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ON NEVANLINNA'S SECONDARY DEFICIENCY

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Abstract. Analogies between the Nevanlinna theory and the theory of heights in number theory have motivated to the determination the precise error term of the second fundamental theorem of the Nevanlinna theory. The Nevanlinna secondary deficiency, introduced by P.M. Wong, gives the error term in a certain sense. We first prove that, from a topological point of view, almost all meromorphic functions have secondary deficiency negative infinity and that the set of meromorphic functions having maximum secondary deficiency is dense in a standard topology. Then we give an improved upper bound on the secondary deficiency for meromorphic functions of finite order and show that the upper bound is sharp.

1. Introduction

Recently there have been a number of papers concerning Nevanlinna's error terms in value distribution theory because of an analogy between the second fundamental theorem and Roth's theorem of number theory. Based on Serge Lang's conjecture concerning Roth's theorem in [5] and [4] and P. Vojta's Nevanlinna-theory and number-theory dictionary in [11], Lang conjectured, broadly speaking, in the one variable case, that for any nonconstant meromorphic function f in $\mathbf{C}$ and q distinct points $a_1, a_2, \dots, a_q$ in $\mathbf{C} \cup \{\infty\}$, we have

$$(1)\quad qT(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{Ram}}(f,r) \le 2T(f,r) + \log T(f,r) + o\big(\log T(f,r)\big)$$

for all large r outside a set of finite Lebesgue measure. This conjecture was proved to be correct by Lang [6] and Wong [12], while Z. Ye [13] showed that the upper bound is sharp. Moreover, Lang in [6] raised a question of determining whether the inequality (for all large r)

$$(2)\quad qT(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{Ram}}(f,r) \ge 2T(f,r) + \log T(f,r) - o\big(\log T(f,r)\big)$$

holds for almost all (in a suitable sense) meromorphic functions.

The coefficient 2 of the term $T(f,r)$ in (1) is well known as an upper bound of sums of deficient values and indexes of multiplicity of f in Nevanlinna theory.

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Similarly, P.M. Wong [12] introduced the concept of secondary deficiency, which identifies the coefficient of the term $\log T(f,r)$ in (1), or the power index, as its counterpart in Roth theorem. An up-to-date account of these matters appears in [6]. The concept of "almost all" in function spaces was considered by A. Offord (e.g. [9]) from a probabilistic point of view and by P. Gauthier and W. Hengartner in [1] from a topological point of view.

In this paper, we first prove the set of entire (meromorphic) functions having the secondary deficiency one to be dense in a space of entire (meromorphic) functions with τ-topology (τ_χ-topology). Then we show that almost all functions have the secondary deficiency negative infinity. Thus almost all functions fail to satisfy inequality (2). Moreover, we obtain an improved upper bound of Nevanlinna's secondary deficiency for meromorphic functions of finite order and show that the upper bound is sharp.

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2. Preliminaries

Let $M(\mathbf{C})$ and $H(\mathbf{C})$ be the sets of meromorphic functions and entire functions in the complex plane $\mathbf{C}$, respectively. Now we are going to assign topologies to $M(\mathbf{C})$, and hence to $H(\mathbf{C})$, such that $M(\mathbf{C})$ and $H(\mathbf{C})$ become complete metric spaces with the topologies. A detailed discussion can be found in [1].

For any K a compact set in $\mathbf{C}$, any $\varepsilon > 0$ and any $f \in M(\mathbf{C})$, we denote

$$O(f,K,\varepsilon) = \{g \in M(\mathbf{C}) : \|f-g\|_K \equiv \sup_{z\in K} |f(z)-g(z)| \le \varepsilon \text{ and } f-g \in H(K)\}.$$

Clearly all these $O(f,K,\varepsilon)$'s consist of a subbasis of $M(\mathbf{C})$ and generate what we call the τ-topology of uniform convergence on compact sets. It is straightforward to show that $M(\mathbf{C})$ with the τ-topology is a complete metric space. Therefore $(M(\mathbf{C}),\tau)$ is of second Baire category. As usual, we regard $H(\mathbf{C})$ as a subspace of $M(\mathbf{C})$. Thus $H(\mathbf{C})$ is a complete metric space, too, since $H(\mathbf{C})$ is closed in the τ-topology.

