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DEFECT RELATION AND ITS REALIZATION FOR QUASIREGULAR MAPPINGS

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Abstract. From the theory of covering surfaces by L.V. Ahlfors follows a pointwise defect relation for meromorphic functions in the plane. The first part of the article contains a proof of a counterpart of Ahlfors' result for quasiregular mappings in all dimensions. The obtained defect relation is an improvement of an earlier version by the author. In the second part the sharpness of the defect relation is proved in dimension three. In this part the method is a modification of the author's proof of the sharpness of an analog of Picard's theorem.

1. Introduction

Quasiregular mappings have turned out to form the right generalization of the geometric part of the theory of analytic functions of one complex variable to real n-dimensional space. These mappings are defined as quasiconformal mappings, but without the injectivity requirement (see 2.2).

In 1980 a Picard-type theorem on omitted values for quasiregular mappings was proved in the following form:

1.1. Theorem [R5]. *For every dimension $n \geq 3$ and every $K \geq 1$ there exists a positive integer $q = q(n,K)$ such that every K-quasiregular mapping $f\colon R^n \to R^n \setminus \{a_1, \dots, a_q\}$ is constant.*

Proofs of 1.1 different from the one in [R5] has been given in [R7], [R8], [EL], and [L]. For generalizations of Theorem 1.1, see [R8], [HR1], and [HR2]. It is easy to construct a nonconstant quasiregular mapping of R^n into R^n omitting one point [Z]. In dimension three Theorem 1.1 is known to be qualitatively best possible, namely, any number of points can be omitted:

1.2. Theorem [R9]. *For each positive integer p there exists a nonconstant $K(p)$-quasiregular mapping $f\colon R^3 \to R^3$ omitting p points.*

The classical value distribution theory of meromorphic functions of R^2 into $\overline{R^2} = R^2 \cup \{\infty\}$ by R. Nevanlinna [N1] gives a far reaching sharpening of Picard's theorem. One of the consequences is Nevanlinna's defect relation. In 1935 L.V. Ahlfors [A] gave a parallel theory which has a very geometric character. One part

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of Ahlfors' theory is contained in the following relation for the covering numbers of a nonconstant meromorphic function $f\colon R^2 \to \overline{R^2}$. Let $n(r,y)$ be the number of points of $f^{-1}(y)$ in the disk $\overline{B}(r) = \{x \in R^2 : |x| \le r\}$ with multiplicity regarded. Let $A(r)$ be the *spherical average*, i.e., the average of the *counting function* $n(r,y)$ over all y in $\overline{R^2}$ with respect to spherical 2-measure. Then according to [A, p. 189] (see also [N2, p. 350]) there exists a set $E \subset [1,\infty[$ with finite logarithmic measure, i.e.,

$$\int_E \frac{dr}{r} < \infty,$$

such that

$$\limsup_{\substack{r\to\infty \\ r \notin E}} \sum_{j=1}^{q} \Big(1 - \frac{n(r,a_j)}{A(r)}\Big)_+ \le 2 \tag{1.3}$$

whenever $a_1, \ldots, a_q$ are distinct points in $\overline{R^2}$. Here $\alpha_+ = \max(0,\alpha)$ for $\alpha \in R^1$. We will call (1.3) Ahlfors' pointwise defect relation and

$$\delta(r,a_j) = \big(1 - n(r,a_j)/A(r)\big)_+$$

the *defect* of a_j in the disk $\overline{B}(r)$ or the *defect function* of a_j.

This article consists of two parts. In the first part we shall establish the counterpart to (1.3) for quasiregular mappings and improve an earlier version of a defect relation from [R6]. The second part of the paper is devoted to proving in dimension three the inverse of the defect relation. This means that for given defect numbers of a given sequence of points we can construct a quasiregular mapping, whose defect functions tend to these defect numbers, and whose dilatation depends only on the sum of the defect numbers. A major task in the second part is to modify the proof of Theorem 1.2 in [R9] for the case of an arbitrary configuration of the omitted points.

Given a nonconstant quasiregular mapping $f\colon R^n \to \overline{R^n}$ we define the counting function $n(r,y)$ as in the classical case, namely, by

$$n(r,y) = \sum_{x \in f^{-1}(y) \cap \overline{B}(r)} i(x,f), \tag{1.4}$$

where $i(x,f)$ is the local topological index. A nonconstant quasiregular mapping is discrete and open by a theorem of Yu.G. Reshetnyak, and so $i(x,f)$ is a well-defined positive integer. Also now we let $A(r)$ be the average of $n(r,y)$ over $\overline{R^n}$ with respect to the spherical n-measure. The defect relation takes the following form.

1.5. Theorem. *Let $f\colon R^n \to \overline{R^n}$ be a nonconstant K-quasiregular mapping and $n \geq 3$. Then there exists a set $E \subset [1,\infty[$ of finite logarithmic measure and a constant $C(n,K) < \infty$ depending only on n and K such that*

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \sum_{j=1}^{q} \delta(r,a_j) \leq C(n,K) \tag{1.6}$$

whenever $a_1,\dots,a_q$ are distinct points in $\overline{R^n}$.

Theorem 1.1 is clearly a corollary of Theorem 1.5. Next we state the inverse of the defect relation (for dimension three).

1.7. Theorem. *Let $a_1,a_2,\dots$ be a sequence of distinct points in $\overline{R^3}$ and let $\delta_1,\delta_2,\dots$ be numbers such that $0<\delta_j\leq 1$ and*

$$\sum_j \delta_j \leq p+1 \tag{1.8}$$

for some integer p. Then there exists a K-quasiregular mapping $f\colon R^3 \to \overline{R^3}$ with K depending only on p such that

$$\lim_{r\to\infty}\Bigl(1-\frac{n(r,a_j)}{A(r)}\Bigr) = \delta_j, \tag{1.9}$$

$$\lim_{r\to\infty}\Bigl(1-\frac{n(r,y)}{A(r)}\Bigr) = 0 \quad \text{if} \quad y\notin\{a_j : j=1,2,\dots\}. \tag{1.10}$$

A weaker form of Theorem 1.5 was proved in [R6] where (1.6) is replaced by

$$\limsup_{\substack{r\to\infty \\ r\notin E}} q\Bigl(\frac{1}{q}\sum_{j=1}^{q}\ \delta(r,a_j)\Bigr)^{n-1} \leq C(n,K). \tag{1.11}$$

Theorem 1.7 shows for $n=3$ that Theorem 1.5 gives qualitatively the right asymptotic upper bound for the sum of the defect functions outside an exceptional set of radii. This was not true with (1.11).

In dimension two it was a long standing problem whether arbitrary defects with the defect sum at most two can be realized for meromorphic functions (the case $n=2$, $p=1$, $K=1$, in 1.7). The full solution to the inverse problem of Nevanlinna theory was given by D. Drasin in [D].

