Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 87–95

PROPERTIES OF REAL SEWING FUNCTIONS

Juhani V. Vainio

University of Helsinki, Department of Mathematics
P.O. Box 4 (Hallituskatu 15), FIN-00014 Helsinki, Finland

Abstract. This paper, which derives some properties of real sewing functions, is divided into three almost independent sections. In the first, we examine the continuability of uniquely sewing functions. The second section establishes the quasisymmetry of an analytic, strictly increasing function (even at a zero of the derivative) using two different techniques. In the third, a new hyperbolicity condition is given for sewing functions with a singularity.

Introduction

A real sewing function is a homeomorphism φ between two open intervals, such that the condition $\varphi = f_2^{-1} \circ f_1$ holds for the boundary values of some conformal maps f_1, f_2, onto adjacent plane domains. Real sewing functions were studied in [3], [4]; we extend those results here. The present paper is divided into three almost independent sections.

Section 1 shows that the sewing property is continuable when combined with assumptions on uniqueness.

Section 2 focuses on analytic and piecewise analytic functions. It establishes the local quasisymmetry of an analytic, strictly increasing function at a zero of the derivative using two different techniques: directly, and by means of a quasiconformal extension.

Section 3 derives a new hyperbolicity criterion for sewing functions with a singularity. (Hyperbolicity means the lack of the global sewing property.)

1. Continuability

We say that a function φ *sews uniquely* if $\varphi\colon I_1 \to I_2$ is a sewing function (between two intervals) such that all solutions defined in all possible domains are conformally related (the terms coming from [3, Section 1.1]). More precisely, we require that if (f_1, f_2) and (f_3, f_4) are two solutions, the homeomorphic mapping defined by the two compositions $f_{i+2} \circ f_i^{-1}$, analytic in its domain of definition except on the arc $f_1 I_1$, must be analytic on the arc as well.

For brevity, we say that a function $\varphi\colon I_1 \to I_2$ "sews on I" or "sews uniquely on I" (I a subinterval of I_1), meaning that the restriction $\varphi \mid I$ has the property in question. (All intervals in Section 1 can be finite or infinite.)

1991 Mathematics Subject Classification: Primary 30C62; Secondary 30D40, 30F20.

Our first theorem establishes a relation between unique and "ordinary" sewing but does not yet claim that the function φ globally sews uniquely. We need not assume that φ is homeomorphic, since this follows from the other assumptions.

Theorem 1.1. *Let $a < b < c < d$, and let $\varphi\colon\,]a,d[\to \mathbf{R}$ sew on $]a,c[$ and $]b,d[$, uniquely on $]b,c[$. Then φ sews on $]a,d[$.*

Proof. Let φ be as assumed in the theorem. The restrictions $\varphi \mid]a,c[$, $\varphi \mid]b,d[$ sew the lower half-plane H_1 to the upper half-plane H_2; let (f_1,f_2) be a solution for the former function and (f_3,f_4) a solution for the latter. Define a map g by $g = f_{i+2}\circ f_i^{-1}$ in f_iH_i $(i=1,2)$. The two expressions of g agree on the arc $C_0 = f_1]b,c[$. The map is then a homeomorphism of the domain $f_1H_1\cup f_2H_2\cup C_0$, and it is conformal except on C_0. The pairs (f_1,f_2), (f_3,f_4) are solutions for $\varphi \mid]b,c[$. Thus the map g is conformal by the uniqueness assumption. Hence the two solutions introduce a conformal structure into the space obtained by sewing H_1 to H_2 by the identification φ. The Riemann surface obtained is simply connected and hence conformally equivalent to a plane domain. The restrictions of a global map to the half-planes define the desired solution. □

The next two results could be called continuation theorems for unique sewing. In view of Theorem 1.1, one may ask why Theorem 1.2 assumes the global sewing property. Does the property not follow by the the former theorem? It does not, since we do not know whether the uniqueness on an interval is inherited by a subinterval. This also explains why Theorem 1.2 is not a corollary of Theorem 1.3.

