$.

An arc A is called an *η-QS arc* if there is an η-QS homeomorphism $f\colon \mathbf{I}\to A$. A metric characterization for quasisymmetric arcs is given in [TV, 4.9].

The following theorem shows that the concepts defined above are QS invariants.

1.6. Theorem. *Let X and Y be metric spaces and let $f\colon X\to Y$ be η-*QS. *If $p\ge 0$, $c\ge 1$ and the space X is (locally) (c,p)-*BT, *then fX is (locally) $(2\eta(c),p)$-*BT. *The same is also true for the property* CBT.

Proof. Let $h\colon \mathbf{S}^p\to fX$ be continuous. Then the map $f^{-1}\circ h\colon \mathbf{S}^p\to X$ has a continuous extension $g'\colon \overline{\mathbf{B}}^{p+1}\to X$ such that $d(g'\overline{\mathbf{B}}^{p+1})\le cd(f^{-1}h\mathbf{S}^p)$. Define a map $g\colon \overline{\mathbf{B}}^{p+1} \to fX$ by setting $g = f \circ g'$. Then g is a continuous extension of h and the first inequality of [TV, 2.5] (substituting $A \mapsto f^{-1}h\mathbf{S}^p$, $B \mapsto g'\overline{\mathbf{B}}^{p+1}$ and $f \mapsto f$) gives

$$d\big(g\overline{\mathbf{B}}^{p+1}\big) = d\big(fg'\overline{\mathbf{B}}^{p+1}\big) \le 2\eta\left(\frac{d\big(g'\overline{\mathbf{B}}^{p+1}\big)}{d(f^{-1}h\mathbf{S}^p)}\right)d(ff^{-1}h\mathbf{S}^p) \le 2\eta(c)d(h\mathbf{S}^p).$$

Thus the space fX is $\big(2\eta(c),p\big)$-BT.

The proof of the local version is similar, and so is the invariance of the CBT-property. □

Here are some applications of this result.

1.7. Example. If A is a 2-manifold with a spire (see [GV, 10.2]), there exists no QS embedding of A into $\mathbf{R}^2$.

In fact, a neighborhood of the spire cannot satisfy the local $(c,1)$-BT-condition for any $c \ge 1$, and the claim follows from the theorem.

Let $K \ge 1$ and $n \ge 2$. An $(n-1)$-manifold $M \subset \mathbf{R}^n$ is K*-quasiconformally locally flat* if for every $x \in M$ there is a neighborhood V of x and a K-quasiconformal map $f\colon (V, V\cap M) \to (\mathbf{B}^n, \mathbf{B}^{n-1}\times\{0\})$.

A domain $D \subset \dot{\mathbf{R}}^n$ is called a K*-quasiball* if there is a K-quasiconformal map $f\colon (\dot{\mathbf{R}}^n, \mathbf{B}^n) \to (\dot{\mathbf{R}}^n, D)$.

1.8. Theorem. *Let M be an $(n-1)$-manifold in $\mathbf{R}^n$ with $n \ge 2$. If $K \ge 1$ and if M is K-quasiconformally locally flat, then M is locally $\big(c(K),p\big)$-*BT *for every $p \ge 0$.*

Proof. Let $x \in M$ and let U be a neighborhood of x in $\mathbf{R}^n$ such that $(U, U\cap M)$ is equivalent to $(\mathbf{B}^n, \mathbf{B}^{n-1}\times\{0\})$ via a K-QC map $f\colon U \to \mathbf{B}^n$. There is a neighborhood $V \subset U$ containing x such that $fV = B(0,r)$ for some $r > 0$ and $f\,|\,V$ is η-QS with η depending only on K. Then $f^{-1}\,|\,B(0,r)$ is η'-QS, and $B^{n-1}(0,r)\times\{0\}$ is $(1,p)$-BT for all $p \ge 0$. Thus $V\cap M$ is a neighborhood of x satisfying for all p the local (c,p)-BT-condition with $c = c(K)$. □

1.9. Corollary. *If $G \subset \mathbf{R}^n$ is a K-quasiball, then at every finite point ∂G is locally $\big(c(K),p\big)$-*BT *for every $p \ge 0$.* □

2. Packing measures

In this section we define the packing measure in a metric space. It was introduced by C. Tricot in [Tr] for Euclidean spaces and further studied in [TT] and [SRT]. In a metric space it was defined by D. Sullivan in [Su, Section 8] and also by H. Haase in [Ha]. We give a description of the definition in [TT] applied to a metric space, which is a special case of Haase's more general version, and prove some of its properties.

Let X be a metric space. Furthermore, let E be a nonempty subset of X, let $J \subset \mathbf{N}$ and let $\delta > 0$. Suppose that for every index $j \in J$ a point $x_j \in E$ and a number r_j are given such that $0 < r_j \le \delta$ and that $|x_i - x_j| > r_i + r_j$ for all $i \ne j$. Then we say that the collection $\mathscr{P} = \{B(x_j, r_j)\}_{j\in J}$ is a δ-packing of E, and if $p > 0$, we set $s_p(\mathscr{P}) = \sum_{j\in J}(2r_j)^p$, $\tau_\delta^p(\emptyset) = 0$ and

$$\tau_\delta^p(E) = \sup\{s_p(\mathscr{P}) \mid \mathscr{P} \text{ is a } \delta\text{-packing of } E\}.$$

The *p-dimensional packing premeasure of E* is the number

$$\tau^p(E) = \inf_{\delta>0} \tau_\delta^p(E) = \lim_{\delta\to 0+} \tau_\delta^p(E).$$

With this definition τ^p is the same as $\varphi - P$ of [TT] and P^φ of [SRT] if $X = \mathbf{R}^n$ and $\varphi(t) = t^p$.

The set function τ^p is not an outer measure, since it fails to be countably subadditive. For example, every unbounded set has an infinite p-dimensional packing premeasure for every $p > 0$. Using Munroe's Method I the *p-dimensional packing measure $\nu^p(E)$ of E* is defined by

$$\nu^p(E) = \inf\Big\{\sum_{k\in\mathbf{N}} \tau^p(E_k) \mid F \subset \textstyle\bigcup_{k\in\mathbf{N}} E_k\Big\}.$$

Clearly $\nu^p \le \tau^p$, and ν^p is the same as $\varphi - p$ of [TT] and p^φ of [SRT], φ as above.