With the bound $p+1=2$ Theorem 1.7 was proved in [R2] for $n=3$ and the method was extended to all dimensions $n\geq 3$ in [R4]. For general p Theorem 1.7, as well as Theorem 1.2, remains an open question for dimensions $n\geq 4$.

For some other results on value distribution of quasiregular mappings we refer to [R3], [MR], and [S]. For example, in [R3] upper bounds for $n(r,a)$ in terms of the spherical average are given. In [S] such bounds are proved for sums, i.e., inequalities in the opposite direction to (1.6). Such results were obtained for meromorphic functions in the plane by J. Miles [M].

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Part I: Proof of the defect relation

2. Covering averages

2.1. Some notation. The n-ball and $(n-1)$-sphere in R^n with center x and radius r are denoted by $B(x,r)$ and $S(x,r)$. We also write $B(r)=B(0,r)$, $S(r)=S(0,r)$, $B=B(1)$, $S=S(1)$. Sometimes we indicate the dimension and write for example $B^n(x,r)=B(x,r)$. The Euclidean distance in R^n is denoted by d and the standard orthonormal basis vectors by $e_1,\dots,e_n$. The Lebesgue measure in R^n is m and the normalized k-dimensional Hausdorff measure in R^n is $\mathscr{H}^k$. Sometimes we write dx for dm. We set $\omega_{n-1}=\mathscr{H}^{n-1}(S)$. We let $\overline{R^n}=R^n\cup\{\infty\}$ be equipped with the spherical metric via stereographic projection so that $\overline{R^n}$ is isometric to an n-sphere in R^{n+1} with radius $1/2$. The spherical distance in $\overline{R^n}$ is denoted by σ and balls and spheres with respect to σ by $B_\sigma(x,u)$ and $S_\sigma(x,u)$. The set of integers is $\mathbf{Z}$. If $\gamma\colon\Delta\to\overline{R^n}$ is a path, we denote its locus $\gamma\Delta$ by $|\gamma|$. If Γ is a family of nonconstant paths in $\overline{R^n}$, we let $M_n(\Gamma)=M(\Gamma)$ be the (n)-modulus of Γ (see [V, p. 16]). In Part I we shall use the letter b, with subscript, prime, etc., to represent positive constants depending only on the dimension n.

2.2. Quasiregular mappings. Let $n\ge 2$ and let G be a domain in R^n. A continuous mapping $f\colon G\to R^n$ is called *quasiregular* if (1) f belongs to the local Sobolev space $W^1_{n,\mathrm{loc}}(G)$, i.e., f has first order weak partial derivatives which are locally L^n-integrable, and (2) there exists a constant K, $1\le K<\infty$, such that

$$|f'(x)|^n\le KJ_f(x)\qquad\text{a.e.}\tag{2.3}$$

Here $|f'(x)|$ is the supremum norm of the formal derivative $f'(x)$ defined by means of the partial derivatives and $J_f(x)$ is the Jacobian determinant. The smallest K in (2.3) is the outer dilatation $K_O=K_O(f)$, and the smallest K, $1\le K<\infty$, in

$$J_f(x)\le K\inf_{|h|=1}|f'(x)h|^n$$

is the inner dilatation $K_I=K_I(f)$. The number $K(f)=\max\bigl(K_O(f),K_I(f)\bigr)$ is the (maximal) dilatation of f. If f is quasiregular and $K(f)\le K$, f is called K-*quasiregular.* The definition of quasiregularity extends in a straightforward manner to the case $f\colon M \to N$ where M and N are oriented and connected Riemannian n-manifolds. If $M \subset \overline{R^n}$, we call a quasiregular mapping $f\colon M \to \overline{R^n}$ also *quasimeromorphic.* A quasiregular homeomorphism is a *quasiconformal* mapping. For the general theory of quasiregular mappings we refer to [MRV1], [MRV2], [BI], [Re], [Vu].

2.4. Averages of the counting function. Let $n \geq 2$ and let $f\colon R^n \to \overline{R^n}$ be a nonconstant K-quasiregular mapping. If $A \subset R^n$ is any bounded Borel set and $y \in \overline{R^n}$, we write

$$n(A,y) = \sum_{x \in f^{-1}(y) \cap A} i(x,f),$$

where we recall the notation $i(x,f)$ for the local (topological) index of f at x (see [MRV1, p. 6]). If Y is an $(n-1)$-sphere in $\overline{R^n}$, we denote by $\nu(A,Y)$ the average of $n(A,y)$ over Y with respect to the spherical $(n-1)$-measure of Y. We abbreviate $n(r,y) = n\bigl(\overline{B}(r),y\bigr)$, $\nu(r,Y) = \nu\bigl(\overline{B}(r),Y\bigr)$, and $\nu(r,t) = \nu\bigl(\overline{B}(r),S(t)\bigr)$. The average of $n(r,y)$ over $\overline{R^n}$ with respect to the spherical n-measure is denoted by $A(r)$. Sometimes we show the mapping f in the notation; for example, we may write $\nu_f(r,t) = \nu(r,t)$. The following lemma is a slight improvement of [R3, 4.1] and the proof can be found in [R7] or [P].

2.5. Lemma. *If $\theta > 1$ and $r,s,t > 0$, then*

$$\nu(\theta r,t) \geq \nu(r,s) - \frac{K_I \bigl|\log \frac{t}{s}\bigr|^{n-1}}{(\log\theta)^{n-1}}. \tag{2.6}$$

The next lemma is essentially [R6, 2.4] and it relates the average $A(r)$ to averages over spheres.

2.7. Lemma. *Let $\alpha = 2^{-1}(n-1)^{-1}$. There exists a set $E \subset [1,\infty[$ of finite logarithmic measure such that the following holds. For every $\varepsilon > 0$ there exists an increasing function $\omega\colon [0,\infty[\to]0,\infty[$ such that*

$$\Bigl|\frac{\nu(s,Y)}{A(s')} - 1\Bigr| < \varepsilon \tag{2.8}$$

and

$$\frac{\nu(s,Y)}{\nu(s',Y)} \geq 1 - \varepsilon \tag{2.9}$$

whenever Y is an $(n-1)$-sphere in $\overline{R^n}$ with spherical radius $u \leq \pi/4$ and $s' \in [\omega(|\log u|),\infty[\setminus E$, where

$$s' = s + \frac{s}{A(s)^{\alpha}}. \tag{2.10}$$

A basic idea in the proof of Theorem 1.5 (and also of Theorem 1.1) is to produce growth relations for averages of counting functions in terms of the number of omitted values for certain restrictions of the map. The tool to obtain such relations is the following result.