Another topology on $M(\mathbf{C})$ is called spherically uniform convergence on compact sets, denoted by τ_χ. For f and g in $M(\mathbf{C})$, set, for any $n \in \mathbf{Z}^+$,

$$d_n(f,g) = \sup_{|z|\le n} \chi\big(f(z),g(z)\big),$$

where χ denotes the spherical metric on the Riemann sphere $\overline{\mathbf{C}}$, with $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Hence, in the usual way, the

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f,g)}{1+d_n(f,g)}$$

is an induced metric on $M(\mathbf{C})$. Clearly $d(f,g) \le d_n(f,g)$ for any n. If $f_n(z) = n$, then f_n is a Cauchy sequence in the τ_χ topology and is not Cauchy in the τ topology. It follows that $\big(M(\mathbf{C}), d\big)$ is not a complete metric space. However, it is not hard to prove that $\big(M(\mathbf{C}) \cup \{\infty\}, d\big)$ is a complete metric space. So $\big(M(\mathbf{C}) \cup \{\infty\}, d\big)$ is of second Baire category.

Finally we give the definition of "almost all" in the topological sense. Let X be a space of functions. We say that almost all functions in X have a certain property if X is of second category while the set of functions in X not possessing the said property is of category one.

As for Nevanlinna theory, standard references are [2] and [8].

3. Results

Let f be a meromorphic function in $\mathbf{C}$, q be any positive integer and $a_1, a_2, \dots, a_q$ be any q (>0) points in $\overline{\mathbf{C}}$. We write

$$S(f, \{a_j\}_1^q, r) = (q-2)T(f,r) - \sum_{j=1}^{q} N(f, a_j, r) + N_{\mathrm{Ram}}(f,r), \tag{3}$$

where $N_{\mathrm{Ram}}(f,r) = N(f', 0, r) + 2N(f, \infty, r) - N(f', \infty, r)$. Thus the definition of secondary deficiency in [12] can be written as

$$\delta_2(f, \{a_j\}) = \liminf_{r\to\infty} \frac{S(f, \{a_j\}_1^q, r)}{\log T(f,r)}. \tag{4}$$

It is known (e.g. [12]) that $\delta_2(f, \{a_j\}) \le 1$ if all a_j's are distinct in $\overline{\mathbf{C}}$. In this paper we are going to prove the following theorems.

Theorem 1. *The set*

$$\mathbf{H}_1 = \{ f \in H(\mathbf{C}) : \exists \text{ distinct } \{a_1, a_2, \dots, a_q\} \subset \overline{\mathbf{C}} \text{ such that } \delta_2(f, \{a_j\}) = 1 \}$$

is dense in $H(\mathbf{C})$ in the τ-topology.

Theorem 2. *The set*

$$\mathbf{H} = \{ f \in H(\mathbf{C}) : \exists \text{ distinct } \{a_1, a_2, \dots, a_q\} \subset \overline{\mathbf{C}} \text{ such that } \delta_2(f, \{a_j\}) > -\infty \}$$

is of first category in $H(\mathbf{C})$ in the τ-topology.

Remark. In 1972, P. Gauthier and W. Hengartner [1] proved that, for almost all (as defined in Section 2) meromorphic functions f, $\sum_{a\in\mathbf{C}} \delta(f,a) = 0$, where $\delta(f,a)$ is the Nevanlinna deficiency. On the other hand, the proof of our Theorem 1 tells us that the set of entire functions having maximum Nevanlinna deficiency, i.e., $\sum_{a\in\mathbf{C}} \delta(f,a) = 2$, is dense in a space of entire functions in the τ-topology. Clearly, $\mathbf{H}_1 \subset \mathbf{H}$. Hence, $\mathbf{H}$ is dense in $(H(\mathbf{C}), \tau)$ by Theorem 1 and $\mathbf{H}_1$ is of first category by Theorem 2. Furthermore, Theorem 2 states that for almost all (as defined in the previous section) entire functions f and for any q distinct points $\{a_1, a_2, \dots, a_q\} \subset \overline{\mathbf{C}}$, $\delta_2(f, \{a_j\}) = -\infty$. Thus, summarizing this discussion, we have

Corollary. *For almost all functions in* $\big(H(\mathbf{C}),\tau\big)$, *we have*

$$\delta_2(f,\{a_j\}) = -\infty \text{ for any } \{a_j\}_1^q \subset \overline{\mathbf{C}}.$$

The set of entire functions having

$$\delta_2(f,\{a_j\}) > -\infty \text{ for some } \{a_j\}_1^q \subset \overline{\mathbf{C}}$$

is dense in $\big(H(\mathbf{C}),\tau\big)$.

