Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 123–148

STRONGLY UNIFORM DOMAINS AND PERIODIC QUASICONFORMAL MAPS

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Abstract. An m-uniform domain in $\mathbf{R}^n$ is, roughly, a domain such that any two maps from the i-sphere, $0 \le i \le m < n$, into the domain can be homotoped to each other without going too far or too close to the boundary, when seen from the perspective of the images of the two maps. We establish several equivalent definitions for m-uniform domains. We apply the theory by investigating the structure of the complementary domains of the fixed point set of a quasiconformal reflection on the n-sphere $\mathbf{S}^n$. Moreover, we establish the (ordinary) uniformity of the complement of the fixed point set of an arbitrary periodic quasiconformal homeomorphism of $\mathbf{S}^n$.

1. Introduction

A domain D in Euclidean n-space $\mathbf{R}^n$ is uniform if, roughly speaking, any point in D can be joined to any other point in D without (i) going too far, and (ii) going too close to the boundary of D, compared to the location and mutual distance of the two points. Uniform domains were introduced by Martio and Sarvas [MS] in their study of injectivity problems in function theory. There are several equivalent definitions for uniformity and the usefulness of the concept is well established; see, for example, [Ge], [GO], [M1], [V4], [Vu2]. Here we take the following definition: a domain $D \subset \mathbf{R}^n$ is said to be *c-uniform*, or simply *uniform*, if there exists a constant $c \ge 1$ such that each pair of points $x_1, x_2 \in D$ can be joined by a path F in D for which

$$\operatorname{diam}(F) \le c\,|x_1 - x_2| \tag{1.1}$$

and

$$\operatorname{dist}\bigl(x, \{x_1, x_2\}\bigr) \le c\,\operatorname{dist}(x, \partial D) \tag{1.2}$$

1991 Mathematics Subject Classification: Primary 30C65.

(1) Supported by NSF Grant DMS 91-48183 and the Academy of Finland. Sloan Fellow.

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for each $x \in F$. Observe that an equivalent definition results when F is allowed to be an arbitrary continuum [V4]. Here and in what follows, $\operatorname{diam}(\cdot)$ denotes Euclidean diameter and $\operatorname{dist}(\cdot,\cdot)$ denotes Euclidean distance with the convention that $\operatorname{dist}(x,\partial D)=\infty$ if $D=\mathbf{R}^n$.

The boundary of a uniform domain need not be of the same dimension everywhere. For example, $\mathbf{B}^n \setminus [0,e_1]$, the unit ball of $\mathbf{R}^n$ minus a radius, is uniform provided $n \ge 3$. In this paper we suggest a definition for a stronger concept of uniformity that detects such lower dimensional impurities on the boundary. The idea is to replace points in the definition with spheres of higher dimension and paths with homotopies between spheres.

Before stating the definition, we fix some notation observed throughout this paper. We let $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\} \approx \mathbf{S}^n$ denote the one-point compactification of $\mathbf{R}^n$, and unless otherwise stated $n \ge 2$. By a *domain* we mean a connected, open subset of $\mathbf{R}^n$. For a nonnegative integer i, $\mathbf{B}^{i+1}$ stands for the open unit ball of $\mathbf{R}^{i+1}$ with $\mathbf{S}^i = \partial \mathbf{B}^{i+1}$ its boundary. In this paper, a *singular i-sphere* is a pair (Σ^i, g), where Σ^i is a subset of $\mathbf{R}^n$ and g is a continuous map of $\mathbf{S}^i$ onto Σ^i. Then we also say that g is a *representation* of Σ^i. Usually we drop the mapping g from the notation and call simply Σ^i a singular i-sphere; it is important however to understand that every singular i-sphere comes with its representation. We also write $(\Sigma^i, g) \subset X$, meaning that Σ^i is a subset of X. Moreover, if $\Sigma^i = (\Sigma^i, g)$ is a singular i-sphere and h is a continuous map on Σ^i, we write $h(\Sigma^i)$ for the singular i-sphere $\bigl(h(\Sigma^i), h\circ g\bigr)$.

1.3. Definition. Let $\varrho\colon [0,\infty) \to [1,\infty)$ be an increasing function and $m < n$ a nonnegative integer. We say that a domain $D \subset \mathbf{R}^n$ is *(m,ϱ)-uniform* if, for each $0 \le i \le m$, each pair of singular i-spheres $\Sigma^i = (\Sigma^i, g)$ and $\tilde\Sigma^i = (\tilde\Sigma^i, \tilde g)$ in D can be joined by a homotopy F in D such that

$$\operatorname{diam}(F) \le \varrho(t)\, \operatorname{diam}(\Sigma^i \cup \tilde\Sigma^i), \tag{1.4}$$

and

$$\operatorname{dist}(x, \Sigma^i \cup \tilde\Sigma^i) \le \varrho(t)\, \operatorname{dist}(x, \partial D) \tag{1.5}$$

for each $x \in F$, whenever

$$\max\{\operatorname{diam}(\Sigma^i), \operatorname{diam}(\tilde\Sigma^i)\} \le t\, \operatorname{dist}(\Sigma^i \cup \tilde\Sigma^i, \partial D). \tag{1.6}$$

We say that D is *m-uniform* if it is (m,ϱ)-uniform for some ϱ, and D is *(m,c)-uniform* if one can choose $\varrho(t) \equiv c$ for some constant c. We term D *strongly uniform* if it is m-uniform for all $m < n$.

In the above definition, and throughout this paper, by "a homotopy F in D joining Σ^i and $\tilde\Sigma^i$" we mean a continuous map

$$F\colon \mathbf{S}^i \times [0,1] \to D$$

such that

$$F(x,0) = g(x) \quad \text{and} \quad F(x,1) = \tilde{g}(x)$$

for each $x \in \mathbf{S}^i$, where $g\colon \mathbf{S}^i \to \Sigma^i$ and $\tilde{g}\colon \mathbf{S}^i \to \tilde{\Sigma}^i$ are the representations of Σ^i and $\tilde{\Sigma}^i$, respectively. By $x \in F$ we mean that x is a point in the image of $\mathbf{S}^i \times [0,1]$ under the map F. Thus we use F to denote both the map and its image $F\bigl(\mathbf{S}^i \times [0,1]\bigr)$.

We reserve the letter m for a nonnegative integer which is less than n, the dimension of the ambient space.

1.7. Remarks. (a) A domain D is m-uniform for $m = 0$ if and only if it is uniform in the ordinary sense. Indeed, it is immediate that $(0,\varrho)$-uniformity implies c-uniformity with $c = \varrho(0)$. Conversely, suppose that D is c-uniform and let g and $\tilde{g}$ be two continuous maps of $\mathbf{S}^0 = \{-1,1\}$ into D. Then we can find two paths $\gamma_{-1}, \gamma_1\colon [0,1] \to D$ joining $g(-1), \tilde{g}(-1)$ and $g(1), \tilde{g}(1)$, respectively, such that the c-uniformity conditions (1.1) and (1.2) hold when F is replaced with γ_{-1} and γ_1 and c is replaced with $c' = c'(c)$ (see [V4, 2.6]). It is easy to check that the homotopy $F\colon \{-1,1\} \times [0,1] \to D$,

$$F(-1,s) = \gamma_{-1}(s), \qquad F(1,s) = \gamma_1(s),$$

satisfies (1.4) and (1.5) with $\varrho(t) = 3c'$. In particular, an m-uniform domain is always uniform.

(b) An m-uniform domain is always m-connected, *i.e.* its ith homotopy group is trivial for all $0 \le i \le m$. We could have defined a weaker notion of m-uniformity by only demanding that (1.4) and (1.5) hold for $i = m$. This would result in a parallel but somewhat different theory, and we believe that the given definition is more natural.

Clearly, a uniform domain need not be (m,ϱ)-uniform for $m > 0$ even if it is m-connected. For instance, if e is any unit vector in $\mathbf{R}^n$, then for $n \ge 3$ the domain $D = \mathbf{B}^n \backslash [0,e]$ is uniform and contractible but not $(n-2)$-uniform.

(c) The definition for ordinary uniformity is usually extended to domains in $\overline{\mathbf{R}}^n$ by declaring a domain D to be uniform if $D \cap \mathbf{R}^n$ is uniform. The advantages of a similar definition in general are not clear. The spherical metric could be used to deal with the general case, but for simplicity we restrict our study to domains that do not contain the point at infinity. Also note that $\mathbf{R}^n$ is trivially strongly uniform.

(d) While preparing this manuscript, it was brought to our attention that P. Alestalo in his forthcoming dissertation [A] has independently studied domains which he calls (m,c)-uniform. According to Alestalo's definition, a domain $D \subset \mathbf{R}^n$ is (m,c)-uniform if there exists a constant $c \ge 1$ such that every continuous map $f\colon \mathbf{S}^m \to D$ can be extended to a continuous map $F\colon \overline{\mathbf{B}}^{m+1} \to D$ satisfying

(i) $\operatorname{diam} F(\overline{\mathbf{B}}^{m+1}) \le c \operatorname{diam} f(\mathbf{S}^m)$ and (ii) $\operatorname{dist}\bigl(F(x), f(\mathbf{S}^m)\bigr) \le c \operatorname{dist}\bigl(F(x), \partial D\bigr)$ for each $x \in \overline{\mathbf{B}}^{m+1}$. It is obvious that if D is (m,c)-uniform in the sense of 1.3, then D is (i,c) uniform in the sense of Alestalo for all $0 \le i \le m$. Conversely, it is not difficult to show that if a domain is (i,c)-uniform in the sense of Alestalo for all $0 \le i \le m$, then it is $(m,3c)$-uniform.