2.11. Lemma. *Let* $a_1, \dots, a_\lambda$, $\lambda \ge 2$, *be points in* $B(1/2)$, *let*

$$\sigma_0 = \frac{1}{4} \min_{j \neq k} |a_j - a_k|,$$

and let $0 < \sigma < \sigma_0$. *Suppose that* $F_1, \dots, F_\lambda$ *are disjoint continua in* R^n *each connecting* S *and* $S(3/2)$ *and such that* $fF_j \subset B(a_j, \sigma)$, $1 \le j \le \lambda$. *Then there exist* a_j *such that*

$$(2.12) \qquad (\lambda^{1/(n-1)} - b'K^2)\Big(\log \frac{\sigma_0}{\sigma}\Big)^{n-1} \le b''K\nu\big(7/4, S(a_j, \sigma_0)\big).$$

Proof. Let Γ_j be the family of paths in $B(3/2) \setminus \overline{B}$ connecting F_j and $F_j^* = \bigcup_{k \neq j} F_k$. It is proved in [R6, pp. 186–188] that there exists $j \in \{1, \dots, \lambda\}$ such that

$$(2.13) \qquad M(\Gamma_j) \ge b_0 \lambda^{1/(n-1)}.$$

Suppose $a_j = 0$ and define a function ϱ' by

$$\varrho'(y) = \frac{1}{\big(\log \frac{\sigma_0}{\sigma}\big)|y|} \quad \text{if} \quad \sigma < |y| < \sigma_0,$$
$$\varrho'(y) = 0 \quad \text{elsewhere.}$$

Set

$$L(x, f) = \limsup_{|h| \to 0} \frac{|f(x+h) - f(x)|}{|h|}.$$

Then (see [MRV1, 3.2])

$$(2.14) \quad b_0 \lambda^{1/(n-1)} \le M(\Gamma_j) \le \int\limits_{B(3/2)} \varrho'\big(f(x)\big)^n L(x, f)^n \, dx$$
$$\le K \int\limits_{B(3/2)} (\varrho' \circ f)^n J_f \, dm \le \frac{K\omega_{n-1}}{\big(\log \frac{\sigma_0}{\sigma}\big)^n} \int_\sigma^{\sigma_0} \frac{\nu(3/2, t)}{t} \, dt.$$

By Lemma 2.5,

$$\nu(3/2, t) \le \nu(7/4, \sigma_0) + K \frac{\big(\log(\sigma_0/\sigma)\big)^{n-1}}{\big(\log(7/6)\big)^{n-1}}.$$

Substituting this into (2.14) gives (2.12). If $a_j \neq 0$, we get (2.12) by performing an auxiliary quasiconformal mapping. The lemma is proved. □

3. Defect sum and lifts of paths

In this section we shall mainly recall some notation and arguments from [R6]. Let $f\colon R^n \to \overline{R^n}$ be a nonconstant K-quasiregular mapping and let $E \subset [1,\infty[$ be the set given by Lemma 2.7. It will be the exceptional set in 1.5. We shall first prove (1.6) for points $a_1,\dots,a_q$ in $B(1/2)$ and in the very end consider the general case.

We write $\nu(F) = \nu(F,1)$ for any bounded Borel set $F \subset R^n$ and $\nu(r) = \nu(r,1)$. By Lemma 2.7 there exists $\kappa > 1$ such that

$$\Bigl|\frac{A(s')}{\nu(s)} - 1\Bigr| < \frac{1}{q} \tag{3.1}$$

and

$$\nu(s') \le \frac{3}{2}\nu(s) \tag{3.2}$$

whenever $s > 0$ is such that $s' \in [\kappa,\infty[\, \setminus E$. We shall later make the bound κ larger when necessary. Fix such s.

To prove 1.5 it suffices to show

$$\sum_{j=1}^{q}\Bigl(1 - \frac{n(s',a_j)}{A(s')}\Bigr)_+ \le C(n,K) < \infty.$$

Set $J = \{1,\dots,q\}$. We may assume $q \ge 2$ and $1 - n(s',a_j)/A(s') > 0$ for all $j \in J$. Set

$$\Delta_j = 1 - \frac{n(s',a_j)}{\nu(s)}. \tag{3.3}$$

By (3.1) we then get

$$\begin{aligned}\sum_{j=1}^{q}\Bigl(1 - \frac{n(s',a_j)}{A(s')}\Bigr) &= \sum_{j\in J}\Delta_j + \sum_{j\in J}\frac{n(s',a_j)}{A(s')}\Bigl(\frac{A(s')}{\nu(s)} - 1\Bigr)\\ &\le \sum_{j\in J}\Delta_j + q\Bigl|\frac{A(s')}{\nu(s)} - 1\Bigr| \le \sum_{j\in J}\Delta_j + 1.\end{aligned} \tag{3.4}$$

To prove 1.5 it is therefore enough to give $\sum_j \Delta_j$ an upper bound depending only on n and K. Hence we may assume that $\Delta_j > 0$ for all $j \in J$ and that $\sum_j \Delta_j \ge 20$.

The strategy for the proof is the following. Since $n(s',a_j)/\nu(s) < 1$, the point a_j is covered less by $f|\overline{B}(s')$ than a point $y \in S$ by $f|\overline{B}(s)$ on average.

This means that, on average, straight paths which join y and a_j have maximal lifts connecting $f^{-1}(y)\cap\overline{B}(s)$ and $S(s')$, and the number of such lifts increases with Δ_j. If the total sum $\sum_j \Delta_j$ is large, it means that one has such lifts for many j's, again on average. The goal is to arrive at a point where Lemma 2.11 can be applied to produce a certain amount of growth on averages of the counting function. For this we decompose $\overline{B}(s)$ into sets U_i, $i=1,\dots,p$, by taking a Whitney-type decomposition for $B(s')$ and restricting it to $\overline{B}(s)$. In Lemma 2.11 $\overline{U}_i$ will then correspond to $\overline{B}(1/2)$ and a set W_i will correspond to $\overline{B}(2)$. The sets W_i will stay inside $\overline{B}(s')$ and they will not overlap too much. We shall obtain lower bounds for the ratios $\nu(W_i)/\nu(U_i)$, which in turn give a lower bound for $\nu(s')$ in terms of $\nu(s)$. If $\sum_j \Delta_j$ is too large, we contradict (3.2). To apply Lemma 2.11 we have to study in detail what happens to the lifts described above.

3.5. Decomposition of $\overline{B}(s)$. We start by giving precise conditions on the decomposition of $\overline{B}(s)$. Set $d_0 = s'-s$. We decompose $\overline{B}(s)$ into disjoint Borel sets U_i, $i\in I=\{1,\dots,p\}$, such that

(1) $0<A_n\le \varrho_0(U_i)\le B_n<\infty$, where $\varrho_0(U_i)$ is the diameter of U_i in the hyperbolic metric $4s'^2\,dx^2/(s'^2-|x|^2)^2$ of the ball $B(s')$ and where A_n and B_n depend only on n.

