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ELEMENTS GENERATING BALAYAGES

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Abstract. We consider balayages in H-cones. Formerly balayages were characterized in terms of sets but a new approach looks at the elements instead of sets. Earlier, for example, we have proved an explicit formula for a balayage in an H-cone possessing a certain type of unit in terms of mixed envelopes formed relative to two partial orderings. Our problem is to describe those elements that generate a balayage. We state a necessary and sufficient condition for an element to generate a balayage in an H-cone possessing a special type of unit. We also give a relation between balayages and extreme points of a convex set of elements dominated by a fixed element.

Introduction

The theory of balayages is an integral part of potential theory. We consider balayages in an H-cone which is an axiomatic model of a convex cone of positive superharmonic functions on a harmonic space. A balayage is a mapping from an H-cone into itself which is additive, left order continuous, contractive and idempotent (see Section 2). Originally a balayage $\hat{R}_u^A$ of a superharmonic function s on a subset A of a harmonic space X is given by

$$\hat{R}_u^A(x) = \liminf_{y\to x} \inf\{\, v(y) \mid v \geq u \quad \text{on } A,\ v \quad \text{is superharmonic}\,\}.$$

If A is open then the mapping $u \mapsto \widehat{R}_u^A$ is a balayage. Moreover, if a harmonic space X satisfies the axiom of polarity [6, Theorem 9.1.1] then the mapping $u \mapsto \widehat{R}_u^A$ is a balayage for any $A \subset X$.

Previously balayages were characterized in terms of sets. In [9, Theorem 2.9] we present an explicit formula for a balayage in terms of mixed envelopes defined relative to two partial orderings. Some related characterizations of balayages are also given by Popa [10]. In this paper we study which elements generate a balayage? Our main theorem gives a necessary and sufficient condition for an element to generate a balayage in an H-cone possessing a special type of unit (Theorem 2.9). We also present a relation between balayages and extreme points of a convex set of elements dominated by a fixed element (Theorem 2.13). We prove an interesting result that the set of balayages is an abelian semigroup with respect to some special addition. This result gives a new formula for the least upper bound of two balayages.

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1. Preliminaries

We use the following definition of an H-cone which is equivalent with the original one ([8, Theorem 1.3]).

Definition 1.1. Let E be an ordered vector space and S be a convex subcone of E such that $S \subset E^+$ and $E = S - S$. The cone S is called an *H-cone* if it possesses the following properties:

(A_1) any upward directed and dominated subset F of S has a least upper bound in E denoted by $\vee F$ and $\vee F \in S$,

(A_2) any subset F of S has a greatest lower bound in E denoted by $\wedge F$ and $\wedge F \in S$,

(A_3) for any elements s and t of S, the greatest lower bound of the set $\{ u \in S \mid s - t \leq u \}$, denoted by $R(s-t)$, satisfies the conditions $R(s-t) \in S$ and $s - R(s-t) \in S$.

A partial order called *specific order*, denoted by $\preccurlyeq$, is defined in an H-cone by

$$s \preccurlyeq t \quad \text{if and only if } t = s + s' \quad \text{for some } s' \in S.$$

Any pair of elements in an H-cone has mixed envelopes introduced by Arsove and Leutwiler in algebraic potential theory ([2]).

Theorem 1.2. *Let S be an H-cone. Then for any elements s and t in S there exist a mixed lower envelope*

$$s \curlywedge t = \max\{ x \in S | x \preccurlyeq s,\ x \leq t \} = s - R(s-t)$$

and a mixed upper envelope

$$s \curlyvee t = \min\{ x \in s | x \succcurlyeq s,\ x \geq t \} = s + R(t-s)$$

satisfying the equality

$$s \curlywedge t + t \curlyvee s = s + t.$$

Proof. See [2, Theorem 2.5]. □

We recall the definitions of special units which are important in the theory of H-cones.

