Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, pages 175–178

TRANSLATION-INVARIANT FUNCTION ALGEBRAS ON COMPACT ABELIAN GROUPS

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Abstract. Characterisations are given of certain translation-invariant function algebras on compact abelian groups.

Let A be a function algebra on a compact Hausdorff space G and set $\bar{A} = \{\bar{\varphi} \mid \varphi \in A\}$ and $\mathrm{Re}(A) = \{\mathrm{Re}(\varphi) \mid \varphi \in A\}$. Recall that A is said to be *Dirichlet* if $A + \bar{A}$ is uniformly dense in $C(G)$—equivalently, $\mathrm{Re}(A)$ is uniformly dense in $\mathrm{Re}\big(C(G)\big)$—and that A is *logmodular* if every real-valued continuous function on G can be uniformly approximated by functions of the form $\log|\varphi|$, where φ is an invertible element of A. Dirichlet algebras are necessarily logmodular, but not conversely.

If G is a compact group, one can also consider subalgebras of $C(G)$ which are translation-invariant, that is, together with every function they contain, they also contain all of its translates. A natural question to ask is, what are the translation-invariant Dirichlet algebras? In the case of the circle group $\mathbf{T} = \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}$ the answer is well known. For if A is the *disc* algebra, that is, the closure of the polynomials on $\mathbf{T}$, then the only translation-invariant Dirichlet algebras on $\mathbf{T}$ are $C(\mathbf{T})$, A and $\bar{A}$ [3, p. 116]. A related, general result has been given by D. Rider, see Remark 3 below. In this paper we consider other related results.

If A is a function algebra on a compact Hausdorff space G and τ is a character of A, then the Hahn–Banach and Riesz–Kakutani theorems guarantee the existence of a regular probability measure m on G such that $\tau(\varphi) = \int \varphi\, dm$ for all $\varphi \in A$. Any such measure is a called a *representing measure* for τ and the set of such measures is a convex subset of $M(G)$. An extreme point of this convex set will be called an *extremal* representing measure for τ.

If τ admits exactly one representing measure m, then generalised Hardy spaces can be associated to A and m and substantial portions of classical Hardy space theory and Toeplitz operator theory can be extended to this context [1, 3, 4].

If A is a Dirichlet or logmodular algebra, then every character of A admits a unique representing measure.

1991 Mathematics Subject Classification: Primary 06F20; Secondary 46J10.

We need some additional terminology. If A is a set of continuous functions on G, then we say A is *antisymmetric* if $A \cap \bar{A} \subseteq \mathbf{C}1$. If A is a subalgebra of $C(G)$, then a measure $m \in M(G)$ is said to be *multiplicative* on A if $\int \varphi\psi\, dm = \int \varphi\, dm \int \psi\, dm$ for all $\varphi, \psi \in A$.

We now fix some notation which will hold throughout the sequel. We denote by G a compact abelian group, and by m its normalised Haar measure. The Pontryagin dual group of G is denoted by Γ.

A *translation-invariant ordering* on Γ is a total order $\leq$ on Γ such that $\chi_1 \leq \chi_2$ implies $\chi_1 + \chi \leq \chi_2 + \chi$ for all $\chi_1, \chi_2, \chi \in \Gamma$. Such an ordering exists if and only if Γ is torsion-free [6, p. 194]. If $\leq$ is a translation-invariant ordering of Γ, the pair $(\Gamma, \leq)$ is called an *ordered group.* In this case let $D = D(\Gamma, \leq)$ denote the closed linear span in $C(G)$ of the positive cone $\Gamma^+ = \{\chi \mid 1 \leq \chi\}$. We shall be interested in giving conditions on a subalgebra of $C(G)$ which ensure that it is equal to one of the algebras $D(\Gamma, \leq)$.