(2) There exist $b_2>0$ and K_0-quasiconformal mappings $\varphi_i\colon R^n\to R^n$, b_2 and K_0 depending only on n, such that $\varphi_i^{-1}B(1/2)\subset U_i\subset \varphi_i^{-1}\overline{B}(1/2)$, $W_i=\varphi_i^{-1}\overline{B}(2)\subset \overline{B}(s')$, and each point belongs to at most b_2 of the sets W_i.

We obtain such a decomposition easily, for example, by making such a decomposition on a cube and then using radial stretching. It follows from (3.1) that the number of sets U_i has an upper bound of the form

$$p\le b_1(s/d_0)^{n-1}=b_1A(s)^{1/2}\le 2b_1\nu(s)^{1/2}. \tag{3.6}$$

For each $i\in I$ write

$$X_i=\varphi_i^{-1}\overline{B}(3/4),\qquad Y_i=\varphi_i^{-1}\overline{B}(1),\qquad Z_i=\varphi_i^{-1}\overline{B}(3/2).$$

Let

$$\sigma_0=\frac14\min_{j\ne k}|a_j-a_k|$$

and for $j\in J$ and $y\in S$ let $\gamma_y^j\colon[0,1]\to\overline{B}(1)$ be the path $\gamma_y^j(t)=(1-t)y+ta_j$ (note the difference in parametrization in [R6, p. 173]). Set $f_0=f|B(s'+1)$.

3.7. Essentially separate lifts. Let $i\in I$ and $j\in J$. For each $y\in S$ we choose a sequence $\lambda_{y,1},\dots,\lambda_{y,k}$, $k=n(s,y)$, of essentially separate maximal f_0-lifts of γ_y^j starting in $f^{-1}(y)\cap\overline{B}(s)$. This means that each $\lambda_{y,\nu}$ is a maximal (partial) f_0-lift of γ_y^j, $\lambda_{y,\nu}(0)\in f^{-1}(y)\cap\overline{B}(s)$, and

$$\operatorname{card}\{\nu:\lambda_{y,\nu}(t)=x\}\le i(x,f)\quad\text{for all } x \text{ and } t.$$

The existence of such a sequence is proved in [R1]. Let $y \in S$. Those lifts $\lambda_{y,\nu}$ that start in U_i and satisfy $|\lambda_{y,\nu}| \not\subset \overline{B}(s')$ are denoted by $\alpha_1, \dots, \alpha_{\nu_y}$. Set

$$n_i^j(y) = \nu_y. \tag{3.8}$$

By [R6, Lemma 4.6 and p. 181] the choices of the sequences $\lambda_{y,1}, \dots, \lambda_{y,k}$, $y \in S$, can be made so that each function $n_i^j\colon S \to R^1$ is measurable. At least $n(s,y) - n(s',a_j)$ of the lifts $\lambda_{y,1}, \dots, \lambda_{y,k}$ must leave $\overline{B}(s')$, hence

$$\sum_{i \in I} n_i^j(y) \ge n(s,y) - n(s',a_j).$$

For the average we thus get, using (3.3), that

$$\frac{1}{\omega_{n-1}} \int_S \sum_{i \in I} n_i^j(y)\,dy \ge \nu(s) - n(s',a_j) = \nu(s)\Delta_j. \tag{3.9}$$

For $i \in I$ let

$$J_i = \left\{ j : 2 \int_S n_i^j(y)\,dy > \omega_{n-1}\nu(U_i)\Delta_j \right\}. \tag{3.10}$$

Note that summing the inequalities in (3.10) over $i \in I$ gives (3.9) except for the factor 2. From (3.9) it follows that

$$\begin{aligned}
\frac{\omega_{n-1}}{2}\nu(s)\sum_{j \in J}\Delta_j &= \frac{\omega_{n-1}}{2}\sum_{i \in I}\sum_{j \in J}\nu(U_i)\Delta_j \ge \frac{\omega_{n-1}}{2}\sum_{i \in I}\sum_{j \in J \setminus J_i}\nu(U_i)\Delta_j \\
&\ge \sum_{i \in I}\sum_{j \in J \setminus J_i}\int_S n_i^j(y)\,dy = \sum_{i \in I}\sum_{j \in J}\int_S n_i^j(y)\,dy - \sum_{i \in I}\sum_{j \in J_i}\int_S n_i^j(y)\,dy \\
&\ge \omega_{n-1}\nu(s)\sum_{j \in J}\Delta_j - \sum_{i \in I}\sum_{j \in J_i}\int_S n_i^j(y)\,dy.
\end{aligned}$$

We thus get the following lemma.

3.11. Lemma. *The functions n_i^j satisfy*

$$\sum_{i \in I}\sum_{j \in J_i}\int_S n_i^j(y)\,dy \ge \frac{\omega_{n-1}}{2}\nu(s)\sum_{j \in J}\Delta_j. \tag{3.12}$$

Inequality (3.12) gives a first step in the estimation of the number of lifts in terms of the sum $\sum_j \Delta_j$.

4. Completion of the proof of Theorem 1.5

In order to apply Lemma 2.11 we shall next study the positions of the lifts $\alpha_1,\dots,\alpha_{\nu_y}$ of γ_y^j starting in U_i (see 3.7) for certain parameter values. First fix i and j. For $1\le\nu\le\nu_y=n_i^j(y)$ we let $t_{y,\nu},u_{y,\nu},v_{y,\nu}$ be the smallest numbers such that $0\le t_{y,\nu}<u_{y,\nu}<v_{y,\nu}$ and $\alpha_\nu(t_{y,\nu})\in\partial U_i$, $\alpha_\nu(u_{y,\nu})\in\partial X_i$, $\alpha_\nu(v_{y,\nu})\in Y_i$. Set

$$L_i^j(y)=\Bigl\{\nu\in\{1,\dots,\nu_y\}:\frac{1-t_{y,\nu}}{1-u_{y,\nu}}\le\frac{1}{\sigma_0}\Bigr\},\tag{4.1}$$

$$M_i^j(y)=\Bigl\{\nu\in\{1,\dots,\nu_y\}:\frac{1-u_{y,\nu}}{1-v_{y,\nu}}\le\frac{3\sigma_0}{2\sigma_i}\Bigr\},\tag{4.2}$$

where $\sigma_i\in\,]0,\sigma_0]$ is chosen so that the equality

$$\Bigl(\log\frac{\sigma_0}{\sigma_i}\Bigr)^{n-1}=A_i\nu(U_i)\tag{4.3}$$

holds with a nonnegative number A_i to be chosen in (4.7) for each $i\in I$.