We do not know whether an (m,ϱ)-uniform domain is always (m,c)-uniform for some constant c. [See "Added in March 1994" below.]

This work was motivated by our examination of Yang's proof [Y] that "quasiconformal reflection domains are uniform". We believe that a quasiconformal reflection domain satisfies conditions which are much more severe than those required from a uniform domain, and in Section 7 below we verify an observation in this direction: a quasiconformal reflection domain is strongly uniform provided it is 1-uniform. We also study more general periodic quasiconformal maps in $\overline{\mathbf{R}}^n$. We prove that if $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a periodic quasiconformal map and not a reflection, then $D = \overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$ is a uniform domain.

The paper is organized as follows. In Section 2 we give a sufficient condition for m-uniformity from which many examples of strongly uniform domains follow. Section 3 is devoted to an auxiliary geometric characterization of m-uniform domains in terms of plumpness; this is needed in Section 4 where we prove one of the main results in this paper: the compactness characterization for m-uniformity. The idea is similar to Väisälä's in [V4], but one expects extra topological difficulties in this case. By using the compactness characterization, in Section 5 we deliver another geometric criterion for m-uniformity which should be fairly easy to check in practice; this is a quantitative connectivity condition given in terms of the quasihyperbolic metric and resembles Gromov's notion of *k-contractibility* [Gr, p. 139]. In Section 6 we prove that m-uniformity is invariant under quasimöbius maps; in particular, it is invariant under quasiconformal self maps of $\mathbf{R}^n$. This result is again based on compactness. In Section 7 we study the fixed point sets of periodic quasiconformal maps.

We wish to point out that many of the proofs here—some with very few changes—are valid in more general metric spaces. For instance, the compactness characterization holds in any metric space that is homeomorphic to $\mathbf{R}^n$ and possesses an appropriate group of similarities. Important examples of such spaces are the so-called homogeneous groups equipped with a Carnot–Carathéodory metric; for instance, the n-dimensional Heisenberg group for each $n \ge 1$ is such a group. For simplicity of notation, we formulate our results in $\mathbf{R}^n$.

Added in March 1994. While this paper was being written, we learned that P. Alestalo at the University of Helsinki was working on similar ideas. As it turned out, there is little overlap between his and our work. We did find many interesting and new ideas in [A] and now these two works complement each other in a nice way. Most importantly from our point of view, Alestalo shows that an (m,ϱ)-

uniform domain is always (m,c)-uniform for some constant c, thus answering our question in the affirmative. In proving this, he uses Theorem 4.3 of this paper together with a theory of higher order plumpness, developed in [A]. Furthermore, Alestalo's concept of homological uniformity will undoubtedly be useful in the study of periodic quasiconformal maps. We chose to leave the first part of the present paper more or less in the form it was submitted, and intend to return to the consequences of Alestalo's theory to periodic maps in a future note.

Acknowledgements. We wish to thank Fred Gehring for his encouragement and suggestions, Robert Edwards for helpful discussions concerning Lemma 4.15, Jussi Väisälä for bringing Alestalo's work to our attention, Pekka Alestalo for telling us more about his work, and Morton Brown and Gaven Martin for some useful references. Special thanks go to Geoffrey Mess who pointed out an error in an earlier version of this paper (we beg the colleagues who have the flawed version to recycle it) and for valuable related discussions, and again Jussi Väisälä, the referee, for his usual meticulous scrutiny of the manuscript which uncovered errors and led to many improvements in the text.

2. A sufficient condition for strong uniformity

For a domain D we let $k_D(\cdot,\cdot)$ denote the *quasihyperbolic metric* in D, defined by using the density $\operatorname{dist}(x,\partial D)^{-1}\,|dx|$ for $x\in D$. We refer the reader to [GP], [GO], [M1], [TV2, Section 6], or [Vu2] for the precise definition and basic properties of this metric. Gehring and Osgood [GO] proved the existence of *quasihyperbolic geodesics*; that is, for each pair of points $x,y\in D$ there is an arc γ in D from x to y such that the quasihyperbolic length of γ equals $k_D(x,y)$. It is easy to see that such a geodesic need not be unique. We say that D has the *unique quasihyperbolic geodesic property*, or UQHG property, if every pair of points in D can be joined by a unique quasihyperbolic geodesic in D. Generally, we have seen that uniformity does not imply strong uniformity. The next result shows that if D has the UQHG property, then uniformity does imply strong uniformity.

2.1. Theorem. *If a domain D is c-uniform and has the* UQHG *property, then D is (m,c')-uniform for all m, where $c'=c'(c,n)$ is a constant.*

Proof. Let $\Sigma^i=(\Sigma^i,g)$ and $\tilde\Sigma^i=(\tilde\Sigma^i,\tilde g)$ be two singular i-spheres, $i<n$, in D. For each $x\in\mathbf{S}^i$, let $\gamma_x\colon[0,1]\to D$ parametrize the unique quasihyperbolic geodesic joining $g(x)$ and $\tilde g(x)$ in such a way that

$$\gamma_x(0)=g(x),\qquad \gamma_x(1)=\tilde g(x)$$

and

$$k_D\bigl(\gamma_x(0),\gamma_x(t)\bigr)=tk_D\bigl(\gamma_x(0),\gamma_x(1)\bigr)$$

for all $t\in[0,1]$. Next define $F\colon\mathbf{S}^i\times[0,1]\to D$ by

$$F(x,t)=\gamma_x(t)$$

for all $x \in \mathbf{S}^i$ and $t \in [0,1]$. We shall show that F is a desired homotopy between Σ^i and $\tilde{\Sigma}^i$.

To show that F is a homotopy from g to $\tilde{g}$, it suffices to verify the continuity of F. For this we first show that $F(x,t)$ is continuous with respect to x. If this is not the case, then there exist $(x,t) \in \mathbf{S}^i \times [0,1]$ and a sequence (x_ν), $x_\nu \in \mathbf{S}^i$, such that $x_\nu \to x$ and

$$F(x_\nu, t) \to z \neq F(x,t).$$

Then it follows that

$$\begin{aligned} k_D\bigl(F(x,0), z\bigr) &= \lim_{\nu\to\infty} k_D\bigl(F(x_\nu,0), F(x_\nu,t)\bigr) \\ &= \lim_{\nu\to\infty} t k_D\bigl(F(x_\nu,0), F(x_\nu,1)\bigr) \\ &= t k_D\bigl(F(x,0), F(x,1)\bigr). \end{aligned}$$

Similarly,

$$k_D\bigl(z, F(x,1)\bigr) = (1-t) k_D\bigl(F(x,0), F(x,1)\bigr).$$

This shows that there is more than one quasihyperbolic geodesic joining $F(x,0)$ and $F(x,1)$, contradicting the UQHG property of D. Thus $F(x,t)$ is continuous with respect to x. Next, we use [GO, 1.2] and estimate

$$\begin{aligned} |F(x,t) - F(x,s)| &\le \bigl(\exp\bigl\{k_D\bigl(F(x,t), F(x,s)\bigr)\bigr\} - 1\bigr) \operatorname{dist}\bigl(F(x,t), \partial D\bigr) \\ &= \bigl(\exp\bigl\{|t-s| k_D\bigl(F(x,0), F(x,1)\bigr)\bigr\} - 1\bigr) \operatorname{dist}\bigl(F(x,t), \partial D\bigr). \end{aligned}$$

Since the terms on the right are bounded in x, we see that F is uniformly continuous with respect to t. Thus F is a homotopy from g to $\tilde{g}$.

Finally, we prove that F satisfies the uniformity conditions (1.4) and (1.5) with $\varrho = c'$ for some constant $c' = c'(c,n)$. Since D is uniform, by [GO, Corollary 2] there exists $c_0 = c_0(c,n) > 0$ such that for any $(x,t) \in \mathbf{S}^i \times [0,1]$ we have

$$\operatorname{dist}\bigl(\gamma_x(t), \{\gamma_x(0), \gamma_x(1)\}\bigr) \le c_0 \operatorname{dist}\bigl(\gamma_x(t), \partial D\bigr) \tag{2.2}$$

and

$$l(\gamma_x) \le c_0 |\gamma_x(0) - \gamma_x(1)|, \tag{2.3}$$

where $l(\gamma_x)$ is the Euclidean length of the geodesic γ_x. For condition (1.4), choose $(x,t), (y,s) \in \mathbf{S}^i \times [0,1]$ such that

$$\operatorname{diam}(F) = |F(x,t) - F(y,s)|.$$

Then

$$\begin{aligned}\operatorname{diam}(F) &\le |\gamma_x(t)-\gamma_x(0)| + |\gamma_x(0)-\gamma_y(0)| + |\gamma_y(0)-\gamma_y(s)|\\ &\le c_0|\gamma_x(0)-\gamma_x(1)| + \operatorname{diam}(\Sigma^i) + c_0|\gamma_y(0)-\gamma_y(1)|\\ &\le (2c_0+1)\operatorname{diam}(\Sigma^i\cup\tilde\Sigma^i).\end{aligned}$$

For condition (1.5), let $(x,t)\in \mathbf{S}^i\times[0,1]$. Then

$$\operatorname{dist}\bigl(F(x,t),\Sigma^i\cup\tilde\Sigma^i\bigr) \le \operatorname{dist}\bigl(\gamma_x(t),\{\gamma_x(0),\gamma_x(1)\}\bigr) \le c_0\operatorname{dist}\bigl(\gamma_x(t),\partial D\bigr).$$

This completes the proof of Theorem 2.1.