Similarly, for a space of meromorphic functions we have

Theorem 3. *The set*

$$\mathbf{M}_1 = \{f \in M(\mathbf{C}) : \exists \text{ distinct } \{a_1, a_2, \ldots, a_q\} \subset \overline{\mathbf{C}} \text{ such that } \delta_2(f,\{a_j\}) = 1\}$$

is dense in $M(\mathbf{C})$ *in the* τ_χ*-topology.*

Theorem 4. *The set*

$$\mathbf{M} = \{f \in M(\mathbf{C}) : \exists \text{ distinct } \{a_1, a_2, \ldots, a_q\} \subset \overline{\mathbf{C}} \text{ such that } \delta_2(f,\{a_j\}) > -\infty\}$$

is of first category in $M(\mathbf{C})$ *in the* τ_χ*-topology.*

The following theorem is about an upper bound of meromorphic functions of finite order.

Theorem 5. *Let* f *be any transcendental meromorphic function of finite order* ϱ. *Then for any distinct points* $\{a_j\}_1^q \subset \overline{\mathbf{C}}$, *we have*

$$\begin{aligned} &\delta_2(f,\{a_j\}) \le 1 - \frac{1}{\varrho} \qquad \text{if } \varrho > 0, \\ &\delta_2(f,\{a_j\}) = -\infty \qquad \text{if } \varrho = 0. \end{aligned}$$

Moreover, for any given $\varrho \ge 0$, *there is an entire function* f *of order* ϱ *and* $\{a_j\}_{j=1}^q \subset \mathbf{C}$ *such that* f *assumes this maximum secondary deficiency.*

4. Proof of Theorems

We begin by constructing a class of entire functions which play an important role in the proof of our theorems.

Let $\varrho > 0$ and $r_n = (n^2/\varrho)^{1/\varrho}$, $n = 1, 2, 3, \ldots$, and set

$$E_\varrho(z) = \prod_{n=1}^{\infty}\left(1 + \left(\frac{z}{r_n}\right)^n\right). \tag{5}$$

Let $r>0$, with $r\in[r_k,r_{k+1})$. Then

$$(6)\qquad \begin{aligned} n(E_\varrho,0,r) &= k(k+1)/2\big(1+o(1)\big)\varrho r^\varrho/2 \quad \text{and}\\ N(E_\varrho,0,r) &= \big(1+o(1)\big)r^\varrho/2 \end{aligned}$$

for all large r. Moreover, for $|z|=r\in[r_k,r_{k+1})$, we have

$$(7)\qquad \begin{aligned} \log|E_\varrho(z)| &= \sum_{n=1}^{k} n\log\Big|\frac{z}{r_n}\Big| + \sum_{n=1}^{k}\log\Big|\Big(\frac{r_n}{z}\Big)^n+1\Big| + \sum_{n=k+1}^{\infty}\log\Big|1+\Big(\frac{z}{r_n}\Big)^n\Big|\\ &= N(E_\varrho,0,r)+I_1+I_2. \end{aligned}$$

It is obvious from (7) and $k=\sqrt{\varrho}\,r_k^{\varrho/2}$ that, for $|z|=r\in[r_k,r_{k+1})$,

$$(8)\qquad |I_1|\le k\log 2\le \sqrt{\varrho}\,r^{\varrho/2}\log 2.$$

Since $\log(1+x)\le x$ for all $0<x\le 1$ and there is a constant $C=C(\varrho)>0$ such that

$$(9)\quad (2+k)^{2/\varrho}-(1+k)^{2/\varrho}\ge \begin{cases} 2(1+k)^{-1+2/\varrho}/\varrho & \text{if } -1+2/\varrho\ge 0\\ 2(2+k)^{-1+2/\varrho}/\varrho & \text{if } -1+2/\varrho\le 0\end{cases} \ge Ck^{-1+2/\varrho},$$

for some constant $C>0$, we have from (9) that, for $|z|=r\in[r_k,r_{k+1})$,

$$(10)\qquad \begin{aligned} |I_2| &\le \log 2+\sum_{n=k+2}^{\infty}\Big(\frac{r_{k+1}}{r_{k+2}}\Big)^n = \log 2+\frac{r_{k+2}}{r_{k+2}-r_{k+1}}\Big(\frac{r_{k+1}}{r_{k+2}}\Big)^{k+2}\\ &\le \log 2+\frac{(k+2)^{2/\varrho}}{(k+2)^{2/\varrho}-(k+1)^{2/\varrho}} \le \log 2+\frac{(2k)^{2/\varrho}}{Ck^{-1+2/\varrho}}\\ &\le \log 2+Ck\le Cr^{\varrho/2}. \end{aligned}$$

It follows from (6), (7), (8) and (10) that E is an entire function and

$$(11)\quad N(E_\varrho,0,r)\le T(E_\varrho,r)\le \log M(E_\varrho,r)=\log E_\varrho(r)\le \big(1+o(1)\big)N(E_\varrho,0,r)$$

for all large r, and so

$$(12)\qquad m(E_\varrho,0,r)=o\big(T(E_\varrho,r)\big)\quad\text{and}\quad \log n(E_\varrho,0,r)=\log T(E_\varrho,r)+O(1).$$

Proof of Theorem 1. Given $f\in H(\mathbf{C})$, a compact set $K\subset\mathbf{C}$ and an $\varepsilon>0$, we will find an $f_*\in\mathbf{H}_1$ such that $\|f-f_*\|_K\le\varepsilon$.