The following averaging estimates for the cardinalities of the sets $L_i^j(y)$ and $M_i^j(y)$ are from [R6].

4.4. Lemma [R6, 3.8 and 3.16]. *There exist nonnegative measurable functions $l_i^j\colon S\to R^1$ and $m_i^j\colon S\to R^1$ such that*

(1) $\operatorname{card}L_i^j(y)\le l_i^j(y)$ *for* $y\in S$,

(2) $\operatorname{card}\bigl(M_i^j(y)\setminus L_i^j(y)\bigr)\le\widetilde m_i^j(y)=A_im_i^j(y)$ *for* $y\in S$,

and the following estimates hold:

$$\int_S l_i^j(y)\,dy\le b_3K\Bigl(\log\frac{1}{\sigma_0}\Bigr)^{n-1},$$
$$\sum_{j\in J}\int_S m_i^j(y)\,dy\le b_4K\nu(U_i).$$

For $i\in I$ we now define (recall J_i from (3.10))

$$J^i=\Bigl\{j\in J_i:3\int_S l_i^j\ge\int_S n_i^j\text{ or }3\int_S\widetilde m_i^j\ge\int_S n_i^j\Bigr\}.\tag{4.5}$$

Lemma 4.4 gives first

$$\begin{aligned}\sum_{j\in J^i}\int_S n_i^j(y)\,dy&\le3\sum_{j\in J^i}\int_S\bigl(l_i^j(y)+\widetilde m_i^j(y)\bigr)\,dy\\&\le3b_3Kq\Bigl(\log\ \frac{1}{\sigma_0}\Bigr)^{n-1}+3b_4KA_i\nu(U_i).\end{aligned}$$

With Lemma 3.11 and (3.6) we then get

$$
\begin{aligned}
&\sum_{i\in I}\sum_{j\in J_i\setminus J^i}\int_S n_i^j(y)\,dy=\sum_{i\in I}\sum_{j\in J_i}\int_S n_i^j(y)\,dy-\sum_{i\in I}\sum_{j\in J^i}\int_S n_i^j(y)\,dy\\
&\geq\frac{\omega_{n-1}}{2}\nu(s)\sum_{j\in J}\Delta_j-6b_1b_3Kq\Big(\log\frac{1}{\sigma_0}\Big)^{n-1}\nu(s)^{1/2}-3b_4K\sum_{i\in I}A_i\nu(U_i).
\end{aligned}
\tag{4.6}
$$

Up to this point the reasoning has been very similar to the one in [R6]. In fact, (4.6) differs from [R6, (5.12)] only in the last term.

Set $\lambda_i=\operatorname{card}(J_i\setminus J^i)$. As A_i increases in the range $[0,\infty[$, the number λ_i decreases from

$$
\lambda_i^0=\operatorname{card}\Big\{j\in J_i:3\int_S l_i^j<\int_S n_i^j\Big\}
$$

to some value λ_i^∞. We may assume that at the discontinuities of the function $A_i\mapsto\lambda_i$ the jumps are 1. If this is not the case originally, we make small variations in the functions m_i^j for different j's. Then it is clear that we may choose $A_i\geq 0$ such that

$$
\lambda_i-1\leq 9\omega_{n-1}^{-1}b_4KA_i\leq\lambda_i. \tag{4.7}
$$

Let $d=\sum_j\Delta_j/10$ and set

$$
I_1=\big\{i\in I:\lambda_i\leq d\text{ or }\nu(U_i)\leq\nu(s)^{1/4}\big\}. \tag{4.8}
$$

Notice that $d\geq 2$ since we have assumed $\sum_j\Delta_j\geq 20$. We now easily obtain the following basic estimate.

4.9. Proposition. *With the choice* (4.7) *of* A_i *we have*

$$
\sum_{i\in I\setminus I_1}\lambda_i\nu(U_i)\geq\frac{\nu(s)}{8}\sum_{j\in J}\Delta_j \tag{4.10}
$$

provided κ *is large enough.*

Proof. By (4.6) and (4.7) we obtain

$$
\begin{aligned}
\sum_{i\in I}\lambda_i\nu(U_i)&=\sum_{i\in I}\sum_{j\in J_i\setminus J^i}\nu(U_i)\geq\sum_{i\in I}\sum_{j\in J_i\setminus J^i}\omega_{n-1}^{-1}\int_S n_i^j(y)\,dy\\
&\geq\frac{1}{2}\nu(s)\sum_{j\in J}\Delta_j-6b_1b_3\omega_{n-1}^{-1}Kq\Big(\log\frac{1}{\sigma_0}\Big)^{n-1}\nu(s)^{1/2}-\frac{1}{3}\sum_{i\in I}\lambda_i\nu(U_i),
\end{aligned}
$$

from which we first get

$$\sum_{i\in I}\lambda_i\nu(U_i)\geq\frac{\nu(s)}{4}\sum_{j\in J}\Delta_j \tag{4.11}$$

for κ large enough. For the sum over I_1 we get from (3.6) and (4.8) the estimate

$$\sum_{i\in I_1}\lambda_i\nu(U_i)\leq\frac{1}{10}\nu(s)\sum_{j\in J}\Delta_j+2b_1\nu(s)^{1/2}q\nu(s)^{1/4}.$$

With (4.11) this gives (4.10) for κ large enough. □

We are now in a position to apply Lemma 2.11 to each map $f\circ\varphi_i^{-1}$, $i\in I\setminus I_1$. Let $i\in I\setminus I_1$. For each $j\in J_i\setminus J^i$ we have by the definitions (3.10) and (4.5) that

$$\int_S(n_i^j-l_i^j-\widetilde{m}_i^j)(y)\,dy\geq\frac{1}{3}\int_S n_i^j(y)\,dy>\frac{\omega_{n-1}}{6}\Delta_j\nu(U_i)>0,$$

and hence $n_i^j(y)-l_i^j(y)-\widetilde{m}_i^j(y)>0$ for some $y\in S$. By Lemma 4.4 we then have $n_i^j(y)-\operatorname{card}\bigl(L_i^j(y)\cup M_i^j(y)\bigr)>0$. This means with the notation in (4.1) and (4.2) that there is an index $\nu\in\{1,\dots,n_i^j(y)\}$ for which $1-v_{y,\nu}\leq 2\sigma_i(1-t_{y,\nu})/3\leq 2\sigma_i/3$. The corresponding lift α_ν has the following properties for some $t\leq 1$:

(1) The restriction $\alpha_i^j=\alpha_\nu|[v_{y,\nu},t]$ connects ∂Y_i and ∂Z_i.