Definition 1.3. Let S be an H-cone. An element $e \in S$ is called a *weak unit* if $s = \vee_{n \in \mathbf{N}} (ne) \wedge s$ for all $s \in S$. An element $p \in S$ is called a *generator* if $s = \bigvee_{n \in \mathbf{N}} (np) \curlywedge s$ for all $s \in S$. An element $s \in S$ is called a *u-quasi-unit* for $u \in S$ if $s = \bigvee_{n \in \mathbf{N}} (ns) \curlywedge u$.

We apply the following characterization of quasi-units given by Arsove and Leutwiler in [1, p. 2499]:

Theorem 1.4. *Let S be an H-cone and u, s be elements of S. Then the following conditions are mutually equivalent:*

(i) *An element s is a u-quasi-unit.*
(ii) $s = (\alpha s) \leftarrowtail u$ *for all* $\alpha > 1$.
(iii) $s = (\alpha s) \leftarrowtail u$ *for some* $\alpha > 1$.
(iv) $R(s - \alpha u) = (1 - \alpha)u$ *for all* $\alpha < 1$.
(v) $R(s - \alpha u) = (1 - \alpha)u$ *for some* $\alpha < 1$.
(vi) *An element s is an extreme point of the convex set* $C = \{\, t \in S \mid t \le u \,\}$.

Let S be an H-cone and $u \in S$. We recall that an element s of an H-cone S is called *u-continuous* if for any $\varepsilon > 0$ and any upward directed family $F \subset S$ with $s = \vee F$ there exists an element f_ε of F such that $s \le f_\varepsilon + \varepsilon u$. An element s in S is *universally continuous*, if it is u-continuous with respect to all weak units u in S.

Definition 1.5. An H-cone S is called a *standard H-cone* ([4, p. 104]) if it has a weak unit and a countable dense set of universally continuous elements.

For a reference to the theory of H-cones we mention [4].

2. Elements generating a balayage

We consider balayages in H-cones. Recall that a mapping B from an H-cone S into S is called

(a) *left order continuous* if for any $s \in S$ the property $B(s) = \bigvee_{t \in F} B(t)$ holds for all upward directed subsets F of S,
(b) *idempotent* if $B^2 = B$,
(c) *contractive* if $B(s) \le s$ for all $s \in S$.

A *balayage* is a mapping $B\colon S \to S$ which is additive, left order continuous, idempotent and contractive. A potential-theoretic model for a balayage is the mapping $s \mapsto R_s^U$ where s is a positive superharmonic function on a harmonic space, U an open set and R_s^U the so-called reduced function. For further reference see [6, Section 4.2].

In the set of mappings from an H-cone S into itself we use the partial ordering given by $\psi \le \varphi$ if $\psi(s) \le \varphi(s)$ for all $s \in S$.

Balayages have the following important property as proved in [9, Lemma 2.3].

Lemma 2.1. *Let S be an H-cone. If $B\colon S \to S$ is a balayage then*

$$B(u) \leftarrowtail v = B(u) \leftarrowtail B(v) \tag{2.1}$$

for all u and v in S.

The value of a balayage at a point is obtained from its value at a generator [9, Theorem 2.9].

Theorem 2.2. *Let S be an H-cone possessing a generator p, and B a mapping from S into S. Then B is a balayage if and only if B is left order continuous and satisfies the equality*

$$B(x) = \bigvee_{n\in\mathbf{N}} \bigl(nB(p)\bigr) \curlyeqprec x \tag{2.2}$$

for all $x \in S$.

Quasi-units and balayages have a close connection stated next.

Proposition 2.3. *Let S be an H-cone and $u \in S$. If $B\colon S \to S$ is a balayage then the element $B(u)$ is a u-quasi-unit and therefore an extreme point of the convex set $\{ s \in S \mid s \le u \}$.*

Proof. Let $B\colon S \to S$ be a balayage and $u \in S$. Applying Lemma 2.1 we obtain

$$\bigl(2B(u)\bigr) \curlyeqprec u = B(2u) \curlyeqprec u = B(2u) \curlyeqprec B(u) = B(u).$$

Hence $B(u)$ is a u-quasi-unit by Theorem 1.4.