Theorem 1.2. *Let $a < b < c < d$, and let $\varphi\colon\,]a,d[\to \mathbf{R}$ sew on $]a,d[$, uniquely on $]a,c[$ and $]b,d[$. Then φ sews uniquely on $]a,d[$.*

Proof. Let φ be as assumed in the theorem, and let (f_1,f_2), (f_3,f_4) be solutions for the (global) function φ. Define a map g by $g = f_{i+2}\circ f_i^{-1}$ in f_iH_i $(i=1,2)$. The two expressions of g agree on the arc $C = f_1]a,d[$. The map is then a homeomorphism of the domain $f_1H_1\cup f_2H_2\cup C$, and it is conformal except on C. Both pairs (f_1,f_2), (f_3,f_4) are solutions for both restrictions $\varphi \mid]a,c[$, $\varphi \mid]b,d[$. Thus, by the uniqueness assumption, the map g is analytic on the overlapping arcs $f_1]a,c[$ and $f_1]b,d[$ and hence analytic on their union C. The map g is then conformal, and the assertion follows. □

Theorem 1.3. *Let $a < b < c$, and let $\varphi\colon\,]a,c[\to \mathbf{R}$ sew on $]a,c[$, uniquely on $]a,b[$ and $]b,c[$. Then φ sews uniquely on $]a,c[$.*

Proof. Let φ be as assumed in the theorem, and let (f_1,f_2), (f_3,f_4) be solutions for φ. Define $g = f_{i+2}\circ f_i^{-1}$ in f_iH_i $(i=1,2)$. The two expressions of g agree on the arc $C = f_1]a,c[$. The map is then a homeomorphism of the domain $f_1H_1\cup f_2H_2\cup C$, and it is conformal except on C. Both pairs (f_1,f_2), (f_3,f_4) are solutions for both restrictions $\varphi \mid]a,b[$, $\varphi \mid]b,c[$. Thus, by the uniqueness assumption, the map g is analytic on the arcs $f_1]a,b[$ and $f_1]b,c[$ and hence analytic except at the point $f_1(b)$. But the single point is removable, the map g is then conformal, and the assertion follows. ▫

Remarks. A locally quasisymmetric function of an open interval sews uniquely (cf. [3, pp. 12, 40]). Hence we obtain for instance by Theorems 1.1 and 1.3: if φ: $]a,c[\to \mathbf{R}$ is locally quasisymmetric in $]a,b[$ and $]b,c[$, and sews on a neighborhood of b, then it sews uniquely on $]a,c[$. The latter assertion also holds if the words "locally quasisymmetric" are replaced by "a strictly increasing function, analytic"; the reason is that, by Theorem 2.2 of the next section, the two analytic restrictions are locally quasisymmetric.

2. Quasisymmetry

In this section we study the quasisymmetry of strictly increasing analytic and piecewise analytic functions, mainly irrespective of the sewing property (which, of course, is implied by quasisymmetry).

Before stating the theorems, we prove a lemma which provides most of the information of Theorem 2.2. Analyticity in a closed (finite) interval naturally means analytic continuability into a wider open interval.

Lemma 2.1. *An analytic, strictly increasing function of a closed interval whose derivative vanishes at either endpoint but not elsewhere is quasisymmetric.*

Proof. We can restrict ourselves to the special case where φ is an analytic, strictly increasing function of an interval $[0,a]$, with $\varphi(0)=0$, $\varphi'(0)=0$, $\varphi'(x)\neq 0$ for $x\neq 0$, because the general case reduces to this by means of translations and the negation.

If φ'' had a zero-approaching sequence of positive zeros, it would vanish identically, φ' would be constant and hence identically zero, a contradiction. Thus there exists a number $b\in]0,a]$ such that $\varphi''(x)\neq 0$ for $x\in]0,b]$. Because φ'' preserves its sign and φ is increasing, φ'' must be positive in $]0,b]$; hence φ is (downwards) convex in $[0,b]$.

Denote

$$Q(x,t)=\bigl[\varphi(x+t)-\varphi(x)\bigr]/\bigl[\varphi(x)-\varphi(x-t)\bigr],$$

with $t>0$ and $x-t,x,x+t\in[0,a]$. We separate two cases, $x\in]0,b/2]$ and $x\in]b/2,a[$.