In fact, there is a polynomial P such that $\|f-P\|_K\le\varepsilon/2$. Set

$$(13)\qquad f_\lambda(z)=P(z)\exp\Big(\lambda\int_0^z E(\xi)\,d\xi\Big),$$

where $\lambda \in \mathbf{C}$ is a parameter and $E = E_1$ is as in (5) with $\varrho = 1$. Clearly f_λ is an entire function of z. Furthermore, for $|\lambda| < 1$,

$$
\begin{aligned}
\|f - f_\lambda\|_K &\le \|f - P\|_K + \|P\|_K \left\| 1 - \exp\left(\lambda \int_0^z E(\xi)\, d\xi\right) \right\|_K \\
&\le \frac{\varepsilon}{2} + \|P\|_K |\lambda| \exp\left(\left\| \int_0^z E(\xi)\, d\xi \right\|_K \right).
\end{aligned} \tag{14}
$$

It follows from (14) that there exists a small real number $\varepsilon_0 > 0$ such that for any $\lambda \in \mathbf{C}$ with $|\lambda| \le \varepsilon_0$ we have

$$\|f - f_\lambda\|_K \le \varepsilon. \tag{15}$$

On the other hand, by [8, p. 276], there exists $c \in \{\lambda \in \mathbf{C} \setminus \{0\} : |\lambda| \le \varepsilon_0\}$ with

$$\lim_{r\to\infty} \frac{m\big(P'/(PE), -c, r\big)}{T\big(P'/(PE), r\big)} = 0.$$

Since $T\big(P'/(PE), r\big) = \big(1 + o(1)\big) T(E, r)$ for all large r,

$$\lim_{r\to\infty} \frac{m\big(P'/(PE), -c, r\big)}{T(E, r)} = 0. \tag{16}$$

Now set $f_* = f_c$, where c is from (16) and f_c is defined as in (13). Thus (11) implies that

$$\log T(f_*, r) \le \log T(P, r) + \log r + \log M(E, r) + \log 2 = \big(1 + o(1)\big) T(E, r) \tag{17}$$

for all large r. Hence, set $q = 2$, $a_1 = 0$ and $a_2 = \infty$; we then obtain from (13), (17), (12) and (16) that

$$
\begin{aligned}
S(f_*, \{a_j\}_1^q, r) &= -N(f_*, 0, r) - N(f_*, \infty, r) + N_{\mathrm{Ram}}(f_*, r) \\
&= O(\log r) + N(P' + cPE, 0, r) \\
&= T(P' + cPE, 0, r) - m(P' + cPE, 0, r) + O(\log r) \\
&\ge T(E, r) - m(P'/(PE), -c, r) - m(E, 0, r) + O(\log r) \\
&\ge \big(1 + o(1)\big) \log T(f_*, r)
\end{aligned} \tag{18}
$$

for all large r. It follows from (18) and (15) that $\delta_2(f_*, \{a_j\}) = 1$ and $\|f - f_*\|_K \le \varepsilon$. Thus Theorem 1 is proved.

Proof of Theorem 2. Let k, n and q be any positive integers. Set

$$\begin{aligned} S_k &= \{f \in H(\mathbf{C}) : T(f,r) \le k \text{ for } r \ge k\} \qquad \text{and} \\ B_{qnk} &= \big\{f \in H(\mathbf{C}) : \exists \{a_j\}_1^q \subset \mathbf{C} \text{ with } |a_j| \le n, \text{ and} \\ &S(f,\{a_j\}_1^q, r) \ge -n \log T(f,r), \text{ for } r \ge k; \text{ or} \\ &S(f,\{a_j\}_1^q \cup \{\infty\}, r) \ge -n \log T(f,r) \text{ for } r \ge k\big\}, \end{aligned} \tag{19}$$

where some a_j's may be the same point in $\mathbf{C}$. Clearly $\bigcup_k S_k$ is the set of all constant functions. By the definition of $\mathbf{H}$ in Theorem 2,

$$\mathbf{H} \subset \bigcup_{q=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (B_{qnk} \cup S_k)\,. \tag{20}$$