Let D be a half-space in $\mathbf{R}^n$. Then the quasihyperbolic metric in D coincides with the hyperbolic metric in D, and so D has the UQHG property. On the other hand, if D is a ball in $\mathbf{R}^n$, the above proof obviously applies when quasihyperbolic geodesics are replaced by hyperbolic geodesics. Since geodesics in balls and half-spaces satisfy (2.2) and (2.3) with an absolute constant [V6, 6.6, 6.19], we obtain the following corollary:

2.4. Corollary. *Balls and half-spaces in* $\mathbf{R}^n$ *are* (m,c)*-uniform for all* m *with an absolute constant* c.

It is easy to see that if $f\colon D\to D'$ is bi-Lipschitz and D is m-uniform, then D' is m-uniform. In particular, any bi-Lipschitz image of a ball or a half-space is (m,c)-uniform for all m, where the constant c depends only on the bi-Lipschitz constant of f. Thus the bi-Lipschitz Fox–Artin ball in $\mathbf{R}^3$ (see [M1, 3.7]) is $(1,c)$-uniform. Interestingly, since the complement of that Fox–Artin ball can be made non-simply connected, we see that the complement of a bi-Lipschitz 3-cell in $\mathbf{R}^3$ need not be 1-uniform. For the ordinary uniformity no example like this is possible [V5, 5.10]. (See also Example 6.6 below.)

We prove in Section 6 that m-uniform domains are invariant under quasiconformal maps $f\colon \mathbf{R}^n\to\mathbf{R}^n$.

3. Plumpness and (m,b)-pairs

In this section we characterize m-uniform domains in other geometric terms; this result will be needed in the next section. Similar characterization for uniform domains was established by Väisälä in [V4], and we build on his ideas. We let $B(x,r)$ denote an open n-ball with center x and radius r.

3.1. Definition [V4, 2.13]. An open set $U\subset\mathbf{R}^n$ is *a-plump*, $a\ge 1$, if for every $x\in U$ and $0<r<\operatorname{diam}(U)$ there exists $z\in\overline{B}(x,r)$ such that $B(z,r/a)\subset U$.

The following two definitions are modified after [V4, 2.13]:

3.2. Definition. A pair of singular i-spheres Σ^i, $\tilde{\Sigma}^i$ in a domain D is said to be an (i,b)-*pair*, $b \geq 1$, if there exist $r_1, r_2 > 0$ with $1/2 \leq r_1/r_2 \leq 2$ such that

$$\Sigma^i(r_1) \subset D, \qquad \tilde{\Sigma}^i(r_2) \subset D$$

and

$$\operatorname{dist}(\Sigma^i, \tilde{\Sigma}^i) \leq 4b \max\{r_1, r_2\},$$

where $E(s)$ denotes the s-neighborhood of a set E,

$$E(s) = \bigcup_{x \in E} B(x, s).$$

3.3. (i,b,σ)**-condition.** A domain D satisfies an (i,b,σ)-*condition* if there exists a function σ: $[0,\infty) \to [1,\infty)$ such that each (i,b)-pair Σ^i, $\tilde{\Sigma}^i \subset D$ with

$$\max\{\operatorname{diam}(\Sigma^i), \operatorname{diam}(\tilde{\Sigma}^i)\} \leq t \operatorname{dist}(\Sigma^i \cup \tilde{\Sigma}^i, \partial D)$$

are homotopic to each other in D by a homotopy F such that

$$\operatorname{diam}(F) \leq \sigma(t) \operatorname{diam}(\Sigma^i \cup \tilde{\Sigma}^i) \tag{3.4}$$

and

$$\operatorname{dist}(x, \Sigma^i \cup \tilde{\Sigma}^i) \leq \sigma(t) \operatorname{dist}(x, \partial D) \tag{3.5}$$

for each $x \in F$.

3.6. Theorem. *If D is (m,ϱ)-uniform, then D is both a-plump for some constant $a = a(\varrho)$ and satisfies an (i,b,σ)-condition for all $b > 0$ and $0 \leq i \leq m$ with $\sigma = \varrho$.*

Proof. The second assertion is trivial and the first follows from [V4, 2.15] since the (m,ϱ)-uniformity subsumes ordinary uniformity.

Our main goal here is to show that the converse is true. The following theorem for $m = 0$ was proved by Väisälä [V4, 2.15].

3.7. Theorem. *If a domain D is a-plump and satisfies an (i,b,σ)-condition for all $0 \leq i \leq m$, then D is (m,ϱ)-uniform with $\varrho = \varrho(a,b,\sigma)$. Furthermore, if $\sigma(t) \equiv$ constant, then $\varrho(t) \equiv$ constant.*

Proof. Let $t > 0$ and fix a pair of singular i-spheres $\Sigma = \Sigma^i$, $\tilde{\Sigma} = \tilde{\Sigma}^i$ in D such that

$$\max\{\operatorname{diam}(\Sigma), \operatorname{diam}(\tilde{\Sigma})\} \leq t \operatorname{dist}(\Sigma \cup \tilde{\Sigma}, \partial D).$$

Let

$$r = \operatorname{dist}(\Sigma \cup \tilde{\Sigma}, \partial D)$$

and choose $x \in \Sigma$, $y \in \tilde{\Sigma}$ such that

$$\operatorname{dist}(\Sigma, \tilde{\Sigma}) = |x - y|.$$

If $|x - y| \le 4br$, then Σ, $\tilde{\Sigma}$ is a (i, b)-pair in D since $\Sigma(r)$, $\tilde{\Sigma}(r) \subset D$. Thus the homotopy given in the (i, b, σ)-condition satisfies (1.4) and (1.5) with $\varrho = \sigma$.

We may thus assume $|x - y| > 4br$. Because D satisfies a $(0, b, \sigma)$-condition and because D is a-plump, D is c-uniform for some constant $c = c(a, b, \sigma)$ by [V4, 2.15]. Thus there is a homotopy F_0 between x and y such that

$$\operatorname{diam}(F_0) \le c' \, |x - y|$$

and

$$\operatorname{dist}(z, \{x, y\}) \le c' \operatorname{dist}(z, \partial D)$$

for each $z \in F_0$, where $c' = c'(c)$. Clearly Σ and x form an (i, b)-pair with

$$\max\bigl\{\operatorname{diam}(\Sigma), \operatorname{diam}(\{x\})\bigr\} \le t \operatorname{dist}(\Sigma \cup \{x\}, \partial D).$$

Thus there is a homotopy F_1 between Σ and x such that

$$\operatorname{diam}(F_1) \le \sigma(t) \operatorname{diam}(\Sigma \cup \{x\}) \tag{3.8}$$

and

$$\operatorname{dist}(z, \Sigma \cup \{x\}) \le \sigma(t) \operatorname{dist}(z, \partial D) \tag{3.9}$$

for each $z \in F_1$. Similarly, there exists a homotopy $\tilde{F}_1$ between y and $\tilde{\Sigma}$ which satisfies (3.8) and (3.9) with F_1, x and Σ replaced with $\tilde{F}_1$, y and $\tilde{\Sigma}$, respectively. We have a homotopy

$$F = F_1 \cup F_0 \cup \tilde{F}_1 \subset D$$

between Σ and $\tilde{\Sigma}$, and it remains to show that F satisfies conditions (1.4) and (1.5) for some $\varrho = \varrho(t, a, b, \sigma)$.

To show that condition (1.4) holds, we estimate

$$\begin{aligned} \operatorname{diam}(F) &\le \operatorname{diam}(F_1) + \operatorname{diam}(F_0) + \operatorname{diam}(\tilde{F}_1) \\ &\le \sigma(t) \operatorname{diam}(\Sigma) + c'|x - y| + \sigma(t) \operatorname{diam}(\tilde{\Sigma}) \\ &\le \bigl(2\sigma(t) + c'\bigr) \operatorname{diam}(\Sigma \cup \tilde{\Sigma}). \end{aligned}$$

To verify condition (1.5), fix $z \in F$. If $z \in F_1$, then

$$\operatorname{dist}(z, \Sigma \cup \tilde{\Sigma}) \le \operatorname{dist}(z, \Sigma \cup \{x\}) \le \sigma(t) \operatorname{dist}(z, \partial D),$$

and similar inequalities hold if $z \in \tilde{F}_1$. On the other hand, if $z \in F_0$, then

$$\operatorname{dist}(z, \Sigma \cup \tilde{\Sigma}) \le \operatorname{dist}(z, \{x, y\}) \le c' \operatorname{dist}(z, \partial D),$$

and we deduce that (1.5) holds with $\varrho(t) = \max\{\sigma(t), c'\}$. Hence both (1.4) and (1.5) hold with $\varrho(t) = 2\sigma(t) + c'$. Theorem 3.7 follows.

4. Compactness characterization of strong uniformity

In this section we give an essentially different characterization for m-uniform domains based on compactness. This characterization appears to be useful in practice. We equip the compact space $\overline{\mathbf{R}}^n \approx \mathbf{S}^n$ with the spherical metric and let

$$K^n = \{A : \emptyset \neq A \subset \overline{\mathbf{R}}^n,\ A \text{ compact}\}.$$

Then K^n is a compact metric space under the Hausdorff metric; see [V5, Section 2] for some basic properties of K^n. Next, let $\mathscr{S}$ denote the group of similarities of $\mathbf{R}^n$; that is, $\alpha \in \mathscr{S}$ if and only if there is $\lambda > 0$ such that

$$|\alpha(x) - \alpha(y)| = \lambda |x - y|$$

for all $x, y \in \mathbf{R}^n$.