1) Suppose $x\in]0,b/2]$, implying $t\in]0,x]$. By computation, we have

$$\frac{\partial Q}{\partial t}=\frac{\varphi'(x+t)\bigl[\varphi(x)-\varphi(x-t)\bigr]-\varphi'(x-t)\bigl[\varphi(x+t)-\varphi(x)\bigr]}{\bigl[\varphi(x)-\varphi(x-t)\bigr]^2},$$

where the numerator can be written in the form

$$\varphi'(x+t)\varphi'(\xi_1)t-\varphi'(x-t)\varphi'(\xi_2)t,\qquad \xi_1\in]x-t,x[,\ \xi_2\in]x,x+t[.$$

Here, by convexity,

$$\varphi'(x-t) < \varphi'(\xi_1), \qquad \varphi'(\xi_2) < \varphi'(x+t);$$

hence the numerator is positive. It follows that, for a fixed x, the function $Q(x,t)$ increases with t, and it then attains its maximum for $t = x$. This implies

$$Q(x,t) \le Q(x,x) \le \varphi(2x)/\varphi(x).$$

By analyticity, there exist a whole number $p \ge 2$ and positive numbers c_1, c_2 such that

$$c_1 x^p \le \varphi(x) \le c_2 x^p, \qquad x \in]0, b/2].$$

From the two double inequalities one obtains $Q(x,t) \le 2^p c_2/c_1$; on the other hand, by convexity we have $Q(x,t) \ge 1$.

2) Suppose then $x \in]b/2, a[$. This case divides into two subcases.

a) The case $t \in]0, b/4]$ is included in the case $x-t, x, x+t \in]b/4, a]$, and φ is quasi-isometric in the interval $]b/4, a]$, hence quasisymmetric. This implies that $Q(x,t)$ is now bounded (uniformly for all x, t) both above and away from zero.

b) In the case $t > b/4$ it is clear that both the numerator and the denominator of $Q(x,t)$ are bounded (uniformly) above and away from zero, implying the same to hold for the quotient. □

We now cancel the assumption of Lemma 2.1 on the derivative. Our theorem implies, in particular, the sewing property of an analytic, strictly increasing function. (Theorem 1 of [4] already established it through a different approach.)

Theorem 2.2. *An analytic, strictly increasing function of a closed interval is quasisymmetric.*

Proof. If φ is an analytic, strictly increasing function of a closed interval, with $\varphi'(x) \ne 0$ at every point, then φ is quasi-isometric and hence quasisymmetric. If φ' has zeros, their number is finite, and between them φ is locally quasisymmetric. Hence it suffices to prove the quasisymmetry in a neighborhood of a zero of φ'. So we can restrict ourselves to the case where φ is an analytic, strictly increasing function of an interval $[-a, a]$, with $\varphi(0) = 0$, $\varphi'(0) = 0$, $\varphi'(x) \ne 0$ for $x \ne 0$. Now the quasisymmetry of φ follows by a removability theorem of Kelingos ([1, Theorem 3]): the function φ is quasisymmetric in both $[-a, 0]$ and $[0, a]$ by our Lemma 2.1, and, in addition, the ratio $\varphi(t)/|\varphi(-t)|$ is bounded away from both zero and inifinity (the ratio tends to 1 for $t \to 0$ since by analyticity the values $\varphi(x)$ are approximately equal to values of the form cx^p for $x \to 0$). □

If a quasisymmetric function of an interval is continued beyond an endpoint by reflection, a quasisymmetric function is obtained ([1, Corollary 2]). Hence we get the following corollary of Theorem 2.2: if an analytic, strictly increasing function φ of a closed interval is continued beyond an endpoint (where possibly $\varphi' = 0$) by reflection, a quasisymmetric function is obtained. This result, as well as Theorem 2.2, is a special case of Theorem 2.3. In view of the theorem, we recall that parabolicity originally means the global sewing property (i.e., with a removable singularity).

Theorem 2.3. *Let φ be a strictly increasing function defined in the union of two adjacent closed intervals whose restrictions to both intervals are analytic. Then φ is quasisymmetric if and only if it is parabolic.*

Proof. Let φ be as assumed in the theorem. By Theorem 2.2, the restrictions of φ to the two intervals are quasisymmetric. We can again restrict ourselves to the case where the singularity lies at zero, with $\varphi(0) = 0$. By the Corollary of Theorem 3 of [4], the function φ is parabolic at 0 if and only if the zeros of the two restrictions (at the point zero) have the same multiplicity. If φ is parabolic at 0, quasisymmetry follows by Theorem 3 of [1], as in the proof of Theorem 2.2; the ratio $\varphi(t)/|\varphi(-t)|$ tends to a finite non-zero limit for $t \to 0$ since $\varphi(x)$ has approximate expressions $c_1 x^p$, $c_2 x^p$ for $x > 0$, $x < 0$. Otherwise $\varphi(x)$ has approximate expressions $c_1 x^p$, $c_2 x^q$, $p \neq q$, for $x > 0$, $x < 0$, implying that the above ratio tends to either 0 or ∞ for $t \to 0$; then the function φ is not quasisymmetric. □