Now we are going to prove that $B_{qnk} \cup S_k$ is closed and has an empty interior. In fact, for any convergent sequence $\{f_p\}_{p=1}^{\infty} \subset B_{qnk} \cup S_k$ in the τ-topology with, say, f its limit function, we prove that $B_{qnk} \cup S_k$ is closed by showing $f \in B_{qnk} \cup S_k$. If $f \notin S_k$, then $f_p \notin S_k$ for all large p since $T(f_p, r)$ converges to $T(f,r)$. Hence $f_p \in B_{qnk}$ for all large p since $f_p \in B_{qnk} \cup S_k$. Thus for each f_p there are q complex numbers $\{a_j^p\}_{j=1}^q$, with $|a_j^p| \le n$, such that

$$\frac{S(f_p, \{a_j^p\}_{j=1}^q, r)}{\log T(f_p, r)} \ge -n \quad \text{or} \quad \frac{S(f_p, \{a_j^p\}_{j=1}^q \cup \{\infty\}, r)}{\log T(f_p, r)} \ge -n$$

for all $r \ge k$. Since $\{a_j^p\}_{p=1}^{\infty}$ is bounded, for each fixed $0 < j \le q$, by the diagonal method there is a sequence $\{p_s\}_{s=1}^{\infty}$ such that, for each j, $\{a_j^{p_s}\}_{s=1}^{\infty}$ converges to a_j^*, where some of the a_j^*'s may coincide. Since f_p is uniformly convergent to f on any compact set in $\mathbf{C}$, f_p' uniformly converges to f' and $f_{p_s} - a_j^{p_s}$ uniformly converges to $f - a_j^*$ for each j. Hence, by Hurwitz's theorem and noting $f \notin S_k$ for any fixed j, we have for any r with $r \ge k$

$$\lim_{s\to\infty} N(f_{p_s}, a_j^{p_s}, r) = N(f, a^*, r) \quad \text{and} \quad \lim_{s\to\infty} T(f_{p_s}, r) = T(f,r).$$

It turns out that, for any r with $r \ge k$,

$$-n \le \lim_{s\to\infty} \frac{S(f_{p_s}, \{a_j^{p_s}\}_{j=1}^q, r)}{\log T(f_{p_s}, r)} = \frac{S(f, \{a_j^*\}_{j=1}^q, r)}{\log T(f,r)}.$$

We can similarly consider the case $\{a_j^p\}_{j=1}^q \cup \{\infty\}$. Thus $f \in B_{qnk} \cup S_k$, i.e., $B_{qnk} \cup S_k$ is closed.

To prove $\operatorname{int}(B_{qnk} \cup S_k) = \emptyset$, we show that the complement of $B_{qnk} \cup S_k$ is dense in $H(\mathbf{C})$. For any given $f \in H(\mathbf{C})$, a compact set $K \subset \mathbf{C}$ and an $\varepsilon > 0$, let $O(f, K, \varepsilon)$ be an open neighborhood of f; then there is a polynomial P_n of degree $n \ge 1$ such that $\|f - P_n\|_K \le \varepsilon$. Since $T(P_n, r) = n \log r + O(1)$ and

$$S(P_n, \{a_j\}_1^q, r) = \sum m(P_n, a_j, r) - 2T(P_n, r) + N_{\mathrm{Ram}}(P_n, r) + O(1) = -2T(P_n, r),$$

$\delta_2(P_n, \{a_j\}) = -\infty$ for any q with $\{a_j\}_1^q \subset \mathbf{C}$, i.e., $P_n \notin B_{qnk} \cup S_k$. This proves $B_{qnk} \cup S_k$ has an empty interior.

It follows from (20) that $\mathbf{H}$ is contained in a countable union of nowhere dense sets. Therefore $\mathbf{H}$ is of first category and Theorem 2 is proved.

Proof of Theorem 3. It suffices to prove that for any $g \in M(\mathbf{C})$ and any $\varepsilon > 0$ there exists g_* such that $d(g, g_*) \leq \varepsilon$ and $g_* \in \mathbf{M}_1$.

