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NONREMOVABLE CANTOR SETS FOR BOUNDED QUASIREGULAR MAPPINGS

Dedicated to Frederick W. Gehring

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Abstract. Recently T. Iwaniec and G. Martin gave the first results where sets of positive Hausdorff dimension are removable for bounded quasiregular mappings. Here we prove a converse and show that for each $\lambda > 0$ there exists a Cantor set E in R^3 with Hausdorff dimension $\dim_H E \le \lambda$ and a bounded $K(\lambda)$-quasiregular mapping $f\colon R^3 \setminus E \to R^3$ that does not extend continuously to any point of E.

1. Introduction

Let G be a domain in the Euclidean n-space R^n, $n \ge 2$, and $E \subset G$ closed in G. We are interested in the following removability question. When does a K-quasiregular mapping $f\colon G \setminus E \to \bar{R}^n = R^n \cup \{\infty\}$ have an extension to a quasiregular mapping of G? See 2.1 for definitions. At an early stage of the theory it was proved in [MRV2] and [Re] that f can be extended quasiregularly if E is of zero n-capacity and f omits a set of positive n-capacity. For a long time this remained as the only general removability result. Recently a satisfactory solution for bounded quasiregular mappings was obtained in the following form.

1.1. Theorem [IM], [I]. *There exists $\alpha = \alpha(n, K) > 0$ such that every bounded K-quasiregular mapping $f\colon G \setminus E \to R^n$ extends quasiregularly to G provided the Hausdorff dimension $\dim_H E$ is at most α. In particular, E is removable if $\dim_H E = 0$. For even dimension $n = 2\ell$, $\alpha(n, K)$ tends to ℓ as $K \to 1$ with fixed n.*

Theorem 1.1 was first proved for even dimension by T. Iwaniec and G. Martin in [IM]. Their paper is based on ideas from the work [DS] by S. Donaldson and D. Sullivan and the main tool in [IM] is a singular integral operator S which is a counterpart of the Beurling–Ahlfors operator in the planar case. Iwaniec and Martin establish linear relations for certain differential ℓ-forms induced by the mapping f and they give an estimate on $\alpha(n, K)$ in terms of the p-norm $\|S\|_p$ of the operator S. They also give explicit estimates for $\|S\|_p$. The odd dimensional case still remained open until Iwaniec gave a nonlinear counterpart of [IM] in [I].

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For simplifications of the arguments in [I], see [IS]. An alternative approach to [I] is given by J.L. Lewis in [L]. In [KM] P. Koskela and O. Martio improve slightly the removabilty result in [MRV2]. They also study the removability question with the additional assumption that f has an extension to a locally Hölder continuous map of G and they obtain a counterpart to a result by L. Carleson (see [G, p. 78]) for analytic functions in the plane.

P. Järvi and M. Vuorinen [JV] study the removability of Cantor sets for quasiregular mappings omitting a finite number of points and give an interesting result complementary to that in [IM] and [I]. To state it let $q(n,K)$ be the bound in the Picard–Schottky type theorem [R2, 2.4].

1.2. Theorem [JV]. *There exists a self-similar Cantor set E in R^n of Hausdorff dimension $d = d(n,K,\beta) > 0$ such that every K-quasiregular mapping $f\colon R^n \setminus E \to R^n$ omitting $\max\bigl(2^n+2, q(n,K)\bigr)$ points, with mutual chordal distance at least β, extends quasiregularly to R^n.*

In the literature there has been in dimensions $n \geq 3$ a lack of good nonremovability results for quasiregular mappings that are for example bounded. The purpose of this paper is to partially fill this gap and give a qualitative converse to Theorem 1.1. We will prove the following.