If A is a subset of $C(G)$, it is said to be *translation-invariant* if φ_s belongs to A whenever φ belongs to A, where φ_s is the translate of φ by $s \in G$ defined by $\varphi_s(t) = \varphi(t-s)$. Clearly, all of the algebras $D(\Gamma, \leq)$ are translation-invariant, since for any character χ of G we have $\chi_s = \overline{\chi(s)}\chi$.

Recall that the convolution $\varphi * \mu$ of a function φ in $C(G)$ and a measure μ in $M(G)$ is defined by

$$\varphi * \mu(s) = \int \varphi(s-t)\, d\mu(t)$$

and that $\varphi * \mu$ is a continuous function on G.

As is well known, $\varphi * \mu = \int \varphi_s\, d\mu(s)$. If A is a translation-invariant, closed, linear subspace of $C(G)$ and $\varphi \in A$, then $\varphi * \mu \in A$. If $\mu = \psi m$, where $\psi \in C(G)$, then $\varphi * \mu = \varphi * \psi$.

1. Theorem. *Let A be a translation-invariant function algebra on G. The following are equivalent conditions*:

(1) *There is an order relation $\leq$ making Γ an ordered group such that $A = D(\Gamma, \leq)$;*
(2) *A is an antisymmetric Dirichlet algebra;*
(3) *A is logmodular and m is multiplicative on A;*
(4) *m is the unique representing measure for a character of A;*
(5) *m is an extremal representing measure for a character of A.*

Proof. If Condition (1) holds, then the equation $\Gamma = \Gamma^+ \cup \overline{\Gamma^+}$ and density of the linear span of Γ in $C(G)$ imply that $A + \bar{A}$ is dense in $C(G)$. Also, antisymmetry of A follows easily from the fact that Γ^+ is antisymmetric. Thus, the implication (1) $\Rightarrow$ (2) holds.

Now suppose that Condition (2) holds. Since every Dirichlet algebra is logmodular (this follows from the observation that $\mathrm{Re}(\varphi) = \log|e^{\varphi}|$), in particular A is logmodular. To show that m is multiplicative on A we shall first need to show that $M = \Gamma \cap A$ is the positive cone for a translation-invariant order relation on Γ. Clearly, $MM \subseteq M$ and antisymmetry of A implies that $M \cap \bar{M} = \{1\}$, so it remains only to show that $\Gamma = M \cup \bar{M}$, that is, that every character χ of G belongs to A or $\bar{A}$.

Since $A + \bar{A}$ is dense in $C(G)$, therefore for some $\varphi, \psi \in A$, the integral $\int (\varphi + \bar{\psi})\chi\, dm$ is non-zero, and it follows that for some $\varphi \in A$ we have $\widehat{\varphi}(\chi) \neq 0$ or $\widehat{\varphi}(\bar{\chi}) \neq 0$, where $\widehat{\varphi}$ is the Fourier transform of φ. However, $\varphi * \chi = \widehat{\varphi}(\chi)\chi$, so if $\widehat{\varphi}(\chi) \neq 0$, then $\chi \in A$, by translation-invariance of A. Similarly, if $\widehat{\varphi}(\bar{\chi}) \neq 0$, then $\bar{\chi} \in A$. Hence, $\Gamma = M \cup \bar{M}$, as required.

Let $M' = M \setminus \{1\}$. We claim that M' is contained in a proper ideal of A. For if it is not, then we can write $1 = \varphi_1\chi_1 + \cdots + \varphi_n\chi_n$ for some functions $\varphi_j \in A$ and some characters $\chi_j \in M'$. Since Γ is totally ordered, we may suppose that $\chi_1 \leq \cdots \leq \chi_n$. Hence, $\bar{\chi}_1 = \varphi_1\chi_1\bar{\chi}_1 + \cdots + \varphi_n\chi_n\bar{\chi}_1$, and all $\chi_j\bar{\chi}_1 \in A$, so $\bar{\chi}_1 \in A$. By antisymmetry of A, χ_1 and $\bar{\chi}_1$ cannot both belong to A, as $\chi_1 \neq 1$. This contradiction therefore shows that M' is contained in a proper ideal of A, as claimed. Now let μ be a representing measure for the character corresponding to a maximal ideal of A containing M'. Then $\int \chi\, d\mu = 0$ for all $\chi \in M'$. By taking conjugates it follows that μ and m have the same Fourier–Stieltjes transforms, so $\mu = m$. Thus, m is multiplicative on A, as required. Thus, we have shown the implication (2) $\Rightarrow$ (3) holds.