Let us estimate the asymptotic behavior of the function $Q(x,t)$ for an analytic, strictly increasing function φ of an interval $[0,a]$, with $\varphi(0) = 0$, $\varphi'(0) = 0$, $\varphi'(x) \neq 0$ for $x \neq 0$. By the proof of Lemma 2.1, we have

$$1 \leq Q(x,t) \leq Q(x,x) = \varphi(2x)/\varphi(x) - 1$$

for all sufficiently small x and all $t \in]0,x]$. By analyticity, $\varphi(x)$ has an expression $cx^p\bigl(1 + \varepsilon(x)\bigr)$, where $c \neq 0$, $p \geq 1$, $\varepsilon(x) \to 0$ for $x \to 0$. The majorant of $Q(x,t)$ then tends to $2^p - 1$ for $x \to 0$. It follows that φ is ϱ-quasisymmetric in an interval $[0,\alpha]$, where ϱ tends to $2^p - 1$ for $\alpha \to 0$.

We want to extend the above function φ quasiconformally. This function can be extended to a quasisymmetric function ψ of the whole real line, yet with a greater dilatation. The function ψ can be extended to a quasiconformal self-map of the upper half-plane whose maximal dilatation by [2] is less than 8 times the dilatation of ψ. We thus get the value $8(2^p - 1)$. But the following theorem shows that the much smaller value p can be approached. (Of course, if $\varphi'(a) \neq 0$, the function φ has a 1-quasiconformal local extension at the point a.) "A local extension at the point a" means that a restriction of φ is extended into a neighborhood of a. "Asymptotic maximal dilatation" means the limit of the maximal dilatation in $U_r(a)$ for $r \to 0$; in our case it simply turns out to be the limit of the dilatation quotient.

Theorem 2.4. *An analytic, strictly increasing function φ of an open interval which obtains a value $\varphi(a)$ with a multiplicity p has a quasiconformal local extension at the point a with asymptotic maximal dilatation p.*

Proof. It suffices to treat the case $a = \varphi(a) = 0$, φ analytic in $]-b,b[$, $\varphi'(x) > 0$ for $x \neq 0$, $\liminf \varphi' > 0$ at $\pm b$. We extend the function φ into the upper half-disk $U_b(0) \cap H_2$ by the map w defined by

$$w(re^{i\theta}) = Re^{i\Phi}, \qquad \Phi = \theta,\ R = \Bigl(1 - \frac{\theta}{\pi}\Bigr)\varphi(r) + \frac{\theta}{\pi}|\varphi(-r)|.$$

The map w is extended into the lower half-disk $U_b(0) \cap H_1$ by reflection. The map w is a homeomorphism. Standard arguments, omitted here, show that w is locally quasiconformal off the origin. Global quasiconformality follows, as we show that the dilatation quotient D tends to the limit p at the origin. (This also implies the assertion on the asymptotic maximal dilatation.) It suffices to consider D in H_2.

On page 32 of [3] one finds the following formula for the complex dilatation of a locally quasiconformal map:

$$\mu(re^{i\theta}) = e^{2i\theta}\frac{rR_r - R\Phi_\theta + i(rR\Phi_r + R_\theta)}{rR_r + R\Phi_\theta + i(rR\Phi_r - R_\theta)},$$

where R, Φ are the polar coordinates of the image of the point $re^{i\theta}$ and the subscripts indicate partial derivation. For w, we have $\Phi_\theta = 1$, $\Phi_r = 0$ and

$$R_r = \varphi'(r) - \frac{\theta}{\pi}\bigl(\varphi'(r) - \varphi'(-r)\bigr),$$
$$R_\theta = -\frac{1}{\pi}\bigl(\varphi(r) + \varphi(-r)\bigr).$$

The expression of μ gives

$$\bigl|\mu(re^{i\theta})\bigr| = \frac{\sqrt{(rR_r - R)^2 + R_\theta^2}}{\sqrt{(rR_r + R)^2 + R_\theta^2}}.$$