1.3. Theorem. *For each $\lambda > 0$ there exists a compact, totally disconnected set E in R^3 with Hausdorff dimension $\dim_H E \leq \lambda$ and a bounded $K(\lambda)$-quasiregular mapping $f\colon R^3 \setminus E \to R^3$ that does not extend continuously to any point of E. The set E can be constructed as a self-similar Cantor set.*

We have formulated 1.3 for $n = 3$ only. The construction does not generalize as such for $n \geq 4$, see Remark 3.15. It was pointed out to the author by O. Martio and J. Väisälä that with an auxiliary quasiconformal map Theorem 1.3 can be modified to a form where E is contained in a line segment.

For any positive integer m there exists a quasiregular mapping of R^3 into itself omitting m points [R3]. That example gives a converse to 1.2 for $n = 3$ because ∞ is not removable.

Let us compare the above with some related results in the plane. For 1-quasiregular, i.e. (complex) analytic functions, the classical theorem by P. Painlevé and A.S. Besicovitch says that a bounded and analytic f extends to an analytic function of G if the Hausdorff measure $\mathscr{H}^1(E)$ of E is zero. For K-quasiregular mappings the corresponding holds if $\mathscr{H}^{1/K}(E) = 0$. This follows from the representation $f = \varphi \circ h$, where h is K-quasiconformal and φ analytic, and the fact that h is locally Hölder continuous with exponent $1/K$ (see [KM]).

In the plane we obtain a result like Theorem 1.3 directly as follows. First, by a result of Gehring and Väisälä [GV] there exists a quasiconformal mapping ψ of R^2 which maps a Cantor set E with $\dim_H E < \lambda$ onto another Cantor set with $\dim_H \psi E > 1$. There exists a bounded analytic function φ of $R^2 \setminus \psi E$ (see

[G, p. 78]). Then the quasiregular map $f = \varphi \circ \psi \mid R^2 \setminus E$ has no extension to a quasiregular map of R^2.

Theorem 1.2 is a counterpart of a result by L. Carleson [C] in the plane when E is a Cantor set on the real line with positive logarithmic capacity and $f\colon R^2 \setminus E \to R^2$ is an analytic function omitting three points.

2. Preliminary constructions

2.1. Quasiregular mappings. Let G be a domain R^n. A continuous map $f\colon G \to R^n$ is called *quasiregular* if f belongs to the local Sobolev space $W^1_{n,\mathrm{loc}}(G)$, i.e. f is an ACL^n map and

$$|f'(x)|^n \le K J_f(x) \qquad \text{a.e.} \tag{2.2}$$

for some $K \in [1, \infty[$. Here $|f'(x)|$ is the supremum norm of the formal derivative defined by means of the partial derivatives and $J_f(x)$ is the Jacobian determinant. The smallest K in (2.2) is the outer dilatation $K_O(f)$ of f. The smallest $K' \ge 1$ in the inequality

$$J_f(x) \le K' \inf_{|h|=1} |f'(x)h|^n \qquad \text{a.e.} \tag{2.3}$$

is the inner dilatation $K_I(f)$. A map f is called K-quasiregular if f is quasiregular and $K(f) = \max\bigl(K_O(f), K_I(f)\bigr) \le K$. The definition extends to smooth oriented Riemannian n-manifolds in a straightforward manner. A quasiregular homeomorphism is called *quasiconformal*. Often a quasiregular mapping into $\bar{R}^n = R^n \cup \{\infty\}$ is also called *quasimeromorphic*. For the basic theory, see [MRV1] or [R4].

