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ON SOME PROPERTIES OF HAUSDORFF CONTENT RELATED TO INSTABILITY

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Abstract. We shall prove that certain local and global density properties of the α-dimensional Hausdorff content in $\mathbf{R}^n$ are equivalent. Such properties are related to problems in harmonic approximation.

0. Introduction

Let X be a compact subset of $\mathbf{R}^n$ and α a positive number such that $n-2<\alpha<n$. In [M-O] it was established that the following conditions are equivalent:

(1) $M_*^\alpha(B\setminus \mathring{X}) \le C\, M^\alpha(B\setminus X)$ for all open balls B and some positive constant C.

(2) $\displaystyle\limsup_{r\to 0}\frac{M^\alpha\bigl(B(x,r)\setminus X\bigr)}{r^\alpha}>0$ for M_*^α-a.a. $x\in\partial X$.

Here we denote by $\mathring{X}$ the interior of X, by ∂X the boundary of X, and M^α and M_*^α are the α-dimensional Hausdorff content and lower Hausdorff content. See later for precise definitions.

It was shown in [M-O] that the above conditions characterize compacts sets X having a Lipschitz harmonic approximation property. Moreover, the proof in [M-O] that (2) implies (1) was based on solving a particular approximation problem. Here we give an easier proof which works for all indexes $0<\alpha<n$ and it uses only geometric measure theory. Our method also provides the same result with M_*^α replaced by M^α. The proof uses, with $\alpha=\beta$, the following instability property of the Hausdorff content proved by Fernström [Fe].

If $0<\alpha\le\beta$ and $A\subset\mathbf{R}^n$, then for M^β-a.a. $x\in\mathbf{R}^n$ either

$$\limsup_{r\to 0}\frac{M^\alpha\bigl(A\cap B(x,r)\bigr)}{r^\alpha}\ge\frac{1}{6^\alpha}$$

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or

$$\lim_{r\to 0} \frac{M^\alpha\big(A\cap B(x,r)\big)}{r^\beta} = 0.$$

The main technique in Fernström's proof was to define a capacity, equivalent to the Hausdorff content, by means of the fractional maximal function. Here we shall give a shorter proof of the above instability property using only Vitali's covering lemma and Frostman's lemma. Actually, we get a slight generalization of this result.

A measure function is a non-decreasing function $h(r)$, $r \geq 0$, such that $\lim_{r\to 0} h(r) = 0$. The Hausdorff content M^h related to a measure function h is defined for $A \subset \mathbf{R}^n$ by

$$M^h(A) = \inf \sum_i h(\varrho_i),$$

where the infimum is taken over all countable coverings of A by open balls $B(x_i, \varrho_i)$. When $h(r) = r^\alpha$, $\alpha > 0$, $M^h(A) = M^\alpha(A)$ is called the α-dimensional Hausdorff content of A. The lower α-dimensional Hausdorff content of A is defined by

$$M_*^\alpha(A) = \sup M^h(A),$$

the supremum being taken over all measure functions which satisfy $h(r) \leq r^\alpha$ and $\lim_{r\to 0} h(r) r^{-\alpha} = 0$.

1. Instability

We shall now prove the following.

Theorem 1. *Let $A \subset \mathbf{R}^n$, $0 < \alpha < n$ and $h(r) = \varepsilon(r) r^\alpha$ be a measure function satisfying* (i) $0 \leq \varepsilon \leq 1$, (ii) ε *is non-decreasing and* (iii) $h(3r) \leq C_1 h(r)$, *where C_1 is a constant depending on h. Then for M^h-a.a. $x \in \mathbf{R}^n$ one of the following conditions holds:*

$$\limsup_{r\to 0} \frac{M^\alpha\big(A\cap B(x,r)\big)}{r^\alpha} \geq \frac{C_2}{C_1},$$

$$\lim_{r\to 0} \frac{M^\alpha\big(A\cap B(x,r)\big)}{h(r)} = 0,$$

where the constant C_2 depends only on n.

Remark. If $h(r) = r^\beta$, $\alpha \leq \beta$, we can replace C_2/C_1 by $6^{-\alpha}$.

Proof. First we recall a density property of the Hausdorff content (see [F, 2.10.19]). For all subsets F of $\mathbf{R}^n$

$$\limsup_{r\to 0} \frac{M^h\big(F\cap B(x,r)\big)}{h(r)} \geq C_2 > 0 \qquad M^h - \text{a.a. } x \in F,$$

where C_2 is a constant which depends only on n.

Let

$$E = \Big\{ x \in \mathbf{R}^n \,:\, \limsup_{r\to 0} \frac{M^\alpha\big(A\cap B(x,r)\big)}{r^\alpha} < \frac{C_2}{C_1} \quad \text{and} \quad \limsup_{r\to 0} \frac{M^\alpha\big(A\cap B(x,r)\big)}{h(r)} > 0 \Big\}.$$

We want to prove that $M^h(E)=0$. If $M^h(E)>0$, there would exist $y\in E$ and $t>0$ such that

$$M^\alpha\big(A\cap B(y,t)\big) < \frac{C_2}{C_1} t^\alpha, \tag{1.1}$$

$$M^h\big(E\cap B(y,t)\big) > C_2 h(t). \tag{1.2}$$

Condition (1.1) follows from the definition of E and condition (1.2) follows from the above density property.