(2) $|f\circ\alpha_i^j|\subset\overline{B}(a_j,\sigma_i)$.

Then $\varphi_i\circ\alpha_i^j$ connects S and $S(3/2)$. We now apply Lemma 2.11 to the map $g=f\circ\varphi_i^{-1}$ and the continua $F_j=|\varphi_i\circ\alpha_i^j|$, $j\in J_i\setminus J^i$. We obtain that for some $j\in J_i\setminus J^i$,

$$(\lambda_i^{1/(n-1)}-b'K_0^2K^2)\Bigl(\log\frac{\sigma_0}{\sigma_i}\Bigr)^{n-1}\leq b''K_0K\nu_g\bigl(7/4,S(a_j,\sigma_0)\bigr). \tag{4.12}$$

By (4.3), (4.7), and $\lambda_i\geq d\geq 2$ we have

$$\lambda_i\nu(U_i)=\lambda_iA_i^{-1}\Bigl(\log\frac{\sigma_0}{\sigma_i}\Bigr)^{n-1}\leq 18\omega_{n-1}^{-1}b_4K\Bigl(\log\frac{\sigma_0}{\sigma_i}\Bigr)^{n-1}. \tag{4.13}$$

Lemma 2.5 together with a suitable quasiconformal mapping (as at the end of the proof of 2.11) yields

$$\nu(W_i)=\nu_g(2,1)\geq\nu_g\bigl(7/4,S(a_j,\sigma_0)\bigr)-\frac{b'''K\bigl(\log(1/\sigma_0)\bigr)^{n-1}}{\bigl(\log\,(8/7)\bigr)^{n-1}}. \tag{4.14}$$

Since $i \in I \setminus I_1$, by choosing κ originally large enough we can have the average $\nu(W_i)$ ($\geq \nu(U_i)$) exceeds $b'''K\big(\log(1/\sigma_0)\big)^{n-1}\big(\log(8/7)\big)^{1-n}$, which with (4.14) gives

$$\nu_g\big(7/4, S(a_j,\sigma_0)\big) \leq 2\nu(W_i). \tag{4.15}$$

Inequalities (4.12), (4.13), and (4.15) yield

$$(\lambda_i^{1/(n-1)} - b_5K^2)\lambda_i\nu(U_i) \leq b_6K^2\nu(W_i). \tag{4.16}$$

We formulate the conclusion of this as follows.

4.17. Proposition. *Let* b_5 *and* b_6 *be the constants in* (4.16). *Suppose*

$$\sum_{j\in J}\Delta_j \geq \max\big(20, 10(2b_5K^2)^{n-1}\big). \tag{4.18}$$

Then, for $i \in I \setminus I_1$ *and for large enough* κ,

$$b_5\lambda_i\nu(U_i) \leq b_6\nu(W_i). \tag{4.19}$$

Proof. Since $i \in I \setminus I_1$, we have $\lambda_i \geq d = \sum_j \Delta_j/10 \geq (2b_5K^2)^{n-1}$, hence $\lambda_i^{1/(n-1)} - b_5K^2 \geq b_5K^2$ and (4.19) follows from (4.16). Note that $\sum_j \Delta_j \geq 20$ was needed for $\lambda_i \geq d \geq 2$ in (4.16). □

4.20. Remark. Clearly with the same assumption (4.18) the stronger inequality $\lambda_i^{1+1/(n-1)}\nu(U_i) \leq 2b_6K^2\nu(W_i)$ is also true, but we shall need (4.19).

Suppose that (4.18) holds and κ is large enough for Propositions 4.9 and 4.17. Then by (3.2), 3.5(2), 4.9, and (4.19) we obtain

$$\frac{\nu(s)}{8}\sum_{j\in J}\Delta_j \leq \sum_{i\in I\setminus I_1}\lambda_i\nu(U_i) \leq b_5^{-1}b_6\sum_{i\in I}\nu(W_i) \leq b_5^{-1}b_6b_2\nu(s') \leq \frac{3}{2}b_5^{-1}b_6b_2\nu(s).$$

The final conclusion is therefore that

$$\sum_{j\in J}\Delta_j \leq \max\big(20, 10(2b_5K^2)^{n-1}, 12b_5^{-1}b_6b_2\big) = C'(n,K). \tag{4.21}$$

With (3.4) this proves Theorem 1.5 when $a_1,\dots,a_q \in B(1/2)$.

For the general case we replace S by another sphere $Y = S_\sigma(z,u)$, $u \leq \pi/20$, such that $a_1,\dots,a_q \in \overline{R^n} \setminus B_\sigma(z,5u)$. We apply Lemma 2.7 to Y. With $\nu(s)$ replaced by $\nu(s,Y)$ we get (3.4). We may need to increase κ. Let h be a spherical isometry of $\overline{R^n}$ such that $h(z) = \infty$, and let T be a Möbius transformation of $\overline{R^n}$ that keeps 0 and ∞ fixed and takes hY onto S. Then $Th\big(\overline{R^n}\setminus B_\sigma(z,5u)\big) \subset B(1/2)$ and $\nu_f(s,Y) = \nu_\psi(s,1)$ where $\psi = T\circ h\circ f$. Because $K(\psi) = K(f)$, we conclude that the same bound in (4.21) is valid in the general case as well. This completes the proof of Theorem 1.5.

4.22. Remarks. 1. One can formulate and prove a corresponding defect relation for quasiregular mappings of the unit ball into $\overline{R^n}$ (cf. [R6, Remark 6.13]).

2. M. Pesonen [P] has used the method from [R6] in a modified form to give an alternative proof for quasiregular mappings of Ahlfors' sharp result (1.3) for $n = 2$. Ahlfors' article [A] also includes the case of smooth quasiregular mappings for $n = 2$.

Part II: Realization of given defects

5. Background of construction for p omitted points in R^3

We shall now turn to the proof of Theorem 1.7. The main idea is to divide R^3 into sectorlike sets according to the given numbers δ_j and to make the mapping in a large part of each such set omit in $\overline{R^3}$ a certain subset of $A = \{a_1, a_2, \ldots\}$ consisting of $p+1$ points. The main difficulties in carrying out this program are the following. First, the construction in [R9] must be modified to the present situation. In [R9] a quasiregular mapping $f\colon R^3 \to R^3$ is constructed where the omitted points form a special configuration, in particular, the ratios of their mutual distances are bounded by a constant depending only on p. The second difficulty is that we must be able to move from one sector to a neighboring sector with different configurations of omitted sets. To accomplish this we use modifications of ideas from [R2].

In this section we give a general overview of the method in [R9]. Let us first look at the case $p = 2$, i.e., when the quasiregular mapping $f\colon R^3 \to R^3$ omits two points in R^3. This case is carried out in detail in [R9]. For general p the outline is given in [R9, Section 8].