2.4. The Cantor set. We first choose a positive integer $m > 20$ and $\delta > 0$ such that $1/10 \le m\delta \le 1/3$. More precise bounds are described later. Set

$$\gamma = \left(\frac{1 - m\delta}{m + 1}\right)^{1/3}. \tag{2.5}$$

We perform the construction of our Cantor set by starting out from the rectangular box

$$Q = [0, 1] \times [0, \gamma] \times [0, \gamma^2]$$

and then deleting in each step m parallel slices. In the first step we delete the slices

$$\{x \in Q : j\gamma^3 + (j-1)\delta < x_1 < j\gamma^3 + j\delta\}, \qquad j = 1, \dots, m.$$

Let the union of these be D_1. We are left with boxes $Q_{1k} = I_k \times [0, \gamma] \times [0, \gamma^2]$, $k = 1, \dots, m+1$, where the length $|I_k|$ of I_k is γ^3. Each box Q_{1k} is then similar to Q with ratio γ. Let the union of boxes Q_{1k} be E_1. In the next step we delete from each box Q_{1k} similarly the m slices

$$\{x \in Q_{1k} : j\gamma^4 + (j-1)\gamma\delta < x_2 < j\gamma^4 + j\gamma\delta\}, \qquad j = 1, \dots, m,$$

and we are left with $m+1$ boxes of the form $Q_{2k\ell} = I_k \times J_\ell \times [0, \gamma^2]$, $\ell = 1, \dots, m+1$, where $|J_\ell| = \gamma^4$. We denote the union of such slices by D_2 and the union of the boxes $Q_{2k\ell}$ by E_2. We continue similarly and have in the ith step a union D_i of slices and a union E_i of boxes. A slice in D_i we call a D_i-slice and it has width $\gamma^{i-1}\delta$. As a limit we get the Cantor set

$$E = \textstyle\bigcap_{i=1}^{\infty} E_i.$$

2.6. Triangulations. We are going to extend the identity of $R^3 \setminus Q$ successively into the slices to produce a quasiregular mapping $R^3 \setminus E \to R^3$ omitting a ball. In each slice we will define a decomposition into prisms by a certain triangulation. Each prism will be mapped homeomorphically onto a half of a deformed spherical ring which in general varies when we move along the slice.

To describe these triangulations let A be a D_1-slice. Then A is of the form $A = L \times M$ where L is an open interval $]\alpha, \beta[$ of length δ and $M = [0, \gamma] \times [0, \gamma^2]$. Let p and q be integers such that $4(p-1) < \gamma/\delta \le 4p$, $4(q-1) < \gamma^2/\delta \le 4q$. We first divide $[0, \gamma]$ into $4p$ congruent intervals Y_i and $[0, \gamma^2]$ into $4q$ congruent intervals Z_j. Then we triangulate M as shown in Figure 1, where we see four rectangles $Y_i \times Z_j$. The principle is that the boundary vertices in M belong to five triangles and the interior vertices in M belong to an even number of triangles. With a sufficiently small γ we may adjust the vertices slightly so that each vertex v is in $\pi_1 E$ and it is away from the slices according to the condition

$$d(v, G_k^1) \ge \gamma^{k-1}\delta \qquad \text{for all } k \not\equiv 1 \ (\mathrm{mod}\, 3). \tag{2.7}$$

Here π_i is the orthogonal projection forgetting the ith coordinate and

$$G_k^i = \pi_i\big(\cup\{D_j : 1 \le j \le k,\ j \not\equiv i \pmod 3\}\big).$$

Recall that $\gamma^{k-1}\delta$ is the width of the D_k-slices. The triangulation can be performed so that each triangle is a 10-bilipschitz image of a triangle of side lengths $\delta, \delta, \sqrt{2}\,\delta$. For other slices the triangulations are obtained by similarity from the one described for M.

2.8. Extension to slices. The purpose of the special triangulations described in 2.6 is that we are then able to follow the boundary correspondence inherited from earlier extensions when we extend to a certain slice.

Let $A = L \times M$ be a D_1-slice as in 2.4. We will now describe more precisely how each prism $L \times T$ will be mapped where T is a closed triangle of the triangulation of M. The restriction to $\mathrm{int}(L \times T)$ will be K-quasiconformal with a

Figure 1.