4.1. Definition [V4], [V5]. A family $H \subset K^n$ is *stable* if (i) $\mathscr{S}(H) = H$, and (ii) $H^2 = \{A \in H : 0, e_1 \in \partial A\}$ is compact, where $e_1 = (1, 0, \dots, 0)$.

The following theorem is due to Väisälä [V4, 3.6]:

4.2. Theorem. *For $c \geq 1$ let M_c be the family of all $A \in K^n$ such that $\overline{\mathbf{R}}^n \setminus A \subset \mathbf{R}^n$ is a c-uniform domain. Then M_c is compact and stable. Conversely, if $M \subset K^n$ is a stable family such that the open set $\overline{\mathbf{R}}^n \setminus A \subset \mathbf{R}^n$ is connected for each $A \in M$, then $M \subset M_c$ for some $c \geq 1$.*

For given ϱ: $[0, \infty) \to [1, \infty)$ we denote by $M(m, \varrho)$ the family of all $A \in K^n$ such that $\overline{\mathbf{R}}^n \setminus A \subset \mathbf{R}^n$ is an (m, ϱ)-uniform domain. Our purpose is to prove the following analogues of Theorem 4.2:

4.3. Theorem.
(1) $\mathscr{S}\big(M(m, \varrho)\big) = M(m, \varrho)$.
(2) *If $\{A_j : j = 1, 2, \dots\} \subset M(m, \varrho)$ and $A_j \to A$ in K^n, then $A \in M(m, \varrho')$ with $\varrho'(t) = 6\varrho(t + 1)$.*

4.4. Theorem. *If $M \subset K^n$ is a stable family such that the open set $\overline{\mathbf{R}}^n \setminus A \subset \mathbf{R}^n$ is m-connected for each $A \in M$, then $M \subset M(m, \varrho)$ for some ϱ.*

Theorem 4.3 does not imply that $M(m, \varrho)$ is compact because we have to change ϱ to ϱ'. It would be interesting to know whether $M(m, \varrho)$ is compact or even stable for $m > 0$.

Before giving the proofs of the above results, we present a useful corollary:

4.5. Corollary. *Let $D \subset \mathbf{R}^n$ be an m-connected domain with at least two finite boundary points. Then D is not m-uniform if and only if there exists a sequence of similarities (α_j) such that $\{0, e_1\} \subset \alpha_j(\partial D)$, $\overline{\mathbf{R}}^n \setminus \alpha_j(D) \to A$ in K^n, and either $0 \in \operatorname{int} A$ or $\pi_i(\overline{\mathbf{R}}^n \setminus A) \neq 0$ for some $0 \leq i \leq m$.*

Proof. Let $H_m \subset K^n$ be the family of all $A \in K^n$ such that $\overline{\mathbf{R}}^n \setminus A \subset \mathbf{R}^n$ is m-connected. If D is not m-uniform, its complement cannot belong to any stable subfamily of H_m by Theorem 4.4. Since $\mathscr{S}(H_m) = H_m$ and since ∂D contains at least two finite boundary points, we obtain from [V4, 3.3] that the closure of $\mathscr{S}(\overline{\mathbf{R}}^n \setminus D)^2$ in K^n is not contained in H_m^2. This exactly means that there exists a sequence (α_j) of similarities as asserted in the theorem.

To prove the necessity, suppose that there exists a sequence (α_j) as described in the assertion. If D is m-uniform, then D is (m,ϱ)-uniform for some ϱ, and hence $\alpha_j(D)$ is (m,ϱ)-uniform for each j since α_j is a similarity map. It follows from Theorem 4.3 that $\overline{\mathbf{R}}^n \setminus A$ is (m,ϱ')-uniform for some ϱ'; in particular, $\pi_i(\overline{\mathbf{R}}^n \setminus A) = 0$ for each $0 \le i \le m$. On the other hand, the sets $\overline{\mathbf{R}}^n \setminus \alpha_j(D)$ belong to a stable family by Theorem 4.2, and it follows that $0 \notin \operatorname{int} A$. This contradicts our assumption and shows that D is not m-uniform.

Proof of Theorem 4.3. Let $M = M(m,\varrho)$. It is clear that $\mathscr{S}(M) = M$. Next, let

$$A_j \in M, \qquad A_j \to A \in K^n,$$

and write

$$D_j = \overline{\mathbf{R}}^n \setminus A_j, \qquad D = \overline{\mathbf{R}}^n \setminus A.$$

We need to show that D is (m,ϱ')-uniform for $\varrho'(t) = 6\varrho(t+1)$. Fix two singular i-spheres Σ, $\tilde{\Sigma} \subset D$, $0 \le i \le m$, satisfying

$$\max\{\operatorname{diam}(\Sigma), \operatorname{diam}(\tilde{\Sigma})\} \le t \operatorname{dist}(\Sigma \cup \tilde{\Sigma}, A)$$

for some $t > 0$. We may assume that Σ, $\tilde{\Sigma} \subset D_j$ and that

$$\max\{\operatorname{diam}(\Sigma), \operatorname{diam}(\tilde{\Sigma})\} \le (t+1) \operatorname{dist}(\Sigma \cup \tilde{\Sigma}, A_j)$$

for all j. Since D_j is (m,ϱ)-uniform, there is a homotopy F_j between Σ and $\tilde{\Sigma}$ in D_j such that

$$\operatorname{diam}(F_j) \le \varrho(t+1) \operatorname{diam}(\Sigma \cup \tilde{\Sigma})$$

and

$$\operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) \le \varrho(t+1) \operatorname{dist}(x, A_j) \tag{4.6}$$

for all $x \in F_j$. We show that for large j, $F = F_j$ is an appropriate homotopy in D between Σ and $\tilde{\Sigma}$.

Fix $x_0 \in \Sigma$ and $s > 0$ such that $F_j \subset B(x_0, s)$ for all j. If $A \cap B(x_0, 2s) = \emptyset$, then

$$\operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) \le s \le \operatorname{dist}(x, A)$$

for all $x \in F_j$, as desired. Suppose now that $A \cap B(x_0, 2s) \neq \emptyset$. Then

$$r_j = \max_{x \in A \cap \overline{B}(x_0, 2s)} \operatorname{dist}(x, A_j) \to 0, \qquad j \to \infty.$$

Next write

$$r = \operatorname{dist}(\Sigma \cup \tilde{\Sigma}, A) > 0.$$

Pick j so large that $r_j < r/4\varrho(t+1)$ and let $F = F_j$. We want to show that (4.6) holds for each $x \in F$ with ϱ replaced with 6ϱ and A_j with A.

To this end, fix $x \in F$. If

$$\operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) < 2\varrho(t+1) r_j < r/2,$$

then

$$\operatorname{dist}(x, A) \geq r - \operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) > r/2.$$

On the other hand, if

$$\operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) \geq 2\varrho(t+1) r_j,$$

then by (4.6)

$$\begin{aligned} \operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) &\leq \varrho(t+1) \operatorname{dist}(x, A_j) \leq \varrho(t+1)\bigl(\operatorname{dist}\bigl(x, A \cap \overline{B}(x_0, 2s)\bigr) + r_j\bigr) \\ &\leq 3\varrho(t+1) \operatorname{dist}(x, A) + \tfrac{1}{2} \operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) \end{aligned}$$

which yields

$$\operatorname{dist}(x, \Sigma \cup \tilde{\Sigma}) \leq 6\varrho(t+1) \operatorname{dist}(x, A) = 6\varrho(t+1) \operatorname{dist}(x, \partial D).$$

This shows that D is (m, ϱ')-uniform with $\varrho'(t) = 6\varrho(t+1)$, as desired. The theorem follows.

We need another theorem of Väisälä [V4, 3.4]:

4.7. Theorem. *For $a \geq 1$ let L_a be the family of all $A \in K^n$ such that $(\overline{\mathbf{R}}^n \setminus A) \cap \mathbf{R}^n$ is a-plump. Then L_a is compact and stable. Conversely, for every stable family $L \subset K^n$ there is $a \geq 1$ such that $L \subset L_a$.*

Proof of Theorem 4.4. We need to demonstrate the existence of a function $\varrho \colon [0, \infty) \to [1, \infty)$ such that $D = \overline{\mathbf{R}}^n \setminus A$ is (m, ϱ)-uniform for any given $A \in M$. By Theorem 4.7, D is a-plump for some constant a independent of A. Thus in view of Theorem 3.7 it suffices to show that there exist $\sigma \colon [0, \infty) \to [1, \infty)$ and $b > 0$ such that $D = \overline{\mathbf{R}}^n \setminus A$ satisfies the (i, b, σ)-condition defined in 3.3 for each $A \in M$ and $0 \leq i \leq m$.