A δ-covering of a set $E \subset X$ is a countable collection $\{B_i\}_{i\in I}$ of subsets of X such that $E \subset \cup_{i\in I} B_i$ and $d(B_i) \le \delta$ for all $i \in I$. The *p-dimensional Hausdorff measure $\mathscr{H}^p(E)$ of E* is defined by

$$\mathscr{H}^p(E) = \lim_{\delta\to 0+} \inf\Big\{\sum_{i\in\mathbf{N}} d(B_i)^p \mid \{B_i\}_{i\in\mathbf{N}} \text{ is a } \delta\text{-covering of } E\Big\}.$$

For our purposes the following theorem gives the most important properties of the packing measure ν^p.

2.1. Theorem. *The set function ν^p is a regular metric outer measure and corresponds to a countably additive Borel measure. It has the following additional properties.*

1. Always $\nu^p \ge \mathscr{H}^p$.
2. *In $\mathbf{R}^p$ we have $\Omega_p\nu^p = 2^p m_p$.*
3. *If $p > 0$, $M \ge 0$, $f\colon X \to Y$ is M-Lipschitz and $A \subset X$, then $\tau^p(fA) \le M^p\tau^p(A)$ and $\nu^p(fA) \le M^p\nu^p(A)$.*
4. *If A is an arc, then $\nu^1(A) = \tau^1(A) = l(A)$.*

5. *If* $A_1,\dots,A_p$ *are arcs, then*

$$\frac{2^p}{\Omega_p}\prod_{i=1}^{p} l(A_i) \le \nu^p\Big(\prod_{i=1}^{p} A_i\Big) \le \frac{2^p p^{p/2}}{\Omega_p}\prod_{i=1}^{p} l(A_i).$$

Proof. By [Mu, 11.3] Method I gives an outer measure. That ν^p is metric is proved as in [TT, 5.1].

Since ν^p is a metric outer measure, all Borel sets are measurable and [Mu, 12.3.1] implies that ν^p is regular, since in the definition of ν^p we may restrict our attention to closed covers by virtue of the equation $\tau^p\big(\overline{E}\big) = \tau^p(E)$. So the five properties remain to be proved.

1. The proof of this is similar to the corresponding one in $\mathbf{R}^n$ as presented in [SRT, 3.3], once we have a counterpart for their Lemma 2.1 in metric spaces. This is given in Lemma 2.2 below.

Let $E' \subset X$. It suffices to show that $\mathscr{H}^p(E) \le \tau^p(E)$ for all $E \subset X$, since given this, $\mathscr{H}^p(E') \le \sum_{I\in\mathbf{N}} \mathscr{H}^p(E_i) \le \sum_{i\in\mathbf{N}} \tau^p(E_i)$ for every covering $\{E_i\}_{i\in\mathbf{N}}$ of E', and this implies that $\mathscr{H}^p(E') \le \nu^p(E')$.

We may assume that $\mathscr{H}^p(E) > 0$ and $\tau^p(E) < \infty$. Let $H < \mathscr{H}^p(E)$, let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that $H \le \sum_{i\in\mathbf{N}} d(B_i)^p + \varepsilon$ for every δ_1-covering $\{B_i\}_{i\in\mathbf{N}}$ of E. Then choose $\delta_2 > 0$ such that $s_p(\mathscr{P}) \le \tau^p(E) + \varepsilon$ for all δ_2-packings $\mathscr{P}$ of E. Let $\delta = (\delta_1/6) \wedge \delta_2$, and choose, by Lemma 2.2 below, a δ-packing $\mathscr{P} = \{B(x_i,r_i)\}_{i\in I}$ of E satisfying the condition in the lemma. Then $\sum_{i\in I}(2r_i)^p \le \tau^p(E) + \varepsilon$. For each j the collection

$$\mathscr{B}_j = \big\{\overline{B}(x_i,r_i) \mid 1 \le i \le j\big\} \cup \big\{\overline{B}(x_i,3r_i) \mid i > j\big\}$$

is a δ_1-covering of E and therefore

$$H \le \sum_{i=1}^{j}(2r_i)^p + \sum_{i>j}(6r_i)^p + \varepsilon \le \tau^p(E) + 2\varepsilon + 3^p\sum_{i>j}(2r_i)^p.$$

The last series converges, it is at most $3^p s_p(\mathscr{P})$, and approaches zero as j tends to infinity. This implies that $H \le \tau^p(E) + 2\varepsilon$. Since $\varepsilon > 0$ and $H < \mathscr{H}^p(E)$ were arbitrary, this gives $\mathscr{H}^p(E) \le \tau^p(E)$, proving the claim.

2. If $E \subset \mathbf{R}^p$, the inequality $2^p m_p(E) \le \Omega_p \nu^p(E)$ follows from the first property. To prove the converse for τ^p, let $A \subset \mathbf{R}^p$ be a cube, perhaps containing some of its boundary. Let $\delta > 0$ and let $\mathscr{P} = \{B(x_j,r_j)\}_{j\in J}$ be a δ-packing of A. Then $\bigcup_{j\in J} B(x_j,r_j) \subset B(A,\delta)$. It follows that

$$\Omega_p s_p(\mathscr{P}) = 2^p\sum_{j\in J} m_p\big(B(x_j,r_j)\big) = 2^p m_p\big(\cup_{j\in J} B(x_j,r_j)\big) \le 2^p m_p\big(B(A,\delta)\big),$$

and, further, that $\Omega_p \nu^p(A) \le \Omega_p \tau^p(A) \le \Omega_p \tau^p_\delta(A) \le 2^p m_p\bigl(B(A,\delta)\bigr)$ for all δ. This implies that $\Omega_p \nu^p(A) \le 2^p m_p(A)$. An arbitrary open set $U \subset \mathbf{R}^p$ can be expressed as a disjoint union of such cubes, and therefore $\Omega_p \nu^p(U) \le 2^p m_p(U)$. If $E \subset \mathbf{R}^p$ is arbitrary, then $\Omega_p \nu^p(E) \le \Omega_p \nu^p(U) \le 2^p m_p(U)$ for all open sets U containing E, and thus $\Omega_p \nu^p(E) \le 2^p m_p(E)$.