Set $K = \{z \in \mathbf{C}; |z| \leq n_0\}$ for some positive integer n_0. Thus there is a polynomial P_2 such that

$$gP_2 \in H(K) \quad \text{and} \quad |g(z)P_2(z)| + |P_2(z)| \neq 0, \text{ for } z \in K.$$

Let $m = \min_{z\in K}\big(|g(z)P_2(z)|^2 + |P_2(z)|^2\big)^{1/2} > 0$. There is a polynomial P_1 such that

$$|gP_2 - P_1| < \varepsilon m/2 \qquad \text{for all } z \in K.$$

Set

$$g_\lambda(z) = \frac{P_1(z)}{P_2(z)} \exp\bigg(\lambda \int_0^z E(\xi)\, d\xi\bigg),$$

where $\lambda \in \mathbf{C}$ and E is from (13). Thus, for $|\lambda| \leq 1$,

$$\begin{aligned} d(g, g_\lambda) \leq d_{n_0}(g, g_\lambda) &\leq \max_{z\in K} \frac{|g - g_\lambda|}{(1+|g|^2)^{1/2}(1+|g_\lambda|^2)^{1/2}} \\ &\leq \max_{z\in K} \frac{\big|gP_2 - P_1 e^{\lambda\int_0^z E(\xi)\,d\xi}\big|}{(|P_2|^2 + |gP_2|^2)^{1/2}} \\ &\leq \max_{z\in K}\big(|gP_2 - P_1| + |P_1||1 - e^{\lambda\int_0^z E(\xi)\,d\xi}|\big)/m \\ &\leq \frac{\varepsilon}{2} + \frac{|\lambda|}{m}\|P_1\|_K \exp\bigg(\bigg\|\int_0^z E(\xi)\, d\xi\bigg\|_K\bigg). \end{aligned} \tag{21}$$

It follows from the proof of Theorem 1 that there exists g_* such that $d(g, g_*) \leq \varepsilon$ and $g_* \in \mathbf{M}_1$. Thus Theorem 3 is proved.

Proof of Theorem 4. Since f_n may converge in $\big(M(\mathbf{C}) \cup \{\infty\}, \tau_\chi\big)$ to infinity uniformly in any compact set of $\mathbf{C}$, we let

$$S_k = \{f \in M(\mathbf{C}) : T(f, r) \leq k \text{ for } r \geq k\} \cup \{\infty\}.$$

The rest of the proof of Theorem 2 can be carried over here sentence by sentence if $H(\mathbf{C})$, with entire functions and polynomials replaced by $M(\mathbf{C})$, with meromorphic functions and rational functions.

Proof of Theorem 5. For any r and R, $1 < r < R < \infty$, we have (e.g., [3, Lemma 7])

$$S(f, \{a_j\}_1^q, r) \leq \log\bigg\{\frac{R}{r}\frac{T(f, R)}{R - r}\bigg\} + O(1). \tag{22}$$

Furthermore, by a growth lemma (e.g. [3, Lemma 4]), there exists for any constant $C>1$ and any $\varepsilon>0$ a sequence $\{r_j\}$ going to infinity such that

$$T(f,R_j)\le CT(f,r_j) \qquad \text{for all } j,$$

where $R_j=r_j+r_j/T^{\varepsilon}(f,r_j)$. Applying (22) to r_j and R_j, we get

$$S(f,\{a_j\}_1^q,r_j)\le \log\bigl(1+T^{-\varepsilon}(f,r_j)\bigr)+(1+\varepsilon)\log T(f,r_j)-\log r_j+O(1).$$

It follows from $\log T(f,r_j)\le(\varrho+\varepsilon)\log r_j$ that

$$\delta_2(f,\{a_j\})\le 1+\varepsilon-\left(\liminf_{j\to\infty}\frac{\log T(f,r_j)}{\log r_j}\right)^{-1}\le 1+\varepsilon-(\varrho+\varepsilon)^{-1}.$$

By the arbitrariness of ε, the first part of the theorem is proved.

Now we construct some entire functions which indicate the sharpness of the secondary deficiency.

It suffices to prove the second part of the theorem for the case $0<\varrho<+\infty$. Let $f(z)=E_\varrho(z)$ in (5). Then, for $r\in[r_k,r_{k+1})$,

$$\begin{aligned}\frac{zf'(z)}{f(z)}&=\sum_{n=1}^{k}n+\sum_{n=1}^{k-1}\frac{-n}{1+(z/r_n)^n}+\frac{-k}{1+(z/r_k)^k}\\ &\quad+\frac{(k+1)(z/r_{k+1})^{k+1}}{1+(z/r_{k+1})^{k+1}}+\sum_{n=k+2}^{\infty}\frac{n(z/r_n)^n}{1+(z/r_n)^n}\\ &=n(f,0,r)+J_1+\frac{-k}{1+(z/r_k)^k}+\frac{(k+1)(z/r_{k+1})^{k+1}}{1+(z/r_{k+1})^{k+1}}+J_2.\end{aligned}\tag{23}$$