Figure 2.

fixed K. Let us consider a boundary vertex v in the interior of a side s of M. Let $T_1, \dots, T_5$ be the successive triangles that have v as vertex (Figure 2) and let $P_i = L \times T_i$, $i = 1, \dots, 5$, be the corresponding prisms. Figure 3 shows how the various prisms are mapped. Recall that the boundary correspondence on the rim of the slice A is the identity. Let C_i be the relative boundary of T_i. Each part $L \times C_i$ of the boundary of P_i is mapped onto a ring domain in the plane containing $L \times s$. An end of a prism, like the triangle abc, is mapped onto a somewhat deformed half sphere. If the image of a vertex of a prism differs from the vertex, then the image is indicated by a prime. The principle is that when we move to the next prism, there is only one common face with the preceding prism. Therefore we are somewhat free to define the image in other parts of the prism. If v is a corner vertex of M, we modify the above in an obvious way. Since each interior vertex in the triangulation of M belongs to an even number of triangles, we can continue the extension to the whole slice A as a quasiregular mapping. We denote the extended map of $(R^3 \setminus Q) \cup D_1$ by g_1. In addition, we require that for each prism P, $g_1 \mid P$ is a Λ-bilipschitz map followed by similarity, where $K = \Lambda^4$. The map g_1 extends continuously to $(R^3 \setminus \operatorname{int} Q) \cup \bar{D}_1$ and we denote this extension also by g_1.

The next step is to extend g_1 to all D_2-slices. The boundary correspondence that we have to match now is inherited both from the original identity map and the extensions to the D_1-slices. We have chosen the triangulation for the D_1-slices by means of the condition (2.7) so that the boundary correspondence under g_1 on the rim of each D_2-slice is a local homeomorphism and this correspondence can be assumed to be a diffeomorphism with obvious modification at the corners of the rims. Let $A = I_k \times]\alpha, \beta[\times [0, \gamma^2]$ be a D_2-slice, set $M_\alpha = I_k \times \{\alpha\} \times [0, \gamma^2]$, $M_\beta = I_k \times \{\beta\} \times [0, \gamma^2]$ and let $\dot{M}_\alpha$ and $\dot{M}_\beta$ be the relative boundaries of M_α

Figure 3.

and M_β. By the remark just made, the boundary correspondences $g_1 \mid \dot{M}_\alpha$ and $g_1 \mid \dot{M}_\beta$ have the following property. If F is contained in $\dot{M}_\alpha$ or $\dot{M}_\beta$ and has a diameter less than $3|\beta-\alpha|$, $g_1 \mid F$ approaches a similarity map when $\gamma \to 0$. Because of this property we can arrange the construction indicated in Figure 3 so that with a fixed γ, depending on the dilatation bound for our final quasiregular map, g_1 has no effect on the dilatation bound for the extensions into the D_2-slices. This is one point in the construction which needs a different treatment for dimensions $n \geq 4$ because the conclusion of $g_1 \mid F$ approaching a similarity map does not hold anymore. We denote the extended map of $(R^3 \setminus Q) \cup D_1 \cup D_2$ by g_2.

The successive extensions to D_k-slices, $k = 3, 4, \ldots$, are accomplished similarly. Set

$$U_k = (R^3 \setminus Q) \cup \bigcup_{j=1}^{k} D_j$$

and

$$U = \bigcup_{k\geq 1} U_k = R^3 \setminus E.$$

The limit of the extensions $g_k \colon U_k \to R^3$ is a quasiregular mapping $g \colon U \to R^3$. We may require that each g_k, and hence also g, omits a fixed ball, say $V = B^3\bigl((\gamma^2/2, \gamma^2/2, \gamma^2/2), \gamma^2/4\bigr)$. In the next section we will put more constraints on the extensions g_k.

3. Proof of Theorem 1.3

The Hausdorff dimension of E satisfies $\dim_H E < 3$ and $\dim_H E$ depends on m and δ. Given $\lambda > 0$ we can by the method in [GV] map R^3 onto itself by a $K_0(\lambda)$-quasiconformal mapping h so that hE is a Cantor set with $\dim_H hE < \lambda$. Then $g \circ h^{-1}$ followed by a Möbius transformation is a required map for Theorem 1.3 provided we prove that g does not extend to a continuous map at any point of E.