If Condition (3) holds, then so does Condition (4), since for logmodular algebras every character admits a unique representing measure [3, p. 116]. The implication (4) $\Rightarrow$ (5) is trivial.

Finally, let us suppose that Condition (5) holds. The hypothesis on m implies that $A + \bar{A}$ is dense in $L^1(G, m)$ [3, p. 52] and we may now repeat part of the argument given in showing the implication (2) $\Rightarrow$ (3) holds to deduce that for every character χ of G, either χ or $\bar{\chi}$ belongs to A.

If φ is a real-valued function belonging to A and $\lambda = \int \varphi\, dm$, then

$$\int (\varphi - \lambda)^2\, dm = \left(\int \varphi\, dm - \lambda\right)^2 = 0,$$

because m is multiplicative on A. Hence, $\varphi = \lambda$ a.e. with respect to m and since m has support equal to G, therefore $\varphi = \lambda$. Thus, A is antisymmetric.

Combining these results, we conclude that $M = \Gamma \cap A$ is the positive cone for a translation-invariant order relation $\leq$ on Γ. If $D = D(\Gamma, \leq)$, then the containment $D \subseteq A$ is obvious. The reverse containment $A \subseteq D$ follows from the fact that

$$D = \left\{\varphi \in C(G) \mid \int \varphi\bar{\chi}\, dm = 0 \ (\chi \in \Gamma \setminus M)\right\}$$

(for a proof, see [6, p. 217]) and from the fact that m is multiplicative on A. Thus, we have shown that the implication (5) $\Rightarrow$ (1) holds and therefore the theorem is proved. □

2. Remark. The implication (2) $\Rightarrow$ (1) in the theorem is a special case of a result of D. Rider [5] but the proof given here is simpler than Rider's proof. Rider also shows that if on a compact group (not assumed to be abelian) there is defined an antisymmetric, left and right translation-invariant, Dirichlet algebra, then the group is necessarily abelian. This strong result does not appear to be obtainable by the methods of the present paper.

That part of the proof of the implication (2) $\Rightarrow$ (3) which involves showing that m is multiplicative on A is taken from the proof of a related result in [2, p. 166].

Since a torsion-free discrete abelian group has connected dual, the equivalent conditions of the theorem imply that G is connected.

If the assumption of translation-invariance is dropped, no such decisive theory as we obtained here is to be expected. For example, there exists a proper Dirichlet subalgebra of the disc algebra on the circle $\mathbf{T}$ [1, pp. 232–5] which by the remarks at the beginning of this paper cannot be translation-invariant.

References

[1] Browder, A.: Introduction to function algebras. - Benjamin, New York–Amsterdam, 1969.

[2] Gamelin, T.W.: Uniform algebras. - Prentice–Hall, New Jersey, 1969.

[3] Leibowitz, G.: Lectures on complex function algebras. - Scott–Foresman, Illinois, 1970.

[4] Murphy, G.J.: Toeplitz operators on generalised H^2 spaces. - J. Integral Equations Operator Theory, 15, 1992, 825–852.

[5] Rider, D.: Translation-invariant Dirichlet algebras on compact groups. - Proc. Amer. Math. Soc. 17, 1966, 977–983.

[6] Rudin, W.: Fourier analysis on groups. - Interscience, New York–London, 1962.

Received 25 November 1993