Following the notation and terminology in [R9] we choose the omitted points to be $u_2 = -e_3/2$, $u_3 = e_3/2$, set $u_1 = \infty$, and let U_1, U_2, U_3 be the components of $R^3 \setminus (S^2 \cup B^2 \cup \{u_2, u_3\})$ such that $u_j \in \overline{U}_j$, $j = 2, 3$. Here R^2 is identified with $R^2 \times \{0\} \subset R^3$. Each component of $W_j = f^{-1} U_j$ must necessarily tend to ∞. In [R9] the sets W_1 and W_2 consist of one component and W_3 has six components. The sets W_j stick into each other in a complicated way near ∞. In a sense this phenomenon is inevitable.

The construction of f is achieved by first giving an approximation of $f^{-1}(S^2 \cup B^2) = \bigcup_j \partial W_j$ denoted by $|M_\infty|$ [R9, 4.1]. The components of the complement of $|M_\infty|$ are denoted by $V_1, V_2, V_3(h)$, $h = 0, 1, \ldots, 5$, and they are all topological 3-balls. The set V_j is thus an approximation of W_j, $j = 1, 2, 3$, if we set $V_3 = V_3(0) \cup \cdots \cup V_3(5)$. To get an idea how these sets V_j are constructed, consider the half spaces $H^+ = \{x \in R^3 : x_3 > 0\}$ and $H^- = \{x \in R^3 : x_3 < 0\}$ to be the first approximations of V_1 and V_2. Then stick in six infinite cones with branches between H^+ and H^-, which means that H^+ and H^- must be deformed somewhat. The idea is to get an approximation of V_3 which is spread out "between" the common boundary of H^+ and H^-. Let us denote these approximations by V_j^1, $j = 1, 2, 3$. The key goal is to achieve the following condition: For any common boundary point x of two of the sets V_j the third will intersect a neighborhood of x of a specific size. To approach this situation, we next extend V_2^1 by a system of branched cones "between" parts of the common boundary of V_1^1 and V_3^1 and also V_1^1 by a system of branched cones "between" parts of the common boundary of V_2^1 and V_3^1. Next we repeat this with parts of the new common boundaries and continue similarly until a certain size of the smallest cones is reached. This gives a very rough picture about what is going on. In the actual construction we use for the most part PL technique. The branched cones above will correspond to objects called *caves* and the operator to produce caves in the next finer scale is called a *cave refinement* [R9, 2.5].

To get an idea of the map f itself it is illuminating to look at preimages of each half of the sets U_j, for example $f^{-1}(U_1 \cap H^+)$ and $f^{-1}(U_1 \cap H^-)$. These can be described as certain tubes starting from $f^{-1}(S^2 \cup B^2)$ and tending to infinity. As $x \to \infty$ in a tube, $f(x)$ tends to u_1, u_2, or u_3. The restrictions of f to certain level surfaces, which for example in W_1 are preimages of spheres $S(\varrho_i)$ with a sequence (ϱ_i) tending to ∞, are determined by specific triangulations on the surfaces, called *map complexes* [R9, 2.7 and 4.3], in the sense that alternate simplices are mapped onto the upper hemisphere of $S(\varrho_i)$ and the rest onto the lower (Alexander's construction). These level surfaces for example in W_1 are obtained by a small move of ∂V_1 followed by similarity maps [R9, 4.2 and (4.5)]. The triangulations differ topologically when we shift from one level surface to another. To define the map between the level surfaces requires therefore a rather general deformation theory for 2-dimensional discrete open maps [R9, Sections 5 and 6]. For the gluing of different tubes at $f^{-1}(S^2 \cup B^2)$ a still more delicate cave refinement procedure is needed [R9, Section 7].

In the general case of p omitted points in R^3, the construction in [R9] is very similar to the one for $p = 2$. The caves are replaced by what we call *caves with* $p-1$ *passages* [R9, 8.1]. These differ from the earlier caves in that the insides of the earlier caves are now divided by walls into $p-1$ components. The omitted points $u_2, \dots, u_{p+1}$ lie in this order on the x_3-axis in the ball B^3 so that the distances $|u_{j+1} - u_j|$ differ from each other by at most a factor depending only on p.

6. Moving points by quasiconformal mapping

We modify the construction given in [R9] and described in the preceding section to prove the following sharpening of Theorem 1.2.

6.1. Theorem. *For any distinct points $a_1, \dots, a_{p+1}$ in $\overline{R^3}$ there exists a K-quasiregular mapping $F\colon R^3 \to \overline{R^3}$ omitting exactly the points $a_1, \dots, a_{p+1}$, with K depending only on p.*

This modification will be accomplished in Section 8. In this section we shall perform a preliminary quasiconformal mapping which moves the points $a_1, \dots, a_{p+1}$ to the x_3-axis in such a way that the mutual distances are roughly preserved. In doing this we shall make use of the following lemma the proof of which is evident.

6.2. Lemma. *Let $0 < C < 1$ and let $x_1, \dots, x_k, y_1, \dots, y_k$, $k \le p$, be points in $\overline{B^3}(1/2)$ such that*

$$\min_{i \ne j} |x_i - x_j|, \ \min_{j \ne j} |y_i - y_j| \ge C.$$

Then there exists a K-quasiconformal mapping H: $R^3 \to R^3$, K depending only on C and p, such that

(1) *H is the identity in $R^3 \setminus B^3$,*
(2) *H is the translation in $B^3(x_j, C/16)$ onto $B^3(y_j, C/16)$.*

We start by giving a certain ordering on the given points. Let ϱ_0 be the minimum of the spherical distances $\sigma(a_j, a_k)$. We assume that $\varrho_0 = \sigma(a_1, a_2)$. Let h: $\overline{R^3} \to \overline{R^3}$ be a Möbius transformation such that h takes the spherical ball $B_\sigma(a_1, \varrho_0)$ onto $\overline{R^3} \setminus \overline{B^3}(1/2)$, $h(a_1) = \infty$, and $h(a_2) = -e_3/2$. With the notation $b_j = h(a_j)$ we can relabel the points b_j, $j \ge 2$, such that (1) if $B_j = \{b_2, \dots, b_j\}$, then for $j \ge 3$ the minimum d_j of the distances $|b_k - b_l|$, $b_k, b_l \in B_j$, $k \ne l$, is attained as $d_j = |b_j - c_j|$ for some $c_j \in B_{j-1}$, and (2) $d_3 \ge d_4 \ge \dots \ge d_{p+1}$. Notice that $d_3 = |b_3 - b_2| \ge 1/8$ because $B_\sigma(a_2, \varrho_0)$ does not contain any points a_j, $j \ne 2$. Notice also that $B_{p+1} \subset \overline{B^3}(1/2)$.