To prove that g can be constructed so that it does not extend continuously to E it is enough to show that for every k and any box C of E_k there exist $u, v \in C \cap U$ such that $|g(u) - g(v)| \geq \gamma^2/20$. We may assume $k \geq 3$, $k \equiv 3 \pmod 3$, and that C is bounded by D_j-slices, $j = k-2, k-1, k$. Recall from 2.8 the decomposition of a slice into prisms. For $j = k-2, k-1, k$ set

$$\tau_{C,j} = \min\{d(gP) : P \text{ is a prism in a } D_j\text{-slice and } \bar{P} \cap C \neq \emptyset\}, \tag{3.1}$$

$$\tau_C = \min(\gamma^2 \tau_{C,k-2}, \gamma \tau_{C,k-1}, \tau_{C,k}). \tag{3.2}$$

Given a certain bound for the dilatation of g we may increase the diameter $d(gP)$ of the image of prisms P when we move inside a slice. Depending on the dilatation bound K we find a number $s = s(K) > 1$ which is the allowed average ratio of these sizes for neighboring prisms. More precisely, if P and P_1 are prisms in a slice so that moving from P of P_1 within the slice we have to pass through at least $i \geq 20$ prisms, then the construction given in 2.8 allows a ratio

$$\frac{d(gP_1)}{d(gP)} \geq s^i. \tag{3.3}$$

The idea is to choose m and δ so that we can have $\tau_{C'} \geq 2\tau_C$ for some box $C' \subset C$ of E_{k+3}. Here $\tau_{C'}$ is defined as in (3.2) by shifting the indices by 3.

We first take two neighboring D_{k+1}-slices $A_{k+1} = L \times \ell_2 \times \ell_3$ and $\tilde{A}_{k+1} = \tilde{L} \times \ell_2 \times \ell_3$ in C. The lengths of the line segments are $|L| = |\tilde{L}| = \gamma^k \delta$, $|\ell_2| = \gamma^{k+1}$, and $|\ell_3| = \gamma^{k+2}$. Let ℓ_2^* be the line segment concentric to ℓ_2 with $|\ell_2^*| = |\ell_2|/2$ and similarly ℓ_3^*. Set $X_{k+1} = L \times \ell_2^* \times \ell_3^*$, $\tilde{X}_{k+1} = \tilde{L} \times \ell_2^* \times \ell_3^*$. Next, let $A_{k+2} = \ell_1 \times L' \times \ell_3$ and $\tilde{A}_{k+2} = \ell_1 \times \tilde{L}' \times \ell_3$ be neighboring D_{k+2}-slices touching X_{k+1} and $\tilde{X}_{k+1}$. Finally, let $A_{k+3} = \ell_1 \times \ell_2' \times L''$ and $\tilde{A}_{k+3} = \ell_1 \times \ell_2' \times \tilde{L}''$ be neighboring D_{k+3}-slices between A_{k+1} and $\tilde{A}_{k+1}$ touching X_{k+1}, and then necessarily also $\tilde{X}_{k+1}$. The slices $A_j, \tilde{A}_j$, $j = k+1, k+2, k+3$, bound a box C' of E_{k+3}.