3. For the first inequality it suffices to show that $\tau^p_\delta(fA) \le M^p \tau^p_{\delta/M}(A)$ for all $\delta > 0$. So let $\delta > 0$ and let $\mathscr{P} = \{B(x_j, r_j)\}_{j\in J}$ be a δ-packing of fA. For each $j \in J$ we choose a point $y_j \in f^{-1}(x_j)$. Then $\mathscr{P}_1 = \{B(y_j, r_j/M)\}_{j\in J}$ is a δ/M-packing of E and thus $s_p(\mathscr{P}) = M^p s_p(\mathscr{P}_1) \le M^p \tau^p_{\delta/M}(A)$. Since $\mathscr{P}$ was arbitrary, the inequality follows.

For the second claim we may assume that $\nu^p(A) < \infty$. Let $\{E_k\}_{k\in\mathbf{N}}$ cover A such that $\sum_k \tau^p(E_k) < \infty$. Setting $F_k = fE_k$ for each k we get a covering $\{F_k\}_{k\in\mathbf{N}}$ of fA with $\nu^p(fA) \le \sum_k \tau^p(F_k) \le M^p \sum_k \tau^p(E_k)$, where the first part was used. This implies the second inequality.

4. The inequality $l(A) \le \nu^1(A)$ follows from the first property, since $\mathscr{H}^1$ equals the length for arcs. Because of the inequality $\nu^1 \le \tau^1$, it suffices to show that $\tau^1(A) \le l(A)$. We may assume that $l(A) < \infty$. Let $\lambda = l(A)$ and let $f\colon [0,\lambda] \to A$ be the parametrization of A with respect to arc length; then f is 1-Lipschitz. By the third property it follows that

$$\tau^1(A) = \tau^1(f[0,\lambda]) \le \tau^1([0,\lambda]) = \lambda,$$

where the last equality follows as in the proof the the second property.

5. To simplify notation we only prove the claim for two arcs, and consider first the latter inequality. We may assume that $\lambda_i = l(A_i) < \infty$ for $i = 1,2$. For both i let $f_i\colon [0,\lambda_i] \to A_i$ be the parametrization with respect to the arc length; thus the maps f_i are 1-Lipschitz. Define a map $h\colon [0,\lambda_1] \times [0,\lambda_2] \to A_1 \times A_2$ by setting $h(x,y) = \bigl(f_1(x), f_2(y)\bigr)$. Then we have

$$\begin{aligned}|h(x,y) - h(x',y')| &= |f_1(x) - f_1(x')| + |f_2(y) - f_2(y')| \le |x - x'| + |y - y'|\\ &\le \sqrt{2}\sqrt{|x-x'|^2 + |y-y'|^2}\end{aligned}$$

so that h is $\sqrt{2}$-Lipschitz. From this it follows by the second and third properties that

$$\begin{aligned}\nu^2(A_1 \times A_2) &= \nu^2\bigl(h([0,\lambda_1] \times [0,\lambda_2])\bigr) \le 2\nu^2([0,\lambda_1] \times [0,\lambda_2])\\ &= \frac{2\cdot 2^2}{\Omega_2} m_2([0,\lambda_1] \times [0,\lambda_2]) = \frac{8}{\pi}\lambda_1\lambda_2.\end{aligned}$$

For the converse, let $K_i < \lambda_i$ and choose divisions $\mathscr{P}_1$ of A_1 with points $x_0, \ldots, x_n$ and $\mathscr{P}_2$ of A_2 with points $y_0, \ldots, y_n$ such that $s(A_i, \mathscr{P}_i) \ge K_i$, $i =$

$1,2$. Let $\alpha_i = A_1[x_{i-1}, x_i[$ and $\beta_i = A_2[y_{i-1}, y_i[$, $1 \le i \le n$. Define maps $g_{ij}\colon \alpha_i \times \beta_j \to \mathbf{R}^2$ by setting $g_{ij}(x,y) = (|x - x_{i-1}|, |y - y_{j-1}|)$. Then we have

$$\begin{aligned}|g_{ij}(x,y) - g_{ij}(x',y')| &= \sqrt{(|x - x_{i-1}| - |x' - x_{i-1}|)^2 + (|y - y_{j-1}| - |y' - y_{j-1}|)^2} \\ &\le \sqrt{|x - x'|^2 + |y - y|^2} \le |x - x'| + |y - y'|,\end{aligned}$$

so that each g_{ij} is 1-Lipschitz. Therefore

$$\begin{aligned}K_1K_2 \le s(A_1, \mathscr{P}_2)s(A_2, \mathscr{P}_2) &= \sum_{i,j=1}^{n} |x_i - x_{i-1}||y_j - y_{j-1}| \\ &\le \sum_{i,j=1}^{n} m_2\big(g_{ij}(\alpha_i \times \beta_j)\big) = \frac{\pi}{2^2}\sum_{i,j=1}^{n} \nu^2\big(g_{ij}(\alpha_i \times \beta_j)\big) \le \frac{\pi}{4}\sum_{i,j=1}^{n} \nu^2(\alpha_i \times \beta_j) \\ &= \frac{\pi}{4}\nu^2\Big(\textstyle\bigcup_{i,j=1}^{n}\alpha_i \times \beta_j\Big) \le \frac{\pi}{4}\nu^2(A_1 \times A_2).\end{aligned}$$

Since this holds for all $K_i < l(A_i)$, $i = 1,2$, we get the first inequality.

The proof is now complete. □

Had we used the Euclidean metric in the product of the arcs, the first inequality of the fifth property would hold as an equality, since all the maps g_{ij} and also h would be 1-Lipschitz.

In the proof of the first property we used the following lemma, which replaces [SRT, 2.1].