Let $F=A\cap B(y,t)$ and $G=E\cap B(y,t)$. By (1.1) there are balls B_j of radii $r_j\le t$ such that $F\subset \cup B_j$ and $\sum_j r_j^\alpha < C_2 C_1^{-1} t^\alpha$. On the other hand,

$$\begin{aligned} M^h\big(G\cap(\cup 3B_j)\big) &\le \sum_j h(3r_j) \le C_1 \sum h(r_j) \le C_1 \varepsilon(t) \sum r_j^\alpha \\ &< C_2 \varepsilon(t) t^\alpha = C_2 h(t) < M^h(G), \end{aligned}$$

and consequently $M^h(G\setminus \cup 3B_j)>0$.

Therefore, for some fixed $\delta>0$ there is a compact set $K\subset G\setminus\cup 3B_j$ with $M^h(K)>0$ and satisfying

$$\limsup_{r\to 0} \frac{M^\alpha\big(F\cap B(x,r)\big)}{h(r)} > \delta \qquad \text{for} \quad x\in K.$$

Applying Frostman's lemma (e.g. [C]), one gets a positive Borel measure μ satisfying (i) spt $\mu\subset K$, (ii) $\mu B(x,r)\le h(r)$ and (iii) $\mu(K)\ge \text{const}\, M^h(K)$. Take $\eta>0$ such that

$$\frac{C_1}{\delta} \sum_{r_j<4\eta} r_j^\alpha < \mu(K).$$

By Vitali's covering lemma (e.g. [Z, 1.3.6]), there are disjoint balls $B(x_i,3\varrho_i)$ with $x_i\in K$ and $\varrho_i<\eta$ such that

(a) $\mu\big(K\setminus \cup B(x_i,3\varrho_i)\big)=0$.

(b) $M^\alpha\big(F\cap B(x_i,\varrho_i)\big) > \delta h(\varrho_i)$.

Fixed i, if $B_j\cap B(x_i,\varrho_i)\ne\emptyset$ then $B_j\subset B(x_i,3\varrho_i)$, because $x_i\notin 3B_j$. Hence

$$F\cap B(x_i,\varrho_i) \subset \textstyle\bigcup_{B_j\subset B(x_i,3\varrho_i)} B_j$$

and (b) gives

$$\delta h(\varrho_i) < M^\alpha\big(F \cap B(x_i, \varrho_i)\big) \le \sum_{B_j \subset B(x_i, 3\varrho_i)} r_j^\alpha.$$

Finally, by (a) and using that the balls $B(x_i, 3\varrho_i)$ are disjoint, one has

$$\begin{aligned}\mu(K) &= \sum_i \mu\big(K \cap B(x_i, 3\varrho_i)\big) \le \sum_i h(3\varrho_i) \le C_1 \sum_i h(\varrho_i) \\ &\le \frac{C_1}{\delta} \sum_i \sum_{B_j \subset B(x_i, 3\varrho_i)} r_j^\alpha \le \frac{C_1}{\delta} \sum_{r_j < 4\eta} r_j^\alpha \\ &< \mu(K).\end{aligned}$$

Therefore $M^h(E) = 0$.

In the case $h(r) = r^\beta$, $\alpha \le \beta$, we can replace the constant C_2 in the density property by a better one, $2^{-\beta}$. Moreover, we can use the inequality $\sum r_j^\beta \le \big(\sum r_j^\alpha\big)^{\beta/\alpha}$ to get better estimates. Thus, we would prove the same instability result that Fernström proved. □

Essentially, the same arguments used in the above proof give the next result. It gives some information only when $B \setminus A$ has empty interior.

Theorem 2. *Let B be a ball of radius δ and $A \subset B$ with $M^\alpha(A) < (\delta/6)^\alpha$, where $0 < \alpha < n$. Then there is a subset $C \subset B$ with positive Lebesgue measure satisfying*

$$\lim_{r \to 0} \frac{M^\alpha\big(A \cap B(x,r)\big)}{r^n} = 0, \qquad \text{for} \quad x \in C.$$

2. Equivalent conditions

In this section we show the equivalence of the conditions (1) and (2) of the introduction. More precisely, we prove the following theorems.