A natural question is what values of $B(p)$ in the formula (2.2) produce a balayage? For handling this we define the following concept.

Definition 2.4. Let S be an H-cone. An element $u \in S$ *generates* a balayage if the mapping $B\colon S \to S$ defined by

$$B(x) = \bigvee_{n\in\mathbf{N}} (nu) \curlyeqprec x$$

is a balayage.

Applying [9, Theorem 2.10] we obtain directly the next result.

Proposition 2.5. *An element u of an H-cone S generates a balayage if and only if the condition*

$$u = \bigvee_{\substack{n\in\mathbf{N}\\ f\in F}} (nu) \curlyeqprec f \tag{2.3}$$

holds for any upward directed family F with $\bigvee F = u$.

Note that the condition (2.3) does not hold generally for all u in an H-cone S. Indeed, it is possible that $u = \bigvee F$ for some upward directed family F and $u \curlyeqprec f = 0$ for all $f \in F$. For example this property holds if u is a harmonic function and F the set of potentials with $u = \bigvee F$.

Lemma 2.6. *Let S be an H-cone and u an element of S. If the function $\psi\colon S \to S$ defined by $\psi(x) = u \curlyeqprec x$ for $x \in S$ is left order continuous then the element u generates a balayage.*

Proof. By virtue of the preceding proposition we only have to verify the condition (2.3). Assume that F is directed upwards with $\vee F = u$. Since ψ is left order continuous we have

$$u = \psi(u) = \bigvee_{f \in F} u \curlywedge f \leq \bigvee_{\substack{f \in F \\ n \in \mathbf{N}}} (nu) \curlywedge f \leq u,$$

completing the proof. ▫

An example of elements generating a balayage are v-continuous elements for any element $v \in S$.

Lemma 2.7. *Let S be an H-cone and $v \in S$. If u is v-continuous then u generates a balayage. Moreover, any element $u \in S$ enjoying the property*

$$\bigwedge \{\, R(u - t) \mid t \leq u,\ t \text{ is } v\text{-continuous} \,\} = 0, \tag{2.4}$$

generates a balayage.

Proof. Assume that an element u of S is v-continuous for some $v \in S$. It is enough to prove that the mapping ψ defined by $\psi(x) = u \curlywedge x$ for $x \in S$ is left order continuous. Let a family F be directed upwards with $\bigvee F = u$. Since u is v-continuous for any $\varepsilon > 0$ there exists an element f_ε in F such that $u \leq f_\varepsilon + \varepsilon v$. This implies that $u \leq (u + \varepsilon v) \curlywedge (f_\varepsilon + \varepsilon v)$ and further by [2, p. 16]

$$u \leq u \curlywedge f_\varepsilon + \varepsilon v \leq \bigvee_{f \in F} u \curlywedge f + \varepsilon v.$$

Since ε is arbitrary, the condition (2.3) holds.

Lastly suppose that the condition (2.4) is valid for $u \in S$. Denote by $\mathscr{V}$ the set of v-continuous elements. Let F be directed upward with $\bigvee F = u$. By Theorem 1.2 and (2.4) we obtain

$$u = \bigvee_{t \in \mathscr{V}} u \curlywedge t. \tag{2.5}$$

For any $t \in \mathscr{V}$ with $t \leq u$, there exists for each $\varepsilon > 0$ an element $f_\varepsilon \in F$ such that $t \leq f_\varepsilon + \varepsilon v$. Hence we have

$$u \curlywedge t \leq u \curlywedge f_\varepsilon + \varepsilon v \leq \bigvee_{f \in F} u \curlywedge f + \varepsilon v.$$

Combining this with (2.5) we obtain the condition (2.3). ▫

In standard H-cones even the following stronger result holds.