Fix any real number $b > 0$ and suppose that no such σ exists. Then there exists $t > 0$ and for each positive integer j there exist $A_j \in M$ and an (i,b)-pair of singular i-spheres $\Sigma_j^i = \Sigma_j$, $\tilde{\Sigma}_j^i = \tilde{\Sigma}_j$ in $D_j = \overline{\mathbf{R}}^n \setminus A_j$ such that

$$\max\{\mathrm{diam}(\Sigma_j), \mathrm{diam}(\tilde{\Sigma}_j)\} \le t\,\mathrm{dist}(\Sigma_j \cup \tilde{\Sigma}_j, \partial D_j) \tag{4.8}$$

and that for each homotopy F in D_j between Σ_j and $\tilde{\Sigma}_j$ we have either

$$\mathrm{diam}(F) > j\,\mathrm{diam}(\Sigma_j \cup \tilde{\Sigma}_j) \tag{4.9}$$

or

$$\mathrm{dist}(x, \Sigma_j \cup \tilde{\Sigma}_j) > j\,\mathrm{dist}(x, \partial D_j) \tag{4.10}$$

for some $x \in F$.

Since Σ_j and $\tilde{\Sigma}_j$ form an (i,b)-pair, there exist $r_j, s_j > 0$ with $\frac{1}{2} \le r_j/s_j \le 2$ such that

$$\Sigma_j(r_j) \subset D_j, \qquad \tilde{\Sigma}_j(s_j) \subset D_j,$$

and

$$\mathrm{dist}(\Sigma_j, \tilde{\Sigma}_j) \le 4b\max\{r_j, s_j\}. \tag{4.11}$$

We may assume that

$$r_j = s_j = \mathrm{dist}(\Sigma_j \cup \tilde{\Sigma}_j, A_j). \tag{4.12}$$

Choose $x_j \in \Sigma_j \cup \tilde{\Sigma}_j$ and $y_j \in A_j$ such that

$$r_j = |x_j - y_j|.$$

If

$$\max\{|x_j - z| : z \in \Sigma_j \cup \tilde{\Sigma}_j\} < r_j/2,$$

then

$$\Sigma_j \cup \tilde{\Sigma}_j \subset B(x_j, r_j/2) \subset D_j = \overline{\mathbf{R}}^n \setminus A_j,$$

and in this case it is easy to verify (see Corollary 2.4) that Σ_j and $\tilde{\Sigma}_j$ can be joined by a homotopy F in $B(x_j, r_j/2)$ such that

$$\mathrm{diam}(F) \le c\,\mathrm{diam}(\Sigma_j \cup \tilde{\Sigma}_j) \tag{4.13}$$

and

$$\mathrm{dist}(x, \Sigma_j \cup \tilde{\Sigma}_j) \le c\,\mathrm{dist}(x, \partial D_j) \tag{4.14}$$

for all $x \in F$, where c is independent of j. This contradicts our assumptions (4.9) and (4.10) above. Therefore, we may choose $z_j \in \Sigma_j \cup \tilde{\Sigma}_j$ such that

$$|x_j - z_j| \geq r_j/2.$$

On the other hand, for any $z \in \Sigma_j \cup \tilde{\Sigma}_j$, by (4.8), (4.11) and (4.12)

$$|x_j - z| \leq \operatorname{diam}(\Sigma_j) + \operatorname{dist}(\Sigma_j, \tilde{\Sigma}_j) + \operatorname{diam}(\tilde{\Sigma}_j) \leq (2t + 4b)r_j < c_1 r_j,$$

where $c_1 = 2t + 4b + 1$. Thus we have

$$r_j/2 \leq |x_j - z_j| \leq c_1 r_j$$

and

$$\Sigma_j \cup \tilde{\Sigma}_j \subset B(x_j, c_1 r_j).$$

Next let $\Gamma_j = \partial A_j \cap \overline{B}(x_j, 2c_1 r_j)$. If $\operatorname{diam}(\Gamma_j) \leq r_j/2$, then

$$\Sigma_j \cup \tilde{\Sigma}_j \subset B(x_j, c_1 r_j) \backslash \overline{B}(y_j, 3r_j/4),$$

and

$$A_j \cap \overline{B}(x_j, 2c_1 r_j) \subset \overline{B}(y_j, r_j/2).$$

Since Σ_j and $\tilde{\Sigma}_j$ are homotopic in D_j, hence in the complement of the point y_j, one can choose a homotopy

$$F \subset B(x_j, c_1 r_j) \backslash \overline{B}(y_j, 3r_j/4)$$

between Σ_j and $\tilde{\Sigma}_j$. Then (4.13) and (4.14) hold for large j. This again contradicts our assumptions above, and we may assume $\operatorname{diam}(\Gamma_j) > r_j/2$. Choose $a_j, b_j \in \Gamma_j$ such that

$$r_j/2 \leq |a_j - b_j| \leq 3c_1 r_j$$

and choose a similarity α_j such that

$$\alpha_j(a_j) = 0, \qquad \alpha_j(b_j) = e_1.$$

Then

$$|\alpha_j(x) - \alpha_j(y)| = L_j |x - y|$$

for all $x, y \in \mathbf{R}^n$, where $L_j = |a_j - b_j|^{-1}$. Since

$$0 < \frac{1}{3c_1} \leq L_j r_j = \frac{r_j}{|a_j - b_j|} \leq 2$$

and

$$|\alpha_j(x_j)| = |\alpha_j(x_j) - \alpha_j(a_j)| \le 2c_1 L_j r_j,$$

by passing to a subsequence we may assume that

$$L_j r_j \to r > 0, \qquad \alpha_j(x_j) \to x_0.$$

Then we have

$$\operatorname{dist}\bigl(\alpha_j(\Sigma_j) \cup \alpha_j(\tilde{\Sigma}_j), \alpha_j(A_j)\bigr) = L_j \operatorname{dist}(\Sigma_j \cup \tilde{\Sigma}_j, A_j) = L_j r_j \to r > 0$$

and

$$\alpha_j(\Sigma_j) \cup \alpha_j(\tilde{\Sigma}_j) \subset B\bigl(\alpha_j(x_j), c_1 L_j r_j\bigr) \to B(x_0, c_1 r).$$

Thus, again by passing to a subsequence, we may assume that

$$\alpha_j(A_j) \to A \in K^n,$$

and hence that there exists a compact set E in $D = \overline{\mathbf{R}}^n \backslash A$ containing $\alpha_j(\Sigma_j) \cup \alpha_j(\tilde{\Sigma}_j)$ for all j.

Because $\{0, e_1\} \subset \partial\alpha_j(A_j)$ and because M is stable, $A \in M$; in particular, $D = \overline{\mathbf{R}}^n \setminus A$ is m-connected. By the topological lemma below, there is a compact set E' in D with $E \subset E'$ such that $\alpha_j(\Sigma_j)$ and $\alpha_j(\tilde{\Sigma}_j)$ are homotopic to each other in E'. Obviously, one can assume

$$E' \subset \mathbf{R}^n \backslash \alpha_j(A_j) = \alpha_j(D_j),$$

and

$$\operatorname{dist}\bigl(E', \alpha_j(A_j)\bigr) \ge \delta > 0$$

for all j. Furthermore,

$$\operatorname{diam}\bigl(\alpha_j(\Sigma_j) \cup \alpha_j(\tilde{\Sigma}_j)\bigr) \ge |\alpha_j(x_j) - \alpha_j(z_j)| = (r_j L_j)/2 \to r/2 > 0.$$

Consequently, for all large j we can find a homotopy F' in E' between $\alpha_j(\Sigma_j)$ and $\alpha_j(\tilde{\Sigma}_j)$ such that (4.13) and (4.14) hold for $F = \alpha_j^{-1} \circ F'$ with c independent of j. This contradicts (4.9) and (4.10), thus completing the proof of Theorem 4.4.

In the proof of Theorem 4.4 we required the following lemma:

4.15. Lemma. *Suppose that $D \subset \mathbf{S}^n \approx \overline{\mathbf{R}}^n$ is an open, connected and simply connected set with $\pi_m(D) = 0$ for some $1 \le m < n$. Then for any given compact set $E \subset D$ there is a path connected compact set E', $E \subset E' \subset D$, such that any continuous map $f\colon \mathbf{S}^m \to E$ is homotopic to a constant map in E'.*

Proof. Since $E \subset P \subset D$ for some polyhedron P, we may assume that E itself is a compact, connected (finite) polyhedron in D and, after triangulation, that E is a finite simplicial complex. Then $\pi_1(E)$ is finitely generated with generators $g_j\colon \mathbf{S}^1 \to E$, $j = 1, \ldots, k$. Let X be a space obtained from E by attaching 2-cells by the maps g_j (see [Sp, p. 146]). Then X is compact and it is easy to see that $\pi_1(X) = 0$. Furthermore, X has finitely generated homology groups and it follows from a theorem of Serre [Sp, p. 509, Corollary 16] that X has finitely generated homotopy groups in every dimension. As above, we attach to each generator of $\pi_m(X)$ an $(m+1)$-cell to kill that generator. The resulting space Y is compact and path connected with $\pi_m(Y) = 0$.

Next, since $\pi_1(D) = 0$, the inclusion $i\colon E \to D$ can be extended to a continuous map $g\colon X \to D$. Similarly, since $\pi_m(D) = 0$, g can be extended to a continuous map $h\colon Y \to D$. Then $E' = h(Y) \subset D$ is the desired set.

4.16. Remark. R.D. Edwards and G. Mess have given an example which shows that Lemma 4.15 is not true without the assumption $\pi_1(D) = 0$, at least when n is sufficiently high ($n \geq 7$ will do).

5. m-uniformity and quasihyperbolic metric

In this section we use the compactness criterion (Corollary 4.5) to provide an alternative and perhaps thus far the most appropriate characterization for m-uniform domains.