2.2. Lemma. *Let $\emptyset \ne E \subset X$ and let $\delta > 0$ be such that $\tau_\delta^p(E) < \infty$. Then there is a δ-packing $\mathscr{P} = \{B(x_i, r_i)\}_{i\in I}$ of E, where either $I = \{1, \dots, n\}$ or $I = \mathbf{N}$, satisfying for each $j \in I$ the condition*

$$E \setminus \textstyle\bigcup_{i=1}^{j}\overline{B}(x_i, r_i) \subset \bigcup_{i>j}\overline{B}(x_i, 3r_i).$$

Proof. Let $x_1 \in E$ be arbitrary and set $r_1 = \delta$. Inductively, if $B(x_j, r_j)$ is already chosen and $E_j = E \setminus \bigcup_{i=1}^{j}\overline{B}(x_i, r_i) \ne \emptyset$, we choose a point $x_{j+1} \in E_j$ with

$$|x_{j+1} - x_i| - r_i > \frac{3}{4}\sup_{y\in E}\min\{|y - x_l| - r_l \mid 1 \le l \le j\} = a_{j+1}$$

for every $1 \le i \le j$, and let $r_{j+1} = \delta \wedge a_{j+1}$; the condition $\tau_\delta^p(E) < \infty$ implies E to be bounded, and therefore a_{j+1} is finite. Then the collection $\mathscr{P} = \{B(x_i, r_i)\}_{i\in I}$ is a δ-packing of E. Since $\tau_\delta^p(E) < \infty$, the series $s_p(\mathscr{P})$ converges and thus either $I = \{1, \dots, n\}$ for some $n \ge 1$ or else $I = \mathbf{N}$ and $\lim_{i\to\infty} r_i = 0$. In the former case we have $E \subset \cup_{i=1}^{n}\overline{B}(x_i, r_i)$, and the claimed property is obvious.

Suppose then that the latter case occurs. Let $j \in \mathbf{N}$ and let $x \in E_j$. Then $\varepsilon = d\big(x, \cup_{i=1}^{j}\overline{B}(x_i, r_i)\big) > 0$, and by the construction we can choose the smallest index $k > j$ with $|x - x_k| \le r_k + \varepsilon \wedge \delta$. We claim that $x \in \overline{B}(x_k, 3r_k)$. This is clear if $r_k = \delta$, and we may assume that $r_k < \delta$, i.e. $r_k = a_k$. If $x \notin \overline{B}(x_k, 3r_k)$, then $3r_k < |x - x_k| \le r_k + \varepsilon \wedge \delta$, and hence $\varepsilon \wedge \delta > 2r_k$. Therefore, for some $1 \le s \le k - 1$

$$r_k \ge \frac{3}{4}(|x - x_s| - r_s) > \frac{3}{4}(\varepsilon \wedge \delta + r_s - r_s) > \frac{3}{2} r_k,$$

a contradiction. Thus the claimed property is proved. □

The following regularity property is connected with packing measures of balls in a metric space.

2.3. Definition. Let $p > 0$ and let A be a metric space. We say that A is *c-bounded in p-measure* if for all $x \in A$ and $r > 0$ the inequality

$$\nu^p\bigl(B(x, r)\bigr) \le c\,(2r)^p$$

is valid.

For example, an open set in $\mathbf{R}^p$ is 1-bounded in p-measure, and it is easy to construct examples of manifolds without this property. A c-quasiconvex arc A is c-bounded in 1-measure, but not conversely. In fact, given $x \in A$ and $r > 0$, let y and z be the first and last points of A in $\overline{B}(x, r)$ in some orientation. Then by Theorem 2.1 we have

$$\nu^1\bigl(B(x, r)\bigr) \le \nu^1(A[y, z]) = l(A[y, z]) \le c|y - z| \le 2cr.$$

3. Turning conditions

In this section we study the BT-properties of the cells. The main result is Theorem 3.4.

3.1. Theorem. *Let A be an m-manifold and B an n-manifold, both without boundary. If $f\colon A \times B \to \mathbf{R}^{m+n}$ is η-QS, then A and B are locally (c, p)-BT for all $p \ge 0$ with the constant c depending only on η.*

Proof. Using the invariance of domain for manifolds we see that the set $f(A \times B)$ is open and hence locally $(1, p)$-BT. By Lemma 1.6 the manifold $A \times B$ is locally $\bigl(2\eta'(1), p\bigr)$-BT. Now Lemma 1.5 implies the result. □

The following theorem gives more information in a special case, although the conditions are quite restrictive.

3.2. Theorem. *Let A, B and Y be metric spaces and let $f\colon A \times B \to Y$ be η-QS. If $c \ge 1$ and $f(A \times B)$ is (c, p)-BT for some $p \ge 0$, then both A and B are (c_1, p)-BT with $c_1 = c_1(c, \eta)$.*

Proof. By Lemma 1.5 it is enough to show that $A \times B$ is (c_1, p)-BT, which follows from Theorem 1.6. □

3.3. Theorem. *Let A be an m-string, let B be an n-string and let f: $A \times B \to \mathbf{R}^{m+n}$ be η-QS. Then f is surjective, and for all $p \ge 0$, A and B are (c, p)-BT with $c = c_\eta$.*

Proof. By Theorem 3.2 it suffices to show that f is surjective. This is the case, since by invariance of domain $f(A \times B)$ is open, and by Lemma 1.1 and [TV, 2.24] it is complete and hence also closed. □

The following theorem is a generalization of [Vä$_3$, 3.2].

3.4. Theorem. *Let A be a p-cell, let B be a q-cell and let f: $A \times B \to \mathbf{R}^{p+q}$ be η-QS. Let $H' = \eta^{-1}(1)^{-1}$ and assume that $d(A) < r(B)/H'$. Then A is $(c, p-1)$-CBT with $c = c(\eta)$. If $p = 1$, we can choose $c = 2H'$.*

Proof. Given a p-cell A' in A, we must show that $d(A') \le cd(\partial A')$. We may assume that $A' = A$, since the assumptions hold for A' as well.