Theorem 2.8. *Let S be a standard H-cone. Any balayage B on S is generated by a u-continuous element for some weak-unit u in S. Conversely, a u-continuous element generates a balayage for any weak unit $u \in S$.*

Proof. We only have to prove the first statement. Assume that B is a balayage on a standard H-cone S. Then the set $B(S) = \{ B(s) \mid s \in S \}$ is also a standard H-cone by [4, Corollary 5.2.6] and there exists a countable dense set $(s_i)_{i \in \mathbf{N}}$ of universally continuous elements in $B(S)$. Since S is a standard H-cone it has a generator p by [4, Lemma 4.3.7]. Moreover $B(p)$ is a weak unit (even a generator) in $B(S)$. Indeed, on account of Theorem 2.2 we have

$$B(x) = \bigvee_{n \in \mathbf{N}} B(np) \curlywedge x = \bigvee_{n \in \mathbf{N}} \big(B(np) \curlywedge B(x) \big).$$

By [4, Proposition 4.1.2] for every s_n there exists $\alpha_n \in \mathbf{R}$ such that $s \leq \alpha_n B(p) \leq \alpha_n p$. Set

$$u = \sum_{n \in \mathbf{N}} \frac{s_n}{2^n \alpha_n B(p)}.$$

Elements s_n are p-continuous in S. Indeed, let F be directed upwards with $\vee F = s_n = B(s_n)$ and $\varepsilon > 0$. Since B is left order continuous we have $\bigvee_{f \in F} B(f) = s_n$. As s_n is $B(p)$-continuous we obtain

$$s_n \leq B(f_\varepsilon) + \varepsilon B(p) \leq f_\varepsilon + \varepsilon p$$

for some $f_\varepsilon \in F$. Hence s_n is p-continuous. Applying [4, Proposition 4.1.2] we easily see that u is p-continuous. Moreover, u is clearly a generator in $B(S)$. Using Lemma 2.1 we find that

$$B(x) = \bigvee_{n \in \mathbf{N}} (nu) \curlywedge B(x) = \bigvee_{n \in \mathbf{N}} (nu) \curlywedge x.$$

Thus the balayage B is generated by a p-continuous element. □

Let S be an H-cone possessing a generator p. If an element u in S generates a balayage B, then Proposition 2.3 implies that $B(p)$ is a p-quasi-unit. Combining this observation with Proposition 2.5 we obtain the result.

Theorem 2.9. *Let S be an H-cone possessing a generator p. Then every balayage is generated by some p-quasi-unit. Conversely, a p-quasi-unit u generates a balayage if and only if*

$$u = \bigvee_{\substack{n \in \mathbf{N} \\ f \in F}} (nu) \curlywedge f$$

for any upward directed family F with $u = \vee F$.

In some important cases there is a one to one correspondence between p-quasi-units and balayages. In order to find sufficient conditions we first state two preliminary results.

Proposition 2.10. *Let S be an H-cone in an ordered vector space E and $f \in E$. Then the mapping $B_f \colon S \to S$ defined by*

$$B_f(x) = \bigvee_{n \in \mathbf{N}} R\big(x \wedge (nf)\big)$$

is a balayage and $B_f = B_{f^+}$. Moreover, $B_f\big(R(f)\big) = R(f)$.

Proof. Let $x \in S$ and $f \in E$. Since by [4, Proposition 2.1.1] the set E is a vector lattice we infer

$$R\big(x \wedge (nf)\big) = R\left(\big(x \wedge (nf)\big) \vee 0\right) = R\big(x \wedge (nf^+)\big).$$

Applying [4, Theorem 2.2.9] the mapping $B_f = B_{f^+}$ is a balayage. The second statement follows from $B_f\big(R(f)\big) \in S$ and $R(f) \geq B_f\big(R(f)\big) \geq f$. □

The following result is a stronger form of the result stated by Boboc [5, Lemma 3, p. 74].