We use the following three asymptotic notations when $r \to 0$: $f \sim g$ means that $f/g \to 1$, $o(f)$ means a function g for which $g/f \to 0$, and $\varepsilon(r)$ means a function with limit 0. We now successively obtain

$$\varphi(-r) \sim -\varphi(r), \quad R \sim \varphi(r), \quad R_\theta = o\bigl(\varphi(r)\bigr) = o(R), \quad R_\theta^2 = o(R^2),$$
$$\varphi'(-r) \sim \varphi'(r), \quad R_r \sim \varphi'(r) \sim p\varphi(r)/r, \quad rR_r \sim p\varphi(r) \sim pR,$$
$$rR_r = pR + o(pR) = pR + o(R),$$
$$|\mu| = \frac{R\sqrt{(p-1+\varepsilon(r))^2 + \varepsilon(r)}}{R\sqrt{(p+1+\varepsilon(r))^2 + \varepsilon(r)}} \sim \frac{p-1}{p+1}.$$

Hence $|\mu(re^{i\theta})|$ tends (uniformly for all θ) to the limit $(p-1)/(p+1)$ for $r \to 0$. It follows that $D = (1+|\mu|)/(1-|\mu|)$ tends to p. □

Remarks. If the situation of Theorem 2.4 is altered by allowing a half-open interval containing an endpoint a, the assertion still holds, for the following reasons: the function φ can be continued by reflection beyond the point a (let $a = 0$), whereby we obtain $R = \varphi(r)$, $R_r = \varphi'(r)$, $R_\theta = 0$ and other simplifications in the above proof, while the conclusion remains valid. This modification by reflection applies to the cases with p even and odd, whereas in the original theorem p must be odd. Let us also note that we have established a new proof of Theorem 2.2, because a real boundary value function of a quasiconformal map is known to be locally quasisymmetric.

3. Hyperbolicity

A strictly increasing continuous function φ of an open interval, sewing on both open halves, gives rise to a construction of doubly connected Riemann surfaces. (This is the case of a sewing with a singularity.) The function φ lacks the global sewing property if and only if all the surfaces are of the hyperbolic type (cf. [3, pp. 12–13]); the function φ is then called hyperbolic.

The rather implicit hyperbolicity theorem ([3, Theorem 2.6]) that was proved in [3] yielded the more explicit Theorem 2.8 of [3] (also modified to Theorem 2 of [4]). We now prove another consequence of the same theorem.

Theorem 3.1. *Let $\varphi\colon]-b,b[\to \mathbf{R}$ $(b<\infty)$ be a strictly increasing continuous function, with $\varphi(0)=0$, sewing on both $]-b,0[$ and $]0,b[$, for which $\varphi'(x)\neq 0$ exists for all $x\neq 0$. If there exist real numbers $p>0$ and $q>1$ such that in an interval $]0,a]$, for decreasing x, the values of the ratio $|\varphi(-x)|/x^p$ increase and those of the ratio $\varphi(x)/|\varphi(-x)|^q$ decrease, then φ is hyperbolic.*

Proof. As in Theorem 2.6 of [3], denote

$$\varphi_1(x) = -\varphi^{-1}(-x), \qquad \varphi_2(x) = \varphi\big(\varphi_1(x)\big).$$

The function $|\varphi(-x)|/x^p$ increases for decreasing x if and only if its composition with the increasing function φ_1 has the same property, i.e., if and only if $x/\varphi_1(x)^p$ has it. The latest condition is, in turn, equivalent to the non-positivity of the derivative of $x/\varphi_1(x)^p$; we thus obtain the condition

$$\varphi_1(x)/\varphi_1'(x) \le px,$$

where $\varphi_1'(x)\neq 0$ since $\varphi'(x)\neq 0$. Treating the assumption "$\varphi(x)/|\varphi(-x)|^q$ decreases for decreasing x" in the same manner, one finds the function $\varphi_2(x)/x^q$ to be increasing, implying the condition

$$\varphi_2(x)/\varphi_2'(x) \le x/q,$$

where $\varphi_2'(x) \neq 0$. Since $\varphi_2(x)/x^q$ is increasing, its values are at most $\varphi_2(a)/a^q$ for $x \leq a$, which implies that

$$\varphi_2(x) < x$$

holds for all sufficiently small x, a condition needed in Theorem 2.6 of [3]. There exists a number c such that all the three previous inequalities hold in the interval $]0, c]$.