Suppose first that the $(p+q)$-cell $f(A \times B)$ is locally flat in $\mathbf{R}^{p+q}$. Choose points $a_0, a_1 \in \partial A$ with $|a_1 - a_0| = d(\partial A)$, let $b_0 \in B$ be such that $B(b_0, r(B)) \cap \partial B = \emptyset$, and let $y \in \partial B$. For each $b \in B$ set $A(b) = A \times \{b\}$ and let $z_i = (a_i, b_0)$, $i = 0, 1$. If $z \in A(b_0)$ and $z' \in A(y)$, then

$$|z - z_0| \le d(A) < \frac{r(B)}{H'} \le \frac{1}{H'}|y - b_0| \le \frac{1}{H'}|z' - z_0|.$$

By the quasisymmetry of f we conclude that $|f(z) - f(z_0)| \le |f(z') - f(z_0)|$. Let

$$r = \sup\{|f(z) - f(z_0)| \mid z \in A(b_0)\}$$

and $G = B(f(z_0), r) \subset \mathbf{R}^{p+q}$. Then $fA(b_0) \subset \overline{G}$. We claim that the intersection $G \cap f(A \times \partial B)$ is empty. In fact, if $(a, b) \in A \times \partial B$, then the choice $y = b$ and $z' = (a, b)$ above leads to the inequality $|f(a, b) - f(z_0)| \ge |f(z) - f(z_0)|$, valid for all $z \in A(b_0)$. Hence $|f(a, b) - f(z_0)| \ge r$ and the claim follows. Let $C = A(b_0)$, $\partial C = \partial A \times \{b_0\}$, and let g: $C \to \mathbf{R}^{p+q}$ be an extension of $f \mid \partial C$ defined by $g = k \circ h$, where h: $C \to \overline{\mathbf{B}}^p$ is a homeomorphism such that $h(z_0) = e_1$, and k: $\overline{\mathbf{B}}^p \to \mathbf{R}^{p+q}$ is the conical extension of $f \circ h^{-1} \mid \mathbf{S}^{p-1}$ with vertices e_1 and $f(z_0)$. Then g is continuous and $gC \subset \overline{G}$, and therefore $g \simeq f \mid C$ rel ∂C in $\mathbf{R}^{p+q} \setminus f(A \times \partial B)$. By Lemma 3.6 below we have $\mathrm{pr}_1 f^{-1} gC = A$.

If $u \in f^{-1}gC$, then

$$(1) \qquad |f(u) - f(z_0)| \le d(gC) \le d(fC) \le 2\eta(1)|f(z_1) - f(z_0)|,$$

where the last inequality follows from [TV, 2.5], since $d(C) = |z_1 - z_0|$. This implies that $|u - z_0| \le \eta'(2\eta(1))|z_1 - z_0| = \eta'(2\eta(1))|a_1 - a_0|$, and therefore

$d(A) \le d(f^{-1}gC) \le 2\eta'\bigl(2\eta(1)\bigr)|a_1 - a_0|$. If $p = 1$, then gC is a line segment and obviously (1) is valid without the factor $2\eta(1)$. This proves the case where $f(A \times B)$ is locally flat in $\mathbf{R}^{p+q}$.

If the flatness assumption is not valid, we proceed by approximating the cell $A \times B$ from within and using the previous result as follows. Let $\varphi\colon \mathbf{I}^p \approx A$, let $\psi\colon \mathbf{I}^q \approx B$ and let $\varepsilon > 0$. Choose a number $s \in]0, 1/2[$ such that if $A_1 = \phi[s, 1-s]^p$, then $d(A) \le d(A_1) + \varepsilon$ and $d(\partial A_1) \le d(\partial A) + \varepsilon$. Then choose $t \in]0, 1/2[$ such that $r\bigl(\psi([t, 1-t]^q)\bigr) > H'd(A_1)$ and let $B_1 = \psi([t, 1-t]^q)$. The cell $f(A_1 \times B_1)$ is then locally flat in $\mathbf{R}^{p+q}$, and it follows from the first part of the proof that $d(A_1) \le cd(\partial A_1)$, and further that

$$d(A) \le d(A_1) + \varepsilon \le cd(\partial A_1) + \varepsilon \le cd(\partial A) + (c+1)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the theorem follows from this. □

3.5. Corollary. *Let $A_1, \dots, A_p$ be arcs and let $f\colon \prod_{i=1}^p A_i \to \mathbf{R}^p$ be η-QS. If $H' = \eta^{-1}(1)^{-1}$ and*

$$d(A_1) < \frac{1}{4H'} \min_{2 \le i \le p} d(A_i),$$

then A_1 is $2H'$-BT.

Proof. By Lemma 1.2 we have $r\bigl(\prod_{k>1} A_k\bigr) = \min\{r(A_k) \mid k > 1\} \ge \min\{d(A_k)/4 \mid k > 1\}$ so that $d(A_1) < r(\prod_{k>1} A_k)/H'$, and the assertion follows from the theorem. □

The following topological result was used in the proof of Theorem 3.4.

3.6. Lemma. *Let A and B be as in Theorem 3.4 and let $f\colon A \times B \to \mathbf{R}^n$ be an embedding, $n = p + q$. Let $b_0 \in \operatorname{int} B$, let $C = A \times \{b_0\}$ and let $g\colon C \to \mathbf{R}^n$ be a continuous map such that $g \simeq f \mid C$ rel ∂C in $\mathbf{R}^n \setminus f(A \times \partial B)$, where $\partial C = \partial A \times \{b_0\}$. If the n-cell $Q = f(A \times B)$ is locally flat in $\mathbf{R}^n$, then $\mathrm{pr}_1 f^{-1} gC = A$.*

Proof. We may assume that $A = \overline{\mathbf{B}}^p$, $B = \overline{\mathbf{B}}^q$ and $b_0 = 0$. Using the local flatness of Q and the Shoenflies theorem we can extend f to $\mathbf{R}^n$ and further reduce the situation to the case where $f = \mathrm{id}_{A \times B}$. Then $Q = A \times B$ and we must show that $P = \mathrm{pr}_1\bigl(gC \cap A \times (B \setminus \partial B)\bigr) = A$. To get a contradiction, assume that $a_0 \in A \setminus P$. Since ∂C is fixed by g, we have $\partial A \subset P$, and may thus assume that $a_0 = 0$. Let $V = \dot{\mathbf{R}}^n \setminus (A \times \partial B)$ and choose a retraction $r\colon V \to \dot{\mathbf{R}}^p \times \{0\}$ such that $r^{-1}(0,0) = \{0\} \times \operatorname{int} B$; in fact, the map r can be defined by setting $r(\infty) = \infty$, and for $(x, y) \in V \setminus \{\infty\}$,

$$r(x,y) = \begin{cases} \infty & \text{if } |y| \ge 1, \\ (x, 0) & \text{if } |y| < 1 \text{ and } |x| \le 1, \\ \dfrac{|x| - |y|}{|x|(1 - |y|)}(x, 0) & \text{if } |y| < 1 \text{ and } |x| > 1. \end{cases}$$