Proposition 2.11. *Let S be an H-cone and $p \in S$. Then the following assertions are equivalent:*

(i) *s is a p-quasi-unit;*

(ii) *There exists a decreasing sequence of balayages $(B_n)_{n \in \mathbf{N}}$ such that*

$$s = B_n(s) \qquad \text{for all } n \in \mathbf{N} \text{ and } s = \bigwedge_{n \in \mathbf{N}} \big(B_n(p)\big).$$

(iii) *There exists a sequence of balayages $(B_n)_{n \in \mathbf{N}}$ such that*

$$s = B_n(s) \qquad \text{for all } n \in \mathbf{N} \quad \text{and } s = \bigwedge_{n \in \mathbf{N}} \big(B_n(p)\big).$$

Proof. Assume that s is a p-quasi-unit. Let $\alpha < 1$ and set $f_\alpha = s - \alpha p$. Define a mapping $B_\alpha \colon S \to S$ by

$$B_\alpha(x) = \bigvee_{m \in \mathbf{N}} R\big(x \wedge (mf_\alpha)\big).$$

Then by Proposition 2.10 the mapping B_α is a balayage and

$$B_\alpha\big(R(s - \alpha p)\big) = R(s - \alpha p).$$

Applying Theorem 1.4 we obtain $R(s-\alpha p)=(1-\alpha)s$ and so

$$B_\alpha(s)=\frac{1}{1-\alpha}B_\alpha\bigl(R(s-\alpha p)\bigr)=s.$$

We show that

$$s+(1-\alpha)p\geq B_\alpha(p).$$

According to [4, Lemma 2.2.8] it is enough to prove that

$$\bigl(p-\bigl(s+(1-\alpha)p\bigr)\bigr)\wedge(s-\alpha p)\leq 0.$$

But this is evident, since E is a vector space and therefore

$$(2\alpha p)\wedge(2s)=s\wedge\alpha p+s\wedge\alpha p\leq s+\alpha p.$$

Setting $B_n=B_{1-1/n}$ for $n\in\mathbf{N}$ we obtain (ii). The condition (ii) implies trivially (iii). Assume that the condition (iii) holds. Then by Lemma 2.1 we have

$$s\leq(2s)\curlyeqprec p=B_n(2s)\curlyeqprec p=B_n(2s)\curlyeqprec B_n(p)\leq B_n(p)$$

for all $n\in\mathbf{N}$. Using the condition (iii) we infer that $(2s)\curlyeqprec p=s$ and so by Theorem 1.4 the element s is a p-quasi-unit. □

Lemma 2.12. *Let S be an H-cone and v be u-continuous for some element u in S. If $\varphi\colon S\to S$ is additive, increasing and contractive then the mapping $\tilde\varphi$ defined by*

$$\tilde\varphi(s)=\bigvee_{n\in\mathbf{N}}\varphi\bigl((nv)\curlyeqprec s\bigr)$$

is additive and left order continuous.

Proof. Similarly as in the proof of [9, Proposition 2.7] we deduce that $\tilde\varphi$ is additive. Assume that F is directed upwards and $\vee F=v$. Since v is u-continuous for some element u there exists f_ε for any $\varepsilon>0$ such that $v\leq f_\varepsilon+\varepsilon u$. Hence we have $v=(nv)\curlyeqprec v\leq(nv)\curlyeqprec f_\varepsilon+\varepsilon u$ and therefore $\tilde\varphi(v)=\vee_{f\in F}\tilde\varphi(f)$. Let $s\in S$ and $F\subset S$ be directed upwards with $s=\vee F$. Assume first that $s\preccurlyeq v$. As the set $v-s+F$ is directed upwards towards v, we have

$$\tilde\varphi(s)+\tilde\varphi(v-s)=\tilde\varphi(v)=\bigvee_{f\in F}\bigl(\tilde\varphi(f)\bigr)+\tilde\varphi(v-s).$$