We now estimate the terms $\varphi_n(x)/\varphi_n'(x)$ in the series of the latter theorem, where $\varphi_n = \varphi \circ \varphi_{n-1}$ for $n = 2, 4, \ldots$, and $\varphi_n = \varphi_1 \circ \varphi_{n-1}$ for $n = 3, 5, \ldots$. For n even, the functions φ_n are the iterations of the function φ_2. We show by induction that

$$\varphi_{2k}(x)/\varphi_{2k}'(x) \leq x/q^k$$

for all $k \geq 1$. The basis, with $k = 1$, has already been established. The induction step $k \to k+1$ is:

$$\frac{\varphi_{2k+2}(x)}{\varphi_{2k+2}'(x)} = \frac{\varphi_2\big(\varphi_{2k}(x)\big)}{\varphi_2'\big(\varphi_{2k}(x)\big)} \cdot \frac{1}{\varphi_{2k}'(x)} \leq \frac{\varphi_{2k}(x)}{q} \cdot \frac{1}{\varphi_{2k}'(x)} \leq \frac{x}{q^{k+1}}.$$

It follows that the sum of the even-indexed terms has a converging geometric ($1/q < 1$) majorant series, and hence it converges. Also the sum of the odd-indexed terms converges, since a term with an index $2k+1$ does not exceed a constant multiple of the term with the index $2k$:

$$\frac{\varphi_{2k+1}(x)}{\varphi_{2k+1}'(x)} = \frac{\varphi_1\big(\varphi_{2k}(x)\big)}{\varphi_1'\big(\varphi_{2k}(x)\big)} \cdot \frac{1}{\varphi_{2k}'(x)} \leq p\varphi_{2k}(x) \cdot \frac{1}{\varphi_{2k}'(x)}.$$

We conclude that the total sum is finite (for all $x \in]\varphi_2(c), c]$). By Theorem 2.6 of [3], the function φ does not sew. Hence φ is hyperbolic. □

Corollary 3.2. *Let φ be as in Theorem 3.1, and further suppose $\varphi(x) = x$ for $x \leq 0$. If there exists a number $r \in]0, 1[$ such that*

$$\varphi(x)/\varphi'(x) \leq rx$$

in an interval $]0, a]$, then φ is hyperbolic.

We remark that one can establish the hyperbolicity of a function φ by testing the validity of the conditions of Theorem 3.1 (or Corollary 3.2) not only for φ but for the three functions defined by the expressions $\varphi^{-1}(x)$, $-\varphi(-x)$ and $-\varphi^{-1}(-x)$ in a neighborhood of the point zero; one "positive result" suffices for the hyperbolicity.

Let us compare our Theorem 3.1 with Theorem 2.8 of [3], which also asserts the hyperbolicity of a function; we will show that neither theorem implies the other. A function φ with the properties

$$\begin{aligned}&\varphi(x) = x \qquad \text{for } x \le 0,\\ &\varphi(x) \sim e^{-1/x}, \quad \varphi'(x) \ge sxD(e^{-1/x}) \qquad \text{for } x > 0 \ (s > 1)\end{aligned}$$

(where $\sim$ has the same meaning as before) satisfies the condition of Corollary 3.2, equivalent to the conditions of Theorem 3.1 in the case $\varphi(x) = x,\ x < 0$. It does not necessarily satisfy a condition of the form

$$\varphi'(x) > \frac{\varphi(x)}{x}\Big(\frac{\ln\varphi(x)}{\ln x}\Big)^k, \qquad k > \tfrac{1}{2}$$

(obtained from the latter theorem in the present case $\varphi_2 = \varphi$), for the reason that there may be intervals where

$$\varphi'(x) < 2sxD(e^{-1/x}) = \frac{2s}{x}e^{-1/x};$$

the latter inequality would contradict the previous one where now $\ln\varphi(x) \sim -1/x$. On the other hand, the function

$$\varphi(x) = x \quad \text{for } x \le 0, \qquad \varphi(x) = xe^{-|\ln x|^p} \quad \text{for } x > 0$$

satisfies the previous inequality for $p > 1/2$ (shown in [3, p. 18]), and the other assumptions of Theorem 2.8 of [3] ($\varphi(x) < x$, φ convex) are also true for φ, but it does not satisfy the condition of Corollary 3.2 if $p \in]1/2, 1[$, as an easy computation shows.

References

[1] Kelingos, J.A.: Boundary correspondence under quasiconformal mappings. - Michigan Math. J. 13, 1966, 235–249.

[2] Reed, T.J.: Quasiconformal mappings with given boundary values. - Duke Math. J. 33, 1966, 459–464.

[3] Vainio, J.V.: Conditions for the possibility of conformal sewing. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 53, 1985, 1–43.

[4] Vainio, J.V.: On the type of sewing functions with a singularity. - Ann. Acad. Sci. Fenn. Ser. A I Math. 14, 1989, 161–167.

Received 5 August 1993