Now we will study what constraints the boundary correspondence on ∂C puts on $d(gP)$ for prisms P in various slices listed above. We first consider the move in the slice A_{k+1} from a D_k-slice to X_{k+1}. This distance is $|\ell_3|/4 = \gamma^{k+2}/4$ and we travel through at least

$$\frac{\gamma^{k+2}}{4\gamma^k\delta} = \frac{\gamma^2}{4\delta} = \frac{1}{4}\,\frac{1}{\delta}\left(\frac{1-m\delta}{m+1}\right)^{2/3} > \frac{1}{4}(m+1)^{1/3} \tag{3.4}$$

prisms P in A_{k+1} when going from a D_k-slice to X_{k+1}. Without regarding other constraints we may by (3.3) and (3.4) increase $d(gP)$ in this move by a ratio $s^{\gamma^2/4\delta}$. To apply (3.3) we need a certain lower bound for m. A D_k-slice has width $\gamma^{k-1}\delta$ and $|L| = \gamma^k\delta$, so there is a drop of ratio γ in the height of the prisms when we shift from a D_k-slice to A_{k+1}. Putting these observations and similar ones for $\tilde{X}_{k+1}$ together we obtain

$$d(gP) \ge c_K s^{\gamma^2/4\delta}\gamma\tau_{C,k} \tag{3.5}$$

for prisms P in X_{k+1} or $\tilde{X}_{k+1}$. Here $c_K \in]0,1[$ is a constant depending on K, which takes into account the behavior at the rim of A_{k+1} and $\tilde{A}_{k+1}$. In (3.5) we have not taken into account the effect of the boundary correspondence from other slices.

The effect of the boundary correspondence from the D_{k-1}-slices to X_{k+1} and $\tilde{X}_{k+1}$ is estimated similarly. In place of (3.4) we get

$$\frac{\gamma^{k+1}}{4\gamma^k\delta} = \frac{\gamma}{4\delta} > \frac{1}{4}(m+1)^{2/3}. \tag{3.6}$$

The drop of heights of prisms from a D_{k-1}-slice to A_{k+1} or $\tilde{A}_{k+1}$ now has a ratio γ^2 and in place of (3.5) we have

$$d(gP) \ge c_K s^{\gamma/4\delta}\gamma^2\tau_{C,k-1} \tag{3.7}$$

for prisms P in X_{k+1} or $\tilde{X}_{k+1}$. By (3.5) and (3.7) the constraints from the D_k-slices and D_{k-1}-slices allow thus an estimate

$$\begin{aligned} d(gP) &\ge c_K \min\bigl(s^{\gamma/4\delta}\gamma^2\tau_{C,k-1}, s^{\gamma^2/4\delta}\gamma\tau_{C,k}\bigr) \\ &\ge c_K\gamma s^{\gamma^2/4\delta}\min(\gamma\tau_{C,k-1}, \tau_{C,k}) \\ &\ge c_K\gamma s^{\gamma^2/4\delta}\tau_C > c_K^2\gamma s^{\gamma^2/4\delta}\tau_C \end{aligned} \tag{3.8}$$

for prisms P in X_{k+1} or $\tilde{X}_{k+1}$.

The effect of the boundary correspondence from the D_{k-2}-slices to X_{k+1} and $\tilde{X}_{k+1}$ does not change the estimate (3.8). To see this it is enough to consider X_{k+1} and the case where A_{k+1} is the first D_{k+1}-slice in C from a given D_{k-2}-slice A_{k-2}. Let $\hat{A}_{k+2}$ be a D_{k+2}-slice touching A_{k-2} and X_{k+1}. We travel through at least

$$\frac{\gamma^{k+3}}{\delta\gamma^{k+1}} = \frac{\gamma^2}{\delta} > (m+1)^{1/3} \tag{3.9}$$

prisms in $\hat{A}_{k+2}$ when going from A_{k-2} to X_{k+1}. The drop of heights of prisms from A_{k-2} to $\hat{A}_{k+2}$ has a ratio γ^4 and the drop from X_{k+1} to $\hat{A}_{k+2}$ is γ. The constraint from A_{k-2} allows therefore an estimate

$$d(gP) \geq c_K^2\gamma^{-1}s^{\gamma^2/\delta}\gamma^4\tau_{C,k-2} \geq c_K^2\gamma s^{\gamma^2/\delta}\tau_C \tag{3.10}$$

for prisms P in X_{k+1}. Since this is stronger than (3.8), we conclude that the boundary correspondence on ∂C allows the earlier given estimate (3.8).