5.1. Definition. We say that a domain D is *quasihyperbolically m-connected*, or QHC(m), if there is an increasing function $\psi\colon [0, \infty) \to [0, \infty)$ such that every singular i-sphere $\Sigma^i \subset D$, $0 < i \leq m$, is homotopic to a point in D through a homotopy F with

$$k_D(F) \leq \psi\big(k_D(\Sigma^i)\big). \tag{5.2}$$

Here $k_D(E)$ denotes the quasihyperbolic diameter of a set E and, as always, we identify the map $F\colon \mathbf{S}^i \times [0,1] \to D$ with its image set.

For a connected set $A \subset D$ the quasihyperbolic diameter can conveniently be estimated in terms of the ratio

$$r_D(A) = \frac{\operatorname{diam}(A)}{\operatorname{dist}(A, \partial D)}.$$

More precisely, we have

$$\log\left(1 + \frac{r_D(A)}{2}\right) \leq k_D(A) \leq \tau_n\big(r_D(A)\big), \tag{5.3}$$

where the function $\tau_n\colon [0, \infty) \to [0, \infty)$ is increasing and depends only on n; see [TV2, Lemma 6.9]. It follows that a domain D is QHC(m) if and only if there is a function $\psi'\colon [0,\infty) \to [0,\infty)$ such that every singular i-sphere $\Sigma^i \subset D$, $0 < i \le m$, is homotopic to a point in D through a homotopy F with

$$r_D(F) \le \psi'\big(r_D(\Sigma^i)\big). \tag{5.4}$$

Moreover, the functions ψ and ψ' depend only on each other and n.

5.5. Theorem. *A domain D is m-uniform if and only if it is both uniform and* QHC(m). *The functions ϱ and ψ depend only on each other and n.*

Proof. Suppose that D is (m,ϱ)-uniform and that Σ^i is a singular i-sphere in D for some $1 \le i \le m$. We can find a homotopy $F \subset D$ between Σ^i and a point $x_0 \in \Sigma^i$ such that

$$\operatorname{diam}(F) \le M \operatorname{diam}(\Sigma^i)$$

and

$$\operatorname{dist}(x,\Sigma^i) \le M \operatorname{dist}(x,\partial D)$$

for $x \in F$, where $M = \varrho\big(r_D(\Sigma^i)\big)$. Since also

$$\operatorname{dist}(\Sigma^i,\partial D) \le \operatorname{dist}(x,\Sigma^i) + \operatorname{dist}(x,\partial D) \le (M+1)\operatorname{dist}(x,\partial D)$$

for any $x \in F$, we have

$$r_D(F) = \frac{\operatorname{diam}(F)}{\operatorname{dist}(F,\partial D)} \le \frac{M(M+1)\operatorname{diam}(\Sigma^i)}{\operatorname{dist}(\Sigma^i,\partial D)} = M(M+1)r_D(\Sigma^i).$$

Thus (5.4) holds with $\psi'(t) = \varrho(t)\big(\varrho(t)+1\big)t$, proving that D is QHC(m).

To prove the sufficiency part of the assertion, we apply Corollary 4.5. Suppose that D is not m-uniform. Then there is a sequence of similarities (α_j) such that $\{0, e_1\} \subset \partial D_j$, where $D_j = \alpha_j(D)$, and

$$\overline{\mathbf{R}}^n \setminus D_j = A_j \to A \in K^n$$

such that $\pi_i(\overline{\mathbf{R}}^n \setminus A) \ne 0$ for some $0 \le i \le m$. Note that the possibility that $0 \in \operatorname{int} A$ is ruled out because D is uniform (this is Corollary 4.5 for $m = 0$ which follows from Väisälä's Theorem 4.2); similarly we infer that $D_0 = \overline{\mathbf{R}}^n \setminus A$ is connected, so that $i > 0$.

Let Σ^i be a singular i-sphere in D_0. Because $\overline{\mathbf{R}}^n \setminus D_j \to \overline{\mathbf{R}}^n \setminus D_0$ in the Hausdorff metric, we may assume that

$$k_{D_j}(\Sigma^i) \le M,$$

where M is independent of j. Similarities are isometries in the quasihyperbolic metric, and so

$$k_D\big(\alpha_j^{-1}(\Sigma^i)\big) \le M$$

for all j. Since D is QHC(m) by assumption, each $\alpha_j^{-1}(\Sigma^i)$ is homotopic to a point through a homotopy $F_j \subset D$ such that

$$k_D(F_j) \le M';$$

in particular,

$$k_{D_j}(\alpha_j \circ F_j) \le M',$$

where M' is independent of j. We infer from this and (5.3) that $\alpha_j \circ F_j$ lies in D_0 for all j sufficiently large, whence Σ^i is null-homotopic in D_0. This contradicts our assumption that $\pi_i(D_0) \ne 0$. Clearly, ϱ and ψ depend only on each other and n, and the theorem follows.

5.6. Remarks. (a) In Theorem 5.5 it is not enough to assume that D is QHC(m). For example, the domain $D = \mathbf{R}^2 \times (0,1) \subset \mathbf{R}^3$ is QHC(1) but not 1-uniform. One can also infer that being QHC(m) is invariant under quasiconformal maps while uniformity is not; see next section.

(b) Vuorinen [Vu1, 2.49] and recently Väisälä [V6, 6.8] have studied ψ-*uniform domains* D which satisfy the inequality

$$k_D(x,y) \le \psi\bigl(r_D(\{x,y\})\bigr)$$

for each pair of points $x, y \in D$ and for some homeomorphism ψ: $[0,\infty) \to [0,\infty)$. Interestingly, Väisälä [V6, 6.16] has shown that if D is ψ-uniform with $\psi(t)/t \to 0$ as $t \to \infty$, then D is uniform. In light of this and Theorem 5.5 one may ask whether a slow growth for ψ in (5.2) together with uniformity implies (m,c)-uniformity for some constant c.

6. Quasiconformal maps and strong uniformity

We study the invariance of m-uniform domains under quasiconformal and related maps. We establish that if D is m-uniform and f: $\mathbf{R}^n \to \mathbf{R}^n$ is quasiconformal, then $f(D)$ is m-uniform. Again, the compactness criterion is crucial here. We also show that a contractible quasiconformally homogeneous domain is strongly uniform if and only if it is uniform.

6.1. Quasimöbius and quasisymmetric maps. The *cross-ratio* of four distinct points $a, b, c, d \in \overline{\mathbf{R}}^n$ is the number

$$|a,b,c,d| = \frac{|a-b||c-d|}{|a-c||b-d|}$$

with the usual convention if one of the points is ∞. If $X \subset \overline{\mathbf{R}}^n$, an embedding f: $X \to \overline{\mathbf{R}}^n$ is θ-*quasimöbius* if

$$|f(a), f(b), f(c), f(d)| \le \theta(|a,b,c,d|)$$

for some homeomorphism θ: $[0,\infty) \to [0,\infty)$ and for all quadruples (a,b,c,d) of distinct points in X.

Similarly, if $X \subset \mathbf{R}^n$, an embedding f: $X \to \mathbf{R}^n$ is η-*quasisymmetric* if

$$\frac{|f(a)-f(b)|}{|f(a)-f(c)|} \le \eta\left(\frac{|a-b|}{|a-c|}\right)$$

for some homeomorphism η: $[0,\infty) \to [0,\infty)$ for all triples (a,b,c) of distinct points in X.

The basic theory of quasimöbius and quasisymmetric maps in Euclidean spaces is given in [V2] and [V3]; see also [TV1]. We recall that an η-quasisymmetric map is always $\theta(\eta)$-quasimöbius and, if X is an open set, a θ-quasimöbius map is K-quasiconformal with dilatation $K = K(\theta, n)$. The following theorem explains the close kinship between quasimöbius and quasisymmetric maps and uniform domains; for a proof, see [V3, Theorem 5.6].

6.2. Theorem. *Suppose that D is a uniform domain and that f is a quasiconformal map of D onto D'. Then D' is uniform if and only if f is quasimöbius. The relevant parameters depend only on each other and n.*

We prove the following theorem.

6.3. Theorem. *If f is a θ-quasimöbius homeomorphism of an (m,ϱ)-uniform domain D onto $D' \subset \mathbf{R}^n$, then D' is (m,ϱ')-uniform, where ϱ' depends only on ϱ, θ, and n.*

6.5. Corollary. *Suppose that D is an (m,ϱ)-uniform domain and f is K-quasiconformal self-homeomorphism of $\overline{\mathbf{R}}^n$. If $f(D) \subset \mathbf{R}^n$, then $f(D)$ is an (m,ϱ')-uniform domain with $\varrho' = \varrho(n,K,\varrho)$.*

Combining Theorems 6.3 with 6.2, we obtain an interesting corollary which states that quasiconformal maps cannot destroy strong uniformity without destroying the ordinary uniformity:

6.5. Corollary. *A domain that can be mapped quasiconformally onto a strongly uniform domain is strongly uniform if and only if it is uniform. In particular, a quasiconformal ball is strongly uniform if and only if it is uniform. The relevant parameters depend only on each other and n.*

6.6. Example. Corollary 6.5 immediately provides examples of domains that cannot be mapped quasiconformally onto a ball. We next exhibit a possibly new example [1] by claiming that there is a domain D in $\mathbf{R}^3$ such that

[1] Jussi Väisälä pointed out that Tukia has essentially the same example in [T1, p. 69]; Tukia's argument for the second assertion in (6) is different.