Define a map $g_1\colon A \to \dot{\mathbf{R}}^p$ by the compositions

$$A \xrightarrow{i} A \times \{0\} \xrightarrow{g} V \xrightarrow{r} \dot{\mathbf{R}}^p \times \{0\} \xrightarrow{\mathrm{pr}_1} \dot{\mathbf{R}}^p,$$

where i is the embedding $x \mapsto (x, 0)$. Since $g \simeq f \mid C$ rel ∂C, we have $g_1 \simeq i_A$ rel ∂A, where $i_A\colon A \hookrightarrow \dot{\mathbf{R}}^p$. Therefore the degree of g_1 in $\operatorname{int} A$ is equal to the degree of i_A, which implies that $g_1 A \supset A$. This is impossible, because $0 \in A \setminus P$ and $r^{-1}(0,0) = \{0\} \times \operatorname{int} B$ imply that $0 \notin g_1 A$.

Therefore $P = A$ and the lemma is proved. □

The following example, which is a three-dimensional version of [Vä$_3$, 3.4], shows that the condition involving the inner radius of B cannot be omitted in the previous theorem.

3.7. Example. Let $A = \mathbf{S}^1$, let $B = \mathbf{I}^2$ and let $f\colon A \times B \to \mathbf{R}^3$ be defined by $f\bigl(e, (s,t)\bigr) = (1+t)e - se_3$, where $e_3 = (0,0,1)$. Elementary calculations show that f is $2\sqrt{2}$-bilipschitz and hence η-QS with $\eta(t) = 8t$. However, there is no upper bound for the BT- and quasiconvexity constants of subarcs of A; the condition $d(A) < r(B)/H'$ is not satisfied, since $d(A) = 2$, $r(B) = 1/2$ and $H' = 8$.

4. Rectifiability conditions

In this section we study the rectifiability of the embedded manifolds dealing with quasiconvexity and packing measures. We start with the following generalization of [Vä$_3$, 4.2].

4.1. Theorem. *Let A be an arc, let B be a p-manifold without boundary and let $f\colon A \times B \to \mathbf{R}^{p+1}$ be η-QS. Let a_0 and a_1 be the end points of A, let $B_0 \subset B$ be open and assume that*

$$\beta = \inf\{|f(a_0, b) - f(a_1, b)| \mid b \in B_0\} > 0.$$

Then we have

$$\mathscr{H}^p(B_0) \le \nu^p(B_0) \le c(\eta, p)\frac{m_{p+1}(fQ)}{\beta^{p+1}} l(A)^p,$$

where $Q = \operatorname{int}(A \times B_0)$.

Proof. The first inequality follows from Theorem 2.1. To prove the second one, we may assume that A is rectifiable and that $m(fQ) < \infty$. Suppose first that $\overline{B}_0$ is compact in B. Let $0 < \delta < d(A)$ be such that $\overline{B}\bigl(\overline{B}_0, \delta\bigr)$ is a compact subset of B, and let $\mathscr{P} = \{B(x_j, r_j)\}_{j \in J}$ be a δ-packing of B_0. Fix $j \in J$ and choose a division $P = \{y_0, y_1, \dots, y_{m(j)}\}$ of A such that $\mu_i = |y_i - y_{i-1}| \in [r_j/2, r_j]$ for all $1 \le i \le m(j)$. Set $A_i = A[y_{i-1}, y_i]$ for $1 \le i \le m(j)$. By Theorem 3.4 we may assume δ to be small enough so that the arcs A_i are c_1-BT with $c_1 = c_1(\eta)$.

Let $1 \le i \le m(j)$. Choose a point $y \in A_i$ such that $|y - y_i| = |y - y_{i-1}| \equiv \mu'$. Then we have $\mu' \in [r_j/4, c_1 r_j]$. We set $z = (y, x_j)$, $z' = (y_{i-1}, x_j)$, $z'' = (y_i, x_j)$ and $\beta_{ij} = |f(z'') - f(z')|$, and let $Q_{ij} = \operatorname{int} A_i \times B(x_j, r_j)$. Choose a point $u \in \partial Q_{ij}$ such that $|f(u) - f(z)| = d\bigl(f(z), \partial(fQ_{ij})\bigr) \equiv r$. If $\mu' \le r_j$, then $\mu' = \mu' \wedge r_j \le |u - z|$; otherwise $\mu' \le c_1 r_j = c_1(r_j \wedge \mu') \le c_1|u - z|$. This latter inequality is therefore valid in both cases and, consequently, $|z' - z| = |y - y_{i-1}| = \mu' \le c_1|u - z|$. Since f is η-QS, it follows that $|f(z') - f(z)| \le \eta(c_1)r$. Similarly it can be shown that $|f(z'') - f(z)| \le \eta(c_1)r$. These inequalities imply that $\beta_{ij} \le 2\eta(c_1)r$. As an open set fQ_{ij} is measurable, and we have $m_{p+1}(fQ_{ij}) \ge m_{p+1}\bigl(B(f(z), r)\bigr) = \Omega_{p+1} r^{p+1}$, and hence $\beta_{ij}^{p+1} \le c_2 m_{p+1}(fQ_{ij})$, where $c_2 = \bigl(2\eta(c_1)\bigr)^{p+1}/\Omega_{p+1}$. Using the Hölder inequality we get

$$\begin{aligned}\beta^{p+1} &\le \Bigl(\sum_{i=1}^{m(j)} \beta_{ij}\Bigr)^{p+1} \le \Bigl(\sum_{i=1}^{m(j)} \beta_{ij}^{p+1}\Bigr)\Bigl(\sum_{i=1}^{m(j)} 1\Bigr)^{p} = m(j)^p \sum_{i=1}^{m(j)} \beta_{ij}^{p+1} \\ &\le c_2 m(j)^p m_{p+1}(fQ_j),\end{aligned}$$

where $Q_j = \operatorname{int} A \times B(x_j, r_j)$.