Since, for $1\le n\le k-1$ and $r\in[r_k,r_{k+1})$, we have from the definition of r_k

$$\left|\frac{z}{r_n}\right|^n\ge\left|\frac{k}{n}\right|^{2n/\varrho}\ge 1+\frac{2n}{\varrho}\frac{k-n}{n}\qquad \text{if } n\ge\varrho/2,$$

then, using (23) and (6) and noting $k\le\sqrt{\varrho}\,r^{\varrho/2}$ for all large r,

$$\begin{aligned}|J_1|&\le O(1)+\sum_{n=[\varrho/2]}^{k-1}\frac{n}{(r/r_n)^n-1}\le O(1)+\sum_{n=1}^{k-1}\frac{\varrho n}{2(k-n)}\\ &=O(k\log k)=o\bigl(n(f,0,r)\bigr).\end{aligned}\tag{24}$$

Since, for $n\ge k+2$ and $r\in[r_k,r_{k+1})$, we obtain from $(1-x)\le e^{-x}$

$$\left|\frac{z}{r_n}\right|^n\le\left|\frac{r_{k+1}}{r_n}\right|^n=\left|\frac{k+1}{n}\right|^{2n/\varrho}\le\exp\left(-\frac{n-k-1}{n}\frac{2n}{\varrho}\right)=e^{-2(n-k-1)/\varrho},$$

then, again using (23) and (6) for all large r,

$$(25)\quad |J_2| \le \sum_{n=k+2}^{\infty} \frac{n}{(r_n/r)^n - 1} \le \sum_{n=k+2}^{\infty} \frac{n}{e^{2(n-k-1)/\varrho} - 1} = O(k) = o\bigl(n(f,0,r)\bigr).$$

Moreover, if $|a| \ge 1$ and $a \ne -1$, then $\operatorname{Re}\{1/(1+a)\} \le 1/2$; thus, for $r \in [r_k, r_{k+1})$,

$$(26)\qquad \operatorname{Re}\bigl\{1/\bigl(1+(z/r_k)^k\bigr)\bigr\} \le 1/2 \qquad \text{if } (z/r_k)^k \ne -1.$$

To estimate the remaining term in (23), set $T_k = (r_k + r_{k+1})/2$. Since

$$(1+x)^{k+1} \ge 1 + (k+1)x$$

for $x > 0$, and (9), we find a constant $C > 0$ such that, for $r \in [r_k, T_k]$,

$$(27)\qquad \begin{aligned} \left(\frac{r_{k+1}}{r}\right)^{k+1} &\ge \left(\frac{2r_{k+1}}{r_k + r_{k+1}}\right)^{k+1} = \left(1 + \frac{(k+1)^{2/\varrho} - k^{2/\varrho}}{(k+1)^{2/\varrho} + k^{2/\varrho}}\right)^{k+1} \\ &\ge 1 + (k+1)\frac{Ck^{-1+2/\varrho}}{2(k+1)^{2/\varrho}} \ge 1 + \frac{C}{2^{2/\varrho}} \end{aligned}$$

for all large k. It follows from (23)–(27) that, for $r \in [r_k, T_k]$ and $(z/r_k)^k \ne -1$,

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)}\right| &\ge \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \\ &\ge n(f,0,r) - |J_1| - k \operatorname{Re}\frac{1}{1+(z/r_k)^k} - |J_2| - \left|\frac{(k+1)(z/r_{k+1})^{k+1}}{1+(z/r_{k+1})^{k+1}}\right| \\ &\ge n(f,0,r) + o\bigl(n(f,0,r)\bigr) - \frac{k}{2} + o\bigl(n(f,0,r)\bigr) - \frac{k+1}{|r_{k+1}/z|^{k+1} - 1} \\ &\ge (1+o(1))n(f,0,r). \end{aligned}$$

It follows from (6) that, for $r \in [r_k, T_k]$,

$$(28)\quad \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f'}{f}(re^{i\theta})\right| d\theta \ge \log n(f,0,r) - \log r + O(1) \ge (\varrho - 1)\log r + O(1).$$