Theorem 3. *Let $0 < \alpha < n$ and let X be a compact set in $\mathbf{R}^n$. Then the following conditions are equivalent:*

(1) $M^\alpha(B \setminus \mathring{X}) \le C\, M^\alpha(B \setminus X)$ *for all open balls B and some positive constant C.*

(2) $\displaystyle\limsup_{r \to 0} \frac{M^\alpha\big(B(x,r) \setminus X\big)}{r^\alpha} > 0$ *for M^α-a.a. $x \in \partial X$.*

Theorem 4. *Let $0 < \alpha < n$ and let X be a compact set in $\mathbf{R}^n$. Then the following conditions are equivalent:*

(1) $M_*^\alpha(B \setminus \mathring{X}) \le C\, M^\alpha(B \setminus X)$ *for all open balls B and some positive constant C.*

(2) $\displaystyle\limsup_{r \to 0} \frac{M^\alpha\big(B(x,r) \setminus X\big)}{r^\alpha} > 0$ *for M_*^α-a.a. $x \in \partial X$.*

In Theorem 3, (1) implies (2) follows easily from the density property stated at the beginning of the proof of Theorem 1. An analogous density result for the lower Hausdorff content (see [M-O, Lemma 3.5]) also gives that (1) implies (2) in Theorem 4. Thus, one only has to show (2) $\Longrightarrow$ (1) in both theorems. Since the proofs are similar we only give a complete description of (2) $\Longrightarrow$ (1) in Theorem 3.

First note that the instability property gives that condition (2) is equivalent to

$$\limsup_{r\to 0} \frac{M^\alpha\big(B(x,r)\setminus X\big)}{r^\alpha} \geq 6^{-\alpha} \qquad \text{for } M^\alpha - \text{a.a. } x\in\partial X.$$

Let B be an open ball. We split $B\setminus \mathring{X} = (B\setminus X)\cup(B\cap\partial X)$, and then it is enough to see that

$$M^\alpha(B\cap\partial X)\leq CM^\alpha(B\setminus X).$$

Without loss of generality we can assume that for each $x\in B\cap\partial X$

$$\limsup_{r\to 0}\big(M^\alpha(B(x,r)\setminus X)\big)/r^\alpha \geq 7^{-\alpha}.$$

Then (see [Z, 1.3.1]) there is a family of disjoint balls $B_j = B(x_j, r_j)$ with $x_j\in B\cap\partial X$ such that each B_j is contained in B, $r_j^\alpha \leq 6^\alpha M^\alpha(B_j\setminus X)$ and $B\cap\partial X\subset \cup 5B_j$. By Melnikov's covering lemma [O, p. 72] there is a subfamily B_{jk} satisfying

(a) $M^\alpha(\cup 5B_j)\leq C\sum_k r_{jk}^\alpha$.
(b) $\sum_{B_{jk}\subset D} r_{jk}\leq Cr^\alpha$ for all open balls D of radius r.

Therefore, with C denoting different constants,

$$\begin{aligned} M^\alpha(B\cap\partial X) &\leq M^\alpha(\cup 5B_j)\\ \text{by (a)}\quad &\leq C\sum_k r_{jk}^\alpha\\ \text{by construction}\quad &\leq C\sum_k M^\alpha(B_{jk}\setminus X)\\ \text{by (b)}\quad &\leq CM^\alpha(\cup B_{jk}\setminus X)\\ &\leq CM^\alpha(B\setminus X). \ \square \end{aligned}$$

We finish this note with an example of a planar compact set X which satisfies the equivalent conditions of Theorem 4 but it does not realize the conditions of Theorem 3.

Let Q be the rectangle $[0,1]\times[-1,1]$. Let $\{B_j\}$ be the sequence of open balls with center $\big((2k-1)/(2^{n+1}), 1/2^n\big)$, $1\leq k\leq 2^n$, $n=1,2,\ldots$, and radius 4^{-n}, that is,

$$\bigcup_j B_j = \bigcup_{n=1}^\infty \bigcup_{k=1}^{2^n} B\Big(\Big(\frac{2k-1}{2^{n+1}}, \frac{1}{2^n}\Big), 4^{-n}\Big).$$

Consider the compact set

$$X = Q \setminus \cup B_j.$$

Then

$$\mathring{X} = \mathring{Q} \setminus \big(\cup \overline{B_j} \cup [0,1]\big),$$
$$\partial X = \partial Q \cup (\cup \partial B_j) \cup [0,1].$$

Take an open ball B of radius r with center lying on $[0,1]$. We have $B \setminus \mathring{X} = B \cap \big([0,1] \cup (\cup \overline{B_j})\big)$ and $B \setminus X = B \cap (\cup B_j)$. Then, since $M^1_*([0,1]) = 0$,

$$M^1_*(B \setminus \mathring{X}) = M^1_*\big(B \cap (\cup \overline{B_j})\big) = M^1(B \setminus X).$$

Moreover, given any $\varepsilon > 0$ there is such an open ball B of radius r such that $M^1(B \setminus X) < \varepsilon r$. On the other hand, we get $M^1(B \setminus \mathring{X}) \geq r/2$ because $B \setminus \mathring{X}$ contains $B \cap [0,1]$.

Thus, the compact set X satisfies condition (1) of Theorem 4 but it does not have property (1) of Theorem 3.

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