Now, $r_j \le 2\mu_i$ for all indices $1 \le i \le m(j)$, which implies that

$$m(j) r_j \le 2 \sum_{i=1}^{m(j)} \mu_i \le 2l(A).$$

Therefore,

$$s_p(\mathscr{P}) = 2^p \sum_{j \in J} r_j^p \le \frac{2^{2p} c_2}{\beta^{p+1}} l(A)^p \sum_{j \in J} m_{p+1}(fQ_j) \le c(\eta, p) \frac{m_{p+1}(fQ)}{\beta^{p+1}} l(A)^p,$$

where $c(\eta, p) = 4^p c_2$.

Since the collection $\mathscr{P}$ is an arbitrary δ-packing of B_0, we have $\nu^p(B_0) \le \tau^p(B_0) \le \tau_\delta^p(B_0) \le c(\eta, p) m_{p+1}(fQ) l(A)^p / \beta^{p+1}$.

Now let B_0 be an arbitrary open set in B. By the previous considerations we have $\nu^p(B_1) \le c(\eta, p) m_{p+1}(fQ) l(A)^p / \beta^{p+1}$ for all open sets B_1 such that $\overline{B}_1$ is a compact subset of B_0, and this implies that the above inequality also holds for $\nu^p(B_0)$. □

Theorem 4.1 can be regarded as a generalisation of the following special case. Let $A = [a, b] \subset \mathbf{R}$ be an interval, let $B \subset \mathbf{R}^p$ be a bounded domain, let $f\colon A \times B \to \mathbf{R}^{p+1}$ be η-QS in the Euclidean metric of $A \times B$ and let β be as in Theorem 4.1. Let Γ be the path family of horizontal line segments connecting the sets $\{a\} \times B$ and $\{b\} \times B$ in $Q = A \times B \subset \mathbf{R}^{p+1}$. By [Vä$_1$, 7.3] the modulus of Γ is $m_p(B)/l(A)^p$ and by [Vä$_1$, 7.1] the modulus of the image-family satisfies $M(f\Gamma) \le m_{p+1}(fQ)/\beta^{p+1}$. Since f is $\eta(1)^p$-quasiconformal in $\operatorname{int} Q$, the inequality $M(\Gamma) \le \eta(1)^p M(f\Gamma)$ implies the conclusion of 4.1 with $c = 2^p \eta(1)^p / \Omega_p$.

4.2. Theorem. *Let A, B and f be as in Theorem 4.1 and suppose that $l(A) \geq d(B)$. If $c \geq 1$ and A is c-quasiconvex, then B is c_1-bounded in p-measure with $c_1 = c_1(c, \eta, p)$.*

Proof. Let $x \in B$ and let $r > 0$. We may assume that $r \leq d(B)$. Choose points a_0 and a_1 in A such that $l(A[a_0, a_1]) = r$. Then $r \leq c|a_1 - a_0|$. Setting $A_0 = A[a_0, a_1]$ and $B_0 = B(x, r)$ we have by Theorem 4.1

$$\nu^p(B_0) \leq c_2 \frac{m_{p+1}(fQ)}{\beta^{p+1}} l(A_0)^p = c_2 r^p \frac{m_{p+1}(fQ)}{\beta^{p+1}},$$

where $Q = A_0 \times B_0$, $\beta = \inf\{|f(a_0, b') - f(a_1, b')| \mid b' \in B_0\}$ and $c_2 = c_2(\eta, p)$. To prove the theorem it thus suffices to find an upper bound for $m_{p+1}(fQ)/\beta^{p+1}$.

Let $\beta_1 > \beta$ and let $b_0 \in B_0$ be such that $|f(a_0, b_0) - f(a_1, b_0)| \leq \beta_1$. Set $z_i = (a_i, b_0)$, $i = 0, 1$. If $z = (a, b) \in A_0 \times B_0$, then

$$|z - z_0| = |a - a_0| + |b - b_0| \leq l(A_0) + 2r = 3r \leq 3c|a_1 - a_0| = 3c|z_1 - z_0|,$$

and hence $|f(z) - f(z_0)| \leq \eta(3c)|f(z_1) - f(z_0)| \leq \eta(3c)\beta_1$. From this it follows that $m_{p+1}(fQ) \leq \Omega_{p+1}\eta(3c)^{p+1}\beta_1^{p+1}$, and letting $\beta_1 \to \beta$ we get the theorem with $c_1 = c_2 2^{-p}\Omega_{p+1}\eta(3c)^{p+1}$. □

4.3. Remarks. 1. The assumption of local quasiconvexity of A in the above theorem will be quite natural after Corollary 4.7 below.

2. Suppose that in Theorem 4.2 the manifold B is the interior of a p-cell B_1 such that $d(B_1) < d(A)/4H'$, where $H' = \eta^{-1}(1)^{-1}$. Then combining Lemma 1.2 and Theorems 3.4 and 4.2 we may conclude that

$$\nu^p(B') \leq c_2 d(\partial B')^p$$

for all p-cells $B' \subset B$ and for some $c_2 = c_2(c, \eta, p)$.

Next we turn our attention to the product of more than two arcs. The case of two arcs is contained in Theorem 4.1 by Theorem 2.1.

4.4. Theorem. *Let $A_1, \ldots, A_p$ be arcs, let $f\colon \prod_{k=1}^p A_k \to \mathbf{R}^p$ be η-QS, let a_1 and b_1 be the end points of A_1 and let*

$$\beta_1 = \min\Big\{|f(a_1, y) - f(b_1, y)| \mid y \in \prod_{k=2}^p A_k\Big\}.$$

Then

$$\prod_{k=2}^p l(A_k) \leq c(p, \eta) \frac{m_p(fQ)}{\beta_1^p} l(A_1)^{p-1},$$

where $Q = \prod_{k=1}^p A_k$.