For any $r \in [T_k, r_{k+1})$, applying Jensen's formula to f'/f and noting $N(f,0,r) = N(f,0,T_k)$, we have from (28), (6), (12) and

$$1 \ge T_k/r \ge T_k/r_{k+1} = \bigl(1 + (r_k/r_{k+1})\bigr)/2 \to 1 \qquad (\text{as } k \to \infty)$$

that

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\log\Bigl|\frac{f'}{f}(re^{i\theta})\Bigr|\,d\theta &= N(f'/f,0,r)-N(f'/f,\infty,r)+O(1)\\
&= N(f',0,r)-N(f,0,r)+O(1)\\
&\ge N(f',0,T_k)-N(f,0,T_k)+O(1)\\
&\ge N(f'/f,0,T_k)-N(f'/f,\infty,T_k)+O(1)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\Bigl|\frac{f'}{f}(T_ke^{i\theta})\Bigr|\,d\theta+O(1)\\
&\ge (\varrho-1)\log T_k+O(1)\\
&\ge (\varrho-1)\log r+(\varrho-1)\log(T_k/r)+O(1)\\
&\ge (\varrho-1)\log r+O(1).
\end{aligned}
\tag{29}
$$

It follows from (28) and (29) that, for all large r,

$$
\frac{1}{2\pi}\int_0^{2\pi}\log\Bigl|\frac{f'}{f}(re^{i\theta})\Bigr|\,d\theta \ge (\varrho-1)\log r+O(1). \tag{30}
$$

Thus for all large r we have from Theorem 3, Jensen's formula and (30)

$$
\begin{aligned}
S(f,\{0,\infty\},r) &= N_{\mathrm{Ram}}(f,r)-N(f,0,r)-N(f,\infty,)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log|f'/f|(re^{i\theta})\,d\theta \ge (\varrho-1)\log r+O(1).
\end{aligned}
$$

It turns out from $\log T(f,r)=\varrho\log r+O(1)$ that $\delta_2(f,\{0,\infty\})=1-1/\varrho$.

Thus Theorem 5 is completely proved.

Remark. Functions similar to (5) were considered in [13], [3] and [7]. However, our functions here make (30) hold for all large r rather than some large r. With $\varrho=\infty$ we have shown in Theorem 1 that the set of entire functions having Nevanlinna's secondary deficiency one is dense in $H(\mathbf{C})$.

5. Examples

In this section we calculate Nevanlinna's second deficiency for some classical functions, based on a recent work of L. Sons and Z. Ye in [10].

Example 1. Let f be any rational function; then, by Theorem 5, $\delta_2(f,\{a_j\})=-\infty$ for any $\{a_j\}\subset\mathbf{C}$.

Example 2. Let $f(z)=e^z$. As we showed in [10], $S(e^z,\{a_j\}_1^q,r)=O(1)$ for any $\{a_j\}_1^q\subset\mathbf{C}\setminus\{0\}$. Since $T(e^z,r)=r/\pi$, we have $\delta_2(f,\{a_j\}_1^q)=0$. Therefore the coefficient of term $\log T(e^z,r)$ in (1) is 0. In fact, if the order of f is less than one, then, by Theorem 5, $S(f,\{a_j\}_1^q,r)<0$ for at least a sequence of $\{r_j\}$ which tends to infinity.

Example 3. Let $\wp$ be a Weierstrass function. Since the order of $\wp$ is two and $S(\wp, \{a_j\}_1^q, r) = O(1)$ for any $\{a_j\}_1^q \subset \mathbf{C}$ as we have shown in [10, Theorem 1], it follows that $\delta_2(\wp, \{a_j\}_1^q) = 0$.

Example 4. Let $\zeta(z) = \zeta(z; \omega_1, \omega_2)$ be a Weierstrass ζ-function with $\mathrm{Im}\omega_1/\omega_2 > 0$. Since $S(\zeta, \{a_j\} \cup \{0, \infty\}, r) = -\log r + O(1)$ and $\log T(\zeta, r) = 2\log r + O(1)$ (see [10, Theorem 2]), then $\delta_2(\zeta, \{a_j\} \cup \{0, \infty\}) = -1/2$, for any $\{a_j\} \subset \mathbf{C} \setminus \{0\}$. Furthermore, we know from [10, Theorem 2] that the L-type of ζ is $\psi(r) = r^{-1/2}$. Hence we see the secondary deficiency of ζ to be exactly equal to the power index in its L-type.

Example 5. Let $\vartheta(z)$ be a Weierstrass ϑ-function. Then from [10, Theorem 4] we have $S(\vartheta, \{a_j\} \cup \{0, \infty\}, r) = \log r + O(1)$ and $\log T(\vartheta, r) = 2\log r + O(1)$; thus $\delta_2(\vartheta, \{a_j\} \cup \{0, \infty\}) = 1/2$ for any $\{a_j\} \subset \mathbf{C} \setminus \{0\}$. Again the $1/2$ identifies the power index in the L-type of ϑ function.

There are other functions which were considered in [10]. Similarly, their secondary deficiencies can be easily calculated.

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