(1) both $\overline{D} \subset \overline{\mathbf{R}}^3$ and $\overline{D}^* = \overline{\mathbf{R}}^3 \setminus D$ are homeomorphic to $\overline{B}^3$;
(2) ∂D is smooth except at one point;
(3) ∂D has tangent at each point;
(4) D^* is bi-Lipschitz equivalent to an open half space and hence is a strongly uniform domain;
(5) D is a uniform domain;
(6) D is not 1-uniform, and hence not a quasiconformal ball by (5) and 6.5.

The domain D is a modification of the bi-Lipschitz Fox–Artin ball; we sketch the construction. Let $\{B_j : j = 1, 2, \ldots\}$ be a collection of disjoint open disks in $\mathbf{R}^2 = \mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$ centered at $(0, 2^j, 0)$ with radius $1/2$. To each disk B_j we attach a domain $D_j \subset \{x_3 > 0\}$ which is an appropriate modification of the domain obtained at the jth stage in the construction of the bi-Lipschitz Fox–Artin ball [M1, p. 176]. This construction can be done in such a way that

$$D^* = \{x \in \mathbf{R}^3 : x_3 < 0\} \cup \bigcup_{j=1}^{\infty} D_j \cup \bigcup_{j=1}^{\infty} B_j$$

and $D = \overline{\mathbf{R}}^3 \setminus D^*$ satisfy (1)–(4). That D is uniform, follows from (4) and a theorem of Väisälä [V5, 5.10]. On the other hand, if we fix an appropriate loop in D around D_1, then all homotopies of its translates ($x \mapsto 2^j x$) to a point have to go arbitrarily close to the boundary ∂D as $j \to \infty$, showing that D is not 1-uniform.

6.7. Quasihyperbolic metric and quasiconformal maps. We make use of the following important property of quasiconformal maps: there is a homeomorphism $\varphi_{K,n}$: $[0, \infty) \to [0, \infty)$ such that

$$k_{D'}\bigl(f(x), f(y)\bigr) \le \varphi_{K,n}\bigl(k_D(x, y)\bigr) \tag{6.8}$$

whenever f: $D \to D'$ is K-quasiconformal and $x, y \in D$. See [GO, Theorem 3]. This uniform continuity in the quasihyperbolic metric is sometimes called the *solidity* of a quasiconformal map, cf. [TV2].

Proof of Theorem 6.3. Suppose that the claim is not true. Then there is a sequence of (m, ϱ)-uniform domains D_j and θ-quasimöbius maps f_j: $D_j \to D_j' \subset \mathbf{R}^n$ with the following property: there is no ϱ' such that all the image domains D_j' are (m, ϱ')-uniform. Of course, it may happen that $f_j = f_1$ for all j. Let M be the closure of the family

$$\{\alpha(\overline{\mathbf{R}}^n \setminus D_j') : \alpha \in \mathscr{S},\ j = 1, 2, \ldots\}$$

in K^n. Then $\mathscr{S}(M) = M$. Moreover, since each D_j' is $c(\theta, \varrho, n)$-uniform (see Theorem 6.2), M is a compact subset of a stable family, and hence stable itself by Väisälä's Theorem 4.2. In particular, $\overline{\mathbf{R}}^n \setminus A$ is connected for each $A \in M$. Thus by our contrapositive assumption and by Theorem 4.4, there is $A'_0 \in M$ such that $\pi_i(\overline{\mathbf{R}}^n \setminus A'_0) \neq 0$ for some $1 \leq i \leq m$. After relabeling we may assume that

$$A'_j = \alpha_j(\overline{\mathbf{R}}^n \setminus D'_j) \to A'_0$$

in the Hausdorff metric, where $D'_0 = \overline{\mathbf{R}}^n \setminus A'_0 \subset \mathbf{R}^n$ is a uniform domain.

In the rest of the proof we let $1 \leq C_1, C_2, \dots$ denote any constants that are independent of j. Fix an essential singular i-sphere Σ^i in D'_0. We may assume

$$\operatorname{dist}(\Sigma^i, A'_j) \geq 1/C_1 > 0,$$

so that by (5.3)

$$k_{\alpha_j(D'_j)}(\Sigma^i) \leq C_2.$$

Now using the solidity of quasiconformal maps (6.8) we deduce that

$$k_{D_j}\bigl(g_j^{-1}(\Sigma^i)\bigr) \leq C_3 \tag{6.9}$$

and hence by (5.3) that

$$\operatorname{diam}\bigl(g_j^{-1}(\Sigma^i)\bigr) \leq C_4 \operatorname{dist}\bigl(g_j^{-1}(\Sigma^i), \partial D_j\bigr),$$

where $g_j = \alpha_j \circ f_j$. Next, fix a point $x_0 \in \Sigma^i$. Because D_j is (m, ϱ)-uniform, we can find a homotopy F_j between $g_j^{-1}(\Sigma^i)$ and $g_j^{-1}(x_0)$ such that

$$k_{D_j}(F_j) \leq C_5;$$

see the proof of Theorem 5.5. In consequence, $F'_j = g_j \circ F_j$ is a homotopy between Σ^i and x_0 in $\overline{\mathbf{R}}^n \setminus A'_j$ such that

$$k_{\alpha_j(D'_j)}(F'_j) \leq \varphi_{K,n}(C_5) = C_6 < \infty.$$

This implies that F'_j lies in D'_0 for all j large enough, contradicting the assumption that Σ^i is essential. The theorem follows.

6.10. Quasiconformally homogeneous domains. Recall that a domain D is *homogeneous with respect to a quasiconformal family* if for some K there is a family Γ of K-quasiconformal self-homeomorphisms of D such that for each pair of points $x, y \in D$ there exists an $f \in \Gamma$ with $f(x) = y$. We also say that D is *quasiconformally homogeneous.*

Every quasiconformal ball is quasiconformally homogeneous and the converse is true if the domain possesses an $(n-1)$-tangent at a single finite boundary point [GP]; see also [M2]. However, for all $n \geq 3$ there are topological balls (even interiors of topologically flat n-cells) in $\mathbf{R}^n$ which are quasiconformally homogeneous yet not quasiconformal balls. Tukia [T2] was the first to construct such examples. Tukia's example domain is uniform, and the next theorem shows that it is even strongly uniform.

6.11. Theorem. *An m-connected quasiconformally homogeneous domain is m-uniform if and only if it is uniform. In particular, a contractible quasiconformally homogeneous domain is strongly uniform if and only if it is uniform.*

Proof. We use Corollary 4.5: if D is not m-uniform, there is a sequence (α_j) of similarities such that $\{0, e_1\} \subset \alpha_j(\partial D)$ and $\overline{\mathbf{R}}^n \setminus \alpha_j(D) = A_j \to A$ in K^n with $\pi_i(\overline{\mathbf{R}}^n \setminus A) \neq 0$ for some $1 \leq i \leq m$. Take notice that the assertion is trivial if D has only one finite boundary point, and the possibility that $0 \in \operatorname{int} A$ or $i = 0$ is excluded by Theorem 4.2.

Fix points $x_0 \in D$ and $y_0 \in D_0 = \overline{\mathbf{R}}^n \setminus A$; we may assume $y_0 \in D_j = \overline{\mathbf{R}}^n \setminus A_j$ for all j. Next choose a quasiconformal self-map f_j of D such that $f_j(x_0) = \alpha_j^{-1}(y_0)$. Because D is quasiconformally homogeneous, we may assume that f_j is K-quasiconformal for some fixed K. Then the sequence

$$\{g_j = \alpha_j \circ f_j \colon D \to D_j\}$$

is an equicontinuous family of K-quasiconformal maps by [V1, 19.3]. Since $g_j(x_0) = y_0 \in D_0$, by passing to a subsequence we may assume that $g_j \to g$, where g is a K-quasiconformal map of D onto D_0; see [V1, 21.9, 37.4]. This is a contradiction in light of the assumption $\pi_i(D) = 0 \neq \pi_i(D_0)$. The theorem follows.

7. Periodic quasiconformal maps

Suppose that f is a self-homeomorphism of $\overline{\mathbf{R}}^n$. We call f *periodic* if there is an integer $k > 0$ such that $f^k = \mathrm{id} = \text{identity}$; the *order* of a periodic f is the smallest such k, and is denoted by $\operatorname{ord}(f)$. We let $\operatorname{fix}(f)$ denote the set of all fixed points of f, *i.e.*

$$\operatorname{fix}(f) = \{x \in \overline{\mathbf{R}}^n : f(x) = x\}.$$

It is a classical issue to try to identify the geometry of the fixed point set of a periodic transformation. We record the following well known result of P.A. Smith [Sm] for an easy reference; it will be crucial in what follows.

7.1. Theorem. *Suppose that f is a periodic self-homeomorphism of $\overline{\mathbf{R}}^n$ with prime power order, i.e.* $\operatorname{ord}(f) = p^s$ *for p prime. Then* $\operatorname{fix}(f)$ *is a* mod p *Čech cohomology r-sphere for some $-1 \leq r \leq n$. Moreover, $n - r$ is even when p is odd.*

For a proof of Theorem 7.1, see [Sm], [Br, Chapter III]. We recall that a compact set $A \subset \overline{\mathbf{R}}^n$ is a Čech cohomology r-sphere if the reduced Čech cohomology groups with coefficients in $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$ satisfy

$$\check{\tilde{H}}^i(A; \mathbf{Z}_p) \simeq \begin{cases} \mathbf{Z}_p, & \text{if } i = r \\ 0, & \text{otherwise.} \end{cases}$$

The purpose of this section is to analyze $\operatorname{fix}(f)$, or rather its complement in $\overline{\mathbf{R}}^n$, when f is periodic and quasiconformal. We call f a *reflection* if it is a sense-reversing involution of $\overline{\mathbf{R}}^n$; then f interchanges two domains D and D^* in $\overline{\mathbf{R}}^n$ with $\partial D = \partial D^* = \operatorname{fix}(f)$. It follows from Smith's theorem that (under its assumptions) $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$ is connected except when f is a reflection; moreover, the latter occurs if and only if $r = n-1$ in Theorem 7.1. Also note that the interior of the fixed point set of a nontrivial periodic transformation is always empty by a theorem of Newman [N], [Br, p. 157].