Hence $\tilde\varphi$ is left order continuous for all $s\in S$ such that $s\preccurlyeq nv$ for some $n\in\mathbf{N}$. Assume next that s is an arbitrary element in S. Since $\tilde\varphi$ is left order continuous at $(nv)\curlyeqprec s$ for all $n\in\mathbf{N}$ we infer

$$\varphi\bigl((nv)\curlyeqprec s\bigr)=\tilde\varphi\bigl((nv)\curlyeqprec s\bigr)=\bigvee_{f\in F}\tilde\varphi\Bigl(\bigl((nv)\curlyeqprec s\bigr)\wedge f\Bigr)\leq\bigvee_{f\in F}\tilde\varphi(f)\leq\tilde\varphi(s)$$

for all $n\in\mathbf{N}$. Hence $\tilde\varphi$ is left order continuous. □

Theorem 2.13. *Let S be an H-cone and v be u-continuous for some $u \in S$. Assume that the greatest lower bound in the set of left order continuous additive mappings for any decreasing sequence $(B_n)_{n\in\mathbf{N}}$ of balayages in S is a balayage. Then for any v-quasi-unit s there exists a balayage $B\colon S \to S$ such that $B(v) = s$. Conversely, for any balayage $B\colon S \to S$ the element $B(v)$ is a v-quasi-unit.*

Moreover, if an H-cone S possesses a generator v which is u-continuous for some weak unit u in S there exists a one to one correspondence between balayages and v-quasi-units.

Proof. Assume that v is u-continuous for some $u \in S$. Let $B\colon S \to S$ be a balayage. Then $B(v)$ is a v-quasi-unit by Proposition 2.3. Let s be a v-quasi-unit. Because of Proposition 2.11 there exists a decreasing sequence of balayages B_n such that $s = \bigwedge_{n\in\mathbf{N}}\bigl(B_n(v)\bigr)$. Since the sequence (B_n) is decreasing, the mapping $\varphi\colon S \to S$ defined by $\varphi(t) = \bigwedge_{n\in\mathbf{N}}\bigl(B_n(t)\bigr)$ for $t \in S$ is additive, increasing and contractive. Using Lemma 2.12 we find out that $\tilde{\varphi}$ is left order continuous, additive and $\tilde{\varphi}(v) = \varphi(v) = s$. By the assumption the greatest lower bound $\bigwedge B_n$ in the set of left order continuous additive mappings is a balayage denoted by B. Hence $B(v) = \tilde{\varphi}(v) = s$. Taking into account Theorem 2.2, this correspondence is one to one if v is u-continuous (for some weak unit u) and a generator. □

The condition of the preceding theorem is equivalent with the axiom of polarity in standard H-cones by [7, p. 188].

Theorem 2.14. *Let S be an H-cone possessing a generator p. Let $B\colon S \to S$ be a balayage. Then there exists a lower directed family of functions $f_n \in S - S$ such that $B = \wedge_{n\in\mathbf{N}} B_{f_n}$.*

Proof. Let p be a generator of an H-cone S. Since by Proposition 2.3 the element $B(p)$ is a p-quasi-unit there exists a decreasing sequence of balayages $(B_{f_n})_{n\in\mathbf{N}}$ such that $B(p) = \bigwedge_{n\in\mathbf{N}}\bigl(B_{f_n}(p)\bigr)$. Applying [9, Corollary 2.9] to the inequality $B(p) \le B_{f_n}(p)$ we infer that $B \le B_{f_n}$ for all $n \in \mathbf{N}$. If $s + s' = np$ for some $n \in \mathbf{N}$ and $s,\ s' \in S$ then

$$B(s) + B(s') = B(np) = \bigwedge_{n\in\mathbf{N}} B_{f_n}(s + s') = \bigwedge_{n\in\mathbf{N}} B_{f_n}(s) + \bigwedge_{n\in\mathbf{N}} B_{f_n}(s').$$