Proof. This immediately follows from Theorem 4.1 and the fifth property in 2.1. □

This makes it possible to prove the local quasiconvexity of all the arcs provided that one of them is quasiconvex.

4.5. Theorem. *Let $A_1, \dots, A_p$ be arcs, let $f\colon \prod_{k=1}^p A_k \to \mathbf{R}^p$ be η-QS and let $H' = \eta^{-1}(1)^{-1}$. Suppose that $c \ge 1$, that A_1 is c-quasiconvex and*

$$d(A_2) < \min_{k \neq 2} d(A_k)/4H'.$$

Then A_2 is c_1-quasiconvex with $c_1 = c_1(c, \eta, p)$.

Proof. It follows from Theorem 4.4 that all the arcs are rectifiable. Let a_2 and b_2 be distinct points in A_2. We must show that $l(A_2[a_2, b_2]) \le c_1|a_2 - b_2|$ with a suitable constant $c_1 = c_1(c, \eta, p)$. Since $d(A_k) > d(A_2)$ for all $k \neq 2$, we can choose points a_1 and b_1 in A_1 such that $|a_1 - b_1| = |a_2 - b_2|$, and for all $3 \le k \le p$ points a_k and b_k in A_k such that $d(A_k[a_k, b_k]) = |a_2 - b_2|$. Set $A'_k = A_k[a_k, b_k]$ and $A' = \prod_{k=1}^p A'_k$. By Theorem 4.4 we have

$$\prod_{k=2}^p l(A'_k) \le c_2 \frac{m_p(fA')}{\beta_1^p} l(A'_1)^{p-1},$$

where $c_2 = c_2(\eta, p)$ and β_1 is as in Theorem 4.4.

Choose a point $x_1 \in \prod_{k=2}^p A'_k$ such that $\beta_1 = |f(a_1, x_1) - f(b_1, x_1)|$ and set $z_0 = (a_1, x_1)$ and $z_1 = (b_1, x_1)$. From Corollary 3.5 it follows that the arc A_2 is $2H'$-BT. If $z \in A'$, then

$$\begin{aligned}
|z - z_0| &\le d(A'_1) + d(A'_2) + \dots + d(A'_p) \\
&\le l(A'_1) + 2H'|a_2 - b_2| + |a_2 - b_2| + \dots + |a_2 - b_2| \\
&\le c|a_1 - b_1| + 2H'|a_2 - b_2| + (p-2)|a_2 - b_2| \\
&= c_3|a_1 - b_1| = c_3|z_0 - z_1|,
\end{aligned}$$

where $c_3 = 2H' + c + p - 2$. Hence $|f(z) - f(z_0)| \le \eta(c_3)|f(z_0) - f(z_1)| = \eta(c_3)\beta_1$, and so $m_p(fA') \le \Omega_p \eta(c_3)^p \beta_1^p$. Setting $c_4 = c_2 \Omega_p \eta(c_3)^p$ we get the inequality

$$\prod_{k=2}^p l(A'_k) \le c_4 l(A'_1)^{p-1},$$

so that

$$\begin{aligned}
l(A'_2) &\le c_4 \frac{l(A'_1)^{p-1}}{\prod_{k=3}^p l(A'_k)} \le c_4 c^{p-1} \frac{|a_1 - b_1|^{p-1}}{\prod_{k=3}^p d(A'_k)} = c_4 c^{p-1} \frac{|a_2 - b_2|^{p-1}}{|a_2 - b_2|^{p-2}} \\
&= c_4 c^{p-1} |a_2 - b_2|,
\end{aligned}$$

from which the theorem follows with $c_1 = c_4 c^{p-1}$. □

It is necessary to set an upper bound for the diameter of A_2 in the previous theorem; otherwise we cannot get even a global BT-condition, as is shown in Example 3.7 above.

The following example shows that it is indeed possible to embed a product of locally non-rectifiable arcs quasisymmetrically into the Euclidean space.

4.6. Example. In [Tu, p. 151–152] P. Tukia gives an example of a homeomorphism $f\colon \mathbf{R} \to J$, where J is a BT but locally non-rectifiable 1-string of the snow-flake type. He also shows that for all $x, y \in \mathbf{R}$ the double inequality

$$\frac{1}{M}|x-y|^\alpha \le |f(x)-f(y)| \le M|x-y|^\alpha$$

is true for some constant $M \ge 1$ and $\alpha = \ln 3/\ln 4$. From this it follows that f is η-QS with $\eta(t) = M^2 t^\alpha$. Generalizing this, it is easy to show that the mapping $F = f \times \cdots \times f\colon \mathbf{R}^n \to J^n$ is θ-QS with $\theta(t) = Mnt^\alpha$, for example. Hence $F^{-1}\colon J^n \to \mathbf{R}^n$ is θ'-QS with $\theta'(t) = \theta^{-1}(t^{-1})^{-1}$ for $t > 0$.

In [Tu, Lemma 4] Tukia proved that no QS embedding of $A \times \mathbf{I}^p$ into $\mathbf{R}^{p+1}$ exists if A is a non-rectifiable arc. The following corollary is a generalisation of this and also summarizes the case of several arcs considered above.

4.7. Corollary. *Let $A_1, \dots, A_p$ be arcs and let $f\colon \prod_{k=1}^p A_k \to \mathbf{R}^p$ be η-QS. Then the following assertions are true.*

1. *All the arcs are locally c-BT with $c = c(\eta)$.*
2. *If some arc has a rectifiable subarc, all the arcs are rectifiable.*
3. *If some arc has a c-quasiconvex subarc, all the arcs are locally c_1-quasiconvex with $c_1 = c_1(c, \eta, p)$.*

We close this article by raising the following questions.

4.8. Open problems. 1. Let A be a p-cell and suppose that it is $(c, p-1)$-CBT for some $c \ge 1$. Is it then also $(c_1, p-1)$-BT for some $c_1 = c_1(c)$, or even for $c_1 = c$?

2. In Theorems 4.1 and 4.2 one of the factors is an arc. Is it also possible to obtain similar results for the product of manifolds of arbitrary dimensions?

3. In this last section, the dimension of the measure is always an integer. Can one also prove similar results for other measures?

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