Next we shall form a partition of $\{2, \dots, p+1\}$ into sets Δ_μ according to the sizes of d_j as follows. Let $\Delta_0 = \{2\}$ and write $j_0 = 2$. In [R9] there is chosen an integer $\nu = \nu_p$ depending on p ($\nu_2 = 24000$ [R9, p. 199]). The construction in [R9] works for larger integers as well. Here we choose $\nu > \max(10^5 p, \nu_p)$. Set $c = \exp \nu^3$. Let j_1 be the first $j \ge 3$ such that $d_j / d_{j+1} > c$. Then we define $\Delta_1 = \{3, \dots, j_1\}$. Next we let j_2 be the first $j \ge j_1 + 1$ such that $d_j / d_{j+1} > c$ and set $\Delta_2 = \{j_1 + 1, \dots, j_2\}$, etc.

For each $\mu \ge 1$ and $b_q \in B_{j_{\mu-1}}$ we define $E(\mu, q)$ to be the set of all b_j with $j \in \Delta_\mu$ for which there exist $k_1, \dots, k_m = j$ in Δ_μ such that $c_{k_{l+1}} = b_{k_l}$, $1 \le l \le m-1$, and $c_{k_1} = b_q$. It follows from the definitions that the sets $E(\mu, q)$ are disjoint and their union is B_{p+1}. Notice that $E(\mu, q)$ may be empty. In particular, $E(\mu, 2) = \emptyset$ for all $\mu \ge 2$. Notice also that $E(1, 2) = B_{j_1} \setminus \{b_2\}$. We also write $E^*(\mu, q) = E(\mu, q) \cup \{b_q\}$.

By applying Lemma 6.2 repeatedly we shall move the points $b_2, \dots, b_{p+1}$ to the x_3-axis X_3 into a position according to their mutual distances by a quasiconformal mapping. We start with $E^*(1, 2) = B_{j_1}$. Let $r = r_{1,2} > 0$ be minimal for which $E^*(1, 2) \subset \overline{B^3}(b_2, r)$. The definition of j_1 gives $d_{j_1} \ge d_3 / c^p \ge 1/(8c^p)$. We shall apply Lemma 6.2 to get a K_1-quasiconformal mapping g_1: $R^3 \to R^3$

such that (1) g_1 is the identity outside $B^3(b_2,2r)$, (2) g_1 takes the points b_j of $E^*(1,2)$ to points b_j^1 that are equidistantly distributed on $X_3\cap\overline{B^3}(b_2,r)$, (3) $d\bigl(g_1E^*(1,2)\bigr)=2r$, and (4) $b_2^1=b_2-re_3$. Together with the translation $x\mapsto x+b_2$ and homotheties with factor $2r$ we apply 6.2 with $C=\min(d_{j_1}/2r,1/p)$ to get the required mapping g_1. Then g_1 is the translation $x\mapsto b_j^1-b_j+x$ in each ball $B^3(b_j,Cr/8)$, $b_j\in E^*(1,2)$. Since $d_{j_1}/r\ge 1/(8c^p)$, we can make K_1 depend on p only.

Next let $b_q\in B_{j_1}$ be such that $E(2,q)\ne\emptyset$. We observe

$$E(2,q)\subset B^3(b_q,pd_{j_1+1})$$

and $pd_{j_1+1}\le pd_{j_1}/c\le Cr/16$; hence $E^*(2,q)$ is contained in the ball $B^3(b_q,Cr/16)$ and g_1 is a translation in the concentric ball with double radius. We can therefore repeat the above for $g_1E^*(2,q)$ instead of $E^*(1,2)$. Let $r'=r_{2,q}$ be minimal such that $E^*(2,q)\subset\overline{B^3}(b_q,r')$ or $g_1E^*(2,q)\subset\overline{B^3}(b_q^1,r')$. Now we apply 6.2 with $C=\min(d_{j_2}/2r',1/p)$. We have $d_{j_2}\ge d_{j_1+1}/c^p$ and $r'\le pd_{j_1+1}$, hence d_{j_2}/r' has the lower bound $(pc^p)^{-1}$, which depends only on p. Thus we obtain $K_2\ge K_1$ depending only on p and a K_2-quasiconformal mapping $g_2\colon R^3\to R^3$ which is the identity outside $B^3(b_q^1,2r')$ and takes the points $g_1E^*(2,q)$ to equidistantly distributed points in $X_3\cap\overline{B^3}(b_q^1,r')$ such that $d\bigl(g_2g_1E^*(2,q)\bigr)=2r'$. If $q'\in\Delta_1$ and $q'\ne q$, then $d\bigl(g_1E^*(2,q),g_1E^*(2,q')\bigr)\ge r/p-Cr/8>r/2p$ and the corresponding mapping for q' does not interact with g_2. We can therefore repeat this procedure for all $E(\mu,m)$, $b_m\in B_{j_{\mu-1}}$, in order of increasing μ to obtain a K_2-quasiconformal mapping $\varphi'\colon R^3\to R^3$ that takes the set B_{p+1} into $X_3\cap B^3(b_2,2r)$ and $1/16\le r\le 1$. We write $v_j'=\varphi'(b_j)$, $j=1,\dots,p+1$. Then $v_1'=\infty$ and v_2' has the smallest x_3-coordinate among the points $v_2',\dots,v_{p+1}'$. Finally we compose with the translation $T(x)=x+(-e_3/2-v_2')$ and write $\varphi=T\circ\varphi'$, $v_j=\varphi(b_j)$, $j=1,\dots,p+1$.

7. Level surfaces

In this section we shall give modifications of the construction in [R9] as a preliminary step in the goal of obtaining the mapping omitting the given points $a_1,\dots,a_{p+1}$ in $\overline{R^3}$. With the help of the quasiconformal mapping φ constructed in the preceding section the problem is reduced to the case where the omitted points are $v_1=\infty$, $v_2,\dots,v_{p+1}$, which lie on the x_3-axis X_3. Note that the indexing of v_j's does not correspond to the order on X_3 except that $v_2=-e_3/2$ has the smallest x_3-coordinate of $v_2,\dots,v_{p+1}$. We shall denote by g the map to be constructed and omitting $v_1,\dots,v_{p+1}$. The map of Theorem 6.1 will then be $F=h^{-1}\circ\varphi^{-1}\circ g$, with h as introduced at the beginning of Section 6.

7.1. Background. To start the construction we first recall more of the notation in [R9]. We consider the case of $p+1$ omitted points $u_1=\infty$, $u_2,\dots,u_{p+1}$

in $\overline{R^3}$. In [R9, 8.2] the points $u_2,\dots,u_{p+1}$ lie almost equidistantly on the segment