Since $B(t) \le \bigwedge_{n\in\mathbf{N}} B_{f_n}(t)$ for all $t \in S$ we have $B(s) = \bigwedge_{n\in\mathbf{N}} B_{f_n}(s)$ for all $s \preccurlyeq np$ for some $n \in \mathbf{N}$. Assume now that $\psi\colon S \to S$ is left order continuous, additive and $\psi \le B_{f_n}$ for all $n \in \mathbf{N}$. Then we obtain

$$\psi\bigl((mp)\curlywedge x\bigr) \le \bigwedge_{n\in\mathbf{N}} B_{f_n}\bigl((mp)\curlywedge x\bigr) = B\bigl((mp)\curlywedge x\bigr)$$

for all $m \in \mathbf{N}$. Hence we conclude

$$\psi(x) = \bigvee_{m\in\mathbf{N}} \psi\bigl((mp)\curlywedge x\bigr) \le \bigvee_{m\in\mathbf{N}} B\bigl((mp)\curlywedge x\bigr) = B(x),$$

establishing the assertion. □

Lemma 2.15. *Let S be an H-cone possessing a generator p and elements u_1 and u_2 in S each generate a balayage. Then the element u_1+u_2 generates a balayage, and this balayage is also generated by $(u_1+u_2)\curlyvee p$.*

Proof. Assume that $u_1 \in S$ and $u_2 \in S$ generate a balayage. Set $z = u_1+u_2$ and let F be an arbitrary upward directed set with $\bigvee F = z$. Proposition 2.5 states that it is enough to show that

$$z = \bigvee \{ (nz)\curlyvee f \mid f \in F,\ n \in \mathbf{N} \}. \tag{2.6}$$

Applying [4, Proposition 2.2.3] there exists upward directed families $(g_{1f})_{f\in F}$ and $(g_{2f})_{f\in F}$ such that $f \geq g_{1f}+g_{2f}$ for all $f \in F$ and $\vee_{f\in F} g_{if} = u_i$ for $i=1,2$. Hence we have

$$(nu_1)\curlyvee g_{1f} + (nu_2)\curlyvee g_{2f} \leq \big(n(u_1+u_2)\big)\curlyvee f$$

for all $f \in F$. Since u_1 and u_2 generate a balayage and $\vee_{f\in F} g_{if} = u_i$ for $i=1,2$, we obtain

$$z = u_1+u_2 \leq \bigvee_{n\in\mathbf{N}} \big(n(u_1+u_2)\big)\curlyvee f \leq \bigvee F = z.$$

Thus the element $z = u_1+u_2$ generates a balayage denoted by B.

We still have to prove that $z\curlyvee p$ generates also B. Let $x \in S$ be arbitrary. From $(mp)\curlyvee x \preccurlyeq mp \preccurlyeq np$ for all m, $n \in \mathbf{N}$ with $m \leq n$ it follows by [2, Theorem 3.2] that

$$(nz)\curlyvee(np) \succcurlyeq (nz)\curlyvee\big((mp)\curlyvee x\big).$$

Hence we have

$$\begin{aligned} B(x) &\geq \bigvee_{\substack{f\in F\\ n\in\mathbf{N}}} \big((nz)\curlyvee(np)\big)\curlyvee x \geq \bigvee_{\substack{f\in F\\ m,n\in\mathbf{N}}} (nz)\curlyvee\big((mp)\curlyvee x\big) \\ &= \bigvee_{m\in\mathbf{N}} B\big((mp)\curlyvee x\big) = B(x). \end{aligned}$$

Consequently the elements u_1+u_2 and $(u_1+u_2)\curlyvee p$ generate the same balayage B.

Theorem 2.16. *Let S be an H-cone with a generator p. Denote by $\mathscr{B}$ the set of balayages from S into S. Then $\mathscr{B}$ is an abelian semigroup with respect to the truncated addition defined by*

$$(B_1 \oplus B_2)(x) = \bigvee_{n\in\mathbf{N}} \Big(n\big(B_1(p)+B_2(p)\big)\Big)\curlyvee x$$

for all $x \in S$. Moreover the equality $(B_1\oplus B_2)(x) = (B_1 \vee B_2)(x)$ holds for all $x \in S$.

Proof. Let p be a generator of an H-cone S. Assume that $B_1\colon S \to S$ and $B_2\colon S \to S$ are balayages. By virtue of Lemma 2.15 the truncation addition is well-defined and $B_1 \oplus B_2$ is a balayage generated by $\big(B_1(p) + B_2(p)\big) \curlywedge p$.