We prove the following two theorems.

7.2. Theorem. *Suppose that f is a periodic sense-preserving quasiconformal self-homeomorphism of $\overline{\mathbf{R}}^n$, and suppose that $\infty \in \operatorname{fix}(f)$. Then $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$ is a uniform domain.*

7.3. Theorem. *Suppose that f is a quasiconformal reflection in $\overline{\mathbf{R}}^n$ with $\infty \in \operatorname{fix}(f)$. Then both components of $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$ are uniform; if either of them is 1-uniform, then both are strongly uniform.*

7.4. Remarks. (a) The normalization $\infty \in \operatorname{fix}(f)$ in above theorems is just for convenience, cf. Remarks 1.7 (c).

(b) The first assertion in Theorem 7.3 was proved by Yang [Y]. We do not know whether the assumption “either of them is 1-uniform” in Theorem 7.3 is necessary. The proof will show that it is not necessary provided the complementary components of the fixed point set of a quasiconformal reflection are always simply connected. For diffeomorphisms this is well-known to be true, but not necessarily so for topological reflections; see [Bi].

(c) In general it is not true that the domain $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$ in Theorem 7.2 is m-uniform for $m > 1$, even if it is 1-uniform; in other words, a direct analog of Theorem 7.3 for arbitrary periodic maps is false. To see this, consider, an orthogonal transformation $\mathscr{O}$ in $\mathbf{R}^4$ such that $\operatorname{fix}(\mathscr{O}) = \mathbf{R}^1$. Then $\mathbf{R}^4 \setminus \operatorname{fix}(\mathscr{O})$ is 1-uniform but $\pi_2\big(\mathbf{R}^4 \setminus \operatorname{fix}(\mathscr{O})\big) \neq 0$. We do not know if there is a similar example under the extra assumption that $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$ is m-connected.

(d) It is easy to see that neither of Theorems 7.2 and 7.3 need hold for general topological transformations.

Proof of Theorem 7.2. Let $D = \overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$. The claim is obviously true if ∂D contains only two points in $\overline{\mathbf{R}}^n$ or if $D = \emptyset$, so we may assume that $\{0, e_1\} \subset \partial D$. Next, if f is not of prime power period, f^t has a prime power period for some positive integer $t < \operatorname{ord}(f)$; because $\operatorname{fix}(f) \subset \operatorname{fix}(f^t)$ and because $\operatorname{int}\big(\operatorname{fix}(f^t)\big) = \emptyset$, we see that $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(f)$ is uniform if $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(f^t)$ is uniform (see [V4, Remark 2.12]). Thus we may assume that $\operatorname{ord}(f)$ is a prime power.

Suppose now that the claim is false. Then by Väisälä's Theorem 4.2 (cf. Corollary 4.5) there is a sequence of similarities (α_j) such that $\{0, e_1\} \subset \alpha_j\big(\operatorname{fix}(f)\big)$,

$\alpha_j(\operatorname{fix}(f)) \to A$ in K^n, and either $0 \in \operatorname{int} A$ or $\overline{\mathbf{R}}^n \setminus A$ is not connected. Consider the maps

$$g_j = \alpha_j \circ f \circ \alpha_j^{-1}.$$

Then each g_j is a K-quasiconformal self-homeomorphism of $\overline{\mathbf{R}}^n$ with K independent of j and $\operatorname{ord}(g_j) = \operatorname{ord}(f) = p^s$ for some prime p and integer s. Moreover,

$$g_j(0) = 0, \quad g_j(e_1) = e_1, \quad g_j(\infty) = \infty,$$

and hence by passing to a subsequence we may assume that $g_j \to g$ uniformly in $\overline{\mathbf{R}}^n$, where g is a K-quasiconformal self-homeomorphism of $\overline{\mathbf{R}}^n$ (see [V1, Chapters 19, 29 and 37.3]).

Clearly g is periodic and sense-preserving. Pick $x_0 \in A$. Then $x_0 = \lim \alpha_j(x_j)$ for some $x_j \in \operatorname{fix}(f)$, whence

$$g(x_0) = \lim_{j\to\infty} \alpha_j \circ f \circ \alpha_j^{-1}(\alpha_j(x_j)) = \lim_{j\to\infty} \alpha_j(x_j) = x_0.$$

Thus $A \subset \operatorname{fix}(g)$. If $\operatorname{fix}(g) = \overline{\mathbf{R}}^n$, then g_j converges uniformly to the identity map in $\overline{\mathbf{R}}^n$. This is impossible in view of the fact that each compact Lie group G acting on $\overline{\mathbf{R}}^n$ has an orbit of diameter at least $\varepsilon(G) > 0$ (see [Br, Theorem 9.6, p. 158]). Thus $\operatorname{fix}(g) \neq \overline{\mathbf{R}}^n$, and hence $\operatorname{fix}(g)$ can not have interior points by the aforementioned theorem of Newman. Consequently, A cannot have interior points.

Next we claim that $\operatorname{ord}(g) = \operatorname{ord}(f)$. Indeed, if this is not the case, then $g^q =$identity for some $1 < q < \operatorname{ord}(f) = \operatorname{ord}(g_j)$, and it follows that

$$g_j^q \to g^q = \text{identity}$$

uniformly in $\overline{\mathbf{R}}^n$. Since $g_j^q \neq \mathrm{id}$, we arrive at a contradiction by invoking [Br, Theorem 9.6, p. 158] as above.

In consequence, g is a periodic self-homeomorphism of $\overline{\mathbf{R}}^n$ with $\operatorname{ord}(g) = p^s$ for some prime p. Smith's theorem ascertains that $\operatorname{fix}(g)$ is a cohomology r-sphere for some $1 \le r \le n-1$; note that $r \le 0$ is excluded because $\{0, e_1, \infty\} \subset \operatorname{fix}(g)$. By the Alexander duality

$$\tilde{\check{H}}^i(\operatorname{fix}(g); \mathbf{Z}_p) \simeq \tilde{H}_{n-i-1}(\overline{\mathbf{R}}^n \setminus \operatorname{fix}(g); \mathbf{Z}_p),$$

where we have the reduced Čech cohomology on the left and the reduced singular homology on the right. Thus if $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(g)$ is not connected, we must have

$$\check{H}^{n-1}(\operatorname{fix}(g); \mathbf{Z}_p) \neq 0$$

or $r = n-1$. However, this cannot be the case since g is not a reflection.

It follows that $\overline{\mathbf{R}}^n \setminus \operatorname{fix}(g)$ and hence $\overline{\mathbf{R}}^n \setminus A$ is connected. As we already demonstrated that $\operatorname{int} A = \emptyset$, the theorem follows.

Proof of Theorem 7.3. Let D and D^* denote the two complementary components of $\operatorname{fix}(f)$. By [Y, Theorem 3.1], both D and D^* are uniform. Since f interchanges D and D^*, by Theorem 6.3 it suffices to show that D is m-uniform for all m, provided it is 1-uniform. Suppose that this is false. As before, we may select similarities α_j such that $\{0, e_1\} \subset \alpha_j(\partial D)$, that $\overline{\mathbf{R}}^n \setminus \alpha_j(D) \to A$ in K^n, and that $\pi_i(\overline{\mathbf{R}}^n \setminus A) \neq 0$ for some $2 \leq i \leq m$; because D is 1-uniform, $\overline{\mathbf{R}}^n \setminus A$ is simply connected (see Theorem 4.3). As in [Y, Proof of 3.1] we can assume that $g_j = \alpha_j \circ f \circ \alpha_j^{-1}$ converges uniformly to a sense-reversing quasiconformal involution g with $\operatorname{fix}(g) = \partial A$; moreover, the Smith theory and duality imply that ∂A is a cohomology $(n-1)$-sphere and that $\overline{\mathbf{R}}^n \setminus \partial A$ has exactly two components: $\operatorname{int} A$ and $\overline{\mathbf{R}}^n \setminus A$. Since g interchanges the components of $\overline{\mathbf{R}}^n \setminus \partial A$, we see that also $\operatorname{int} A$ is simply connected.

Next we invoke [Br, Theorem 7.13, p. 146], which implies that the ith Čech cohomology groups with integer coefficients of the quotient space $A = \overline{\mathbf{R}}^n / \{\mathrm{id}, g\}$ are zero for all $0 \leq i \leq n-2$. The Alexander duality then gives

$$H_i(\overline{\mathbf{R}}^n \setminus A; \mathbf{Z}) = 0$$

for $0 < i < n$. Because $\overline{\mathbf{R}}^n \setminus A$ is simply connected, the Hurewicz isomorphism theorem [Sp, p. 398] now gives

$$\pi_i(\overline{\mathbf{R}}^n \setminus A) \simeq H_i(\overline{\mathbf{R}}^n \setminus A; \mathbf{Z}) = 0$$

for $0 < i < n$. This contradicts our assumption, and the theorem follows.

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Received 23 September 1993
Revised 6 April 1994