Our next task is to consider estimates of the type (3.8) for prisms in the part $Y_{k+2} = \ell_1 \times L' \times \ell_3^*$ ($\tilde{Y}_{k+2} = \ell_1 \times \tilde{L}' \times \ell_3^*$) of A_{k+2} ($\tilde{A}_{k+2}$). Similarly as above we conclude that the boundary correspondence from a D_k-slice allows the estimate

$$d(gP) \geq c_K\gamma^2 s^{\gamma/\delta}\tau_{C,k} \tag{3.11}$$

for prisms P in Y_{k+2} or $\tilde{Y}_{k+2}$. The drop of heights of prisms from X_{k+1} and $\tilde{X}_{k+1}$ to Y_{k+2} and $\tilde{Y}_{k+2}$ is γ. Hence the boundary correspondence from X_{k+1} and $\tilde{X}_{k+1}$ allows by (3.8) the estimate

$$d(gP) \geq c_K^2\gamma^2 s^{\gamma^2/4\delta}\tau_C \tag{3.12}$$

for P in Y_{k+2} or $\tilde{Y}_{k+2}$. No further constraints have to be taken into account for Y_{k+2} and $\tilde{Y}_{k+2}$. Since (3.12) is weaker than (3.11), the boundary correspondence on ∂C allows the estimate (3.12).

Finally we study the situation in A_{k+3} and $\tilde{A}_{k+3}$. The only constraints that matter are those coming from X_{k+1}, $\tilde{X}_{k+1}$ and Y_{k+2}, $\tilde{Y}_{k+2}$. The drops of heights of prisms is γ^2 when we enter A_{k+3} from X_{k+1} or $\tilde{X}_{k+1}$, and γ when we enter A_{k+3} from Y_{k+2} or $\tilde{Y}_{k+2}$. By (3.8) and (3.12) we get then that the boundary correspondence on C allows the estimate

$$d(gP) \geq c_K^3\gamma^3 s^{\gamma^2/4\delta}\tau_C \tag{3.13}$$

for prisms P in A_{k+3} or $\tilde{A}_{k+3}$.

In conclusion we get then from (3.8), (3.12), and (3.13) the estimate

$$\tau_{C'} = \min(\gamma^2 \tau_{C',k+1}, \gamma \tau_{C',k+2}, \tau_{C',k+3}) \geq c_K^3 \gamma^3 s^{\gamma^2/4\delta} \tau_C. \tag{3.14}$$

For sufficiently large m we have $c_K^3 \gamma^3 s^{\gamma^2/4\delta} \geq 2$. Recall the connection between δ, m, and γ from 2.4. Suppose $\tau_C < \gamma^2/20$. Applying the inequality $\tau_{C_{i+1}} \geq 2\tau_{C_i}$ successively to pairs $C_i \supset C_{i+1}$ of boxes like C, C', we find a box $C_\ell \subset \operatorname{int} C$ with $\tau_{C_\ell} \geq \gamma^2/20$. This construction can be accomplished so that each g_k omits the ball V. Theorem 1.3 is proved.

3.15. Remarks. 1. Details to obtain Theorem 1.3 for $n \geq 4$ have not been worked out. In 2.8 we used the fact that smooth maps in dimension one do not carry local dilatation. Therefore, to get a canonical construction for $n \geq 4$ in the sense that the dilatation in a previous slice is not inherited to the next slice—and cause a cumulating increase on the dilatation—the method in 2.8 should be modified. Another difference is that boundary correspondences into slices are not locally homeomorphic for $n \geq 4$. This fact would also cause extra complications to the construction.

2. Similar ideas as in the proof of Theorem 1.3 to change the size and position of images of prisms have been used in [R1]. The method presented here would simplify the arguments in [R1].

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