Applying [2, Theorem 11.7] we note that $\big(B_1(p)+B_2(p)\big) \curlywedge p$ is a p-quasi-unit and so $(B_1 \oplus B_2)(p) = \big(B_1(p) + B_2(p)\big) \curlywedge p$. We show that

$$(B_1 \oplus B_2)(p) = (B_1 \vee B_2)(p).$$

Assume that $w \preccurlyeq B_1(p) + B_2(p)$ and $w \le p$. Applying [4, Theorem 2.1.5] we find elements t_1, t_2 such that $w = t_1 + t_2$, $t_1 \preccurlyeq B_1(p)$ and $t_2 \preccurlyeq B_2(p)$. Since $B(S)$ is specifically solid by [9, Lemma 2.3], we have $B_1(t_1) = t_1$ and $B_2(t_2) = t_2$. Hence we obtain $B_1 \vee B_2(t_1) = t_1$ and $B_1 \vee B_2(t_2) = t_2$. Reviewing to [7, Proposition 2.1] we infer

$$\begin{aligned} w &= B_1(t_1) + B_2(t_2) = B_1 \vee B_2(t_1) + B_1 \vee B_2(t_2) \\ &= B_1 \vee B_2(t_1 + t_2) = B_1 \vee B_2(w). \end{aligned}$$

Setting $w = \big(B_1(p) + B_2(p)\big) \curlywedge p$ we see that $\big(B_1(p) + B_2(p)\big) \curlywedge p \le B_1 \vee B_2(p)$. On the other hand by [4, Corollary 2.1.3] we have

$$B_1 \vee B_2(p) = B_1(p) \vee B_2(p) \preccurlyeq B_1(p) + B_2(p).$$

Combining this with the inequality $B_1 \vee B_2(p) \le p$ we note that

$$\big(B_1(p) + B_2(p)\big) \curlywedge p \ge B_1 \vee B_2(p).$$

Now it is clear that $\oplus$ is associative. □

References

[1] Arsove, M., and H. Leutwiler: Infinitesimal generators and quasi-units in potential theory. - Proc. Nat. Acad. Sci. U.S.A. 72, 1975, 2498–2500.

[2] Arsove, M., and H. Leutwiler: Algebraic potential theory. - Mem. Amer. Math. Soc. 23:226, 1980.

[3] Arsove, M., and H. Leutwiler: Quasi-units in mixed lattice structures. - Conference in Potential Theory, Copenhagen 1979. Lecture Notes in Mathematics 787, Springer-Verlag, Berlin–Heidelberg–New York, 1980, 35–54.

[4] Boboc, N., Gh. Bucur, and A. Cornea: Order and convexity in potential theory: H-cones. - Lecture Notes in Mathematics 853, Springer-Verlag, Berlin–Heidelberg–New York, 1981.

[5] Boboc, N., Gh. Bucur, and A. Cornea: Autodual H-cones. - Conference in Potential Theory, Copenhagen 1979. Lecture Notes in Mathematics 787, Springer-Verlag, Berlin–Heidelberg–New York, 1980, 64–77.

[6] Constantinescu, C., and A. Cornea: Potential theory on harmonic spaces. - Springer-Verlag, Berlin–Heidelberg–New York, 1972.

[7] Cornea, A., and H. Höllain: Morphisms and contractions on standard H-cones. - Hokkaido Mathematical Journal Special Issue 10, 1981, 157–212.

[8] Eriksson-Bique, S.-L.: Real-valued duals of H-cones. - Math. Scand. 71, 1992, 243–251.

[9] Eriksson-Bique, S.-L.: Characterizations of balayages. - Ann. Acad. Sci. Fenn. Ser. A I Math. 19, 1994, 59–66.

[10] Popa, E.: Balayages and morphisms in H-cones. - An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) 36, 1990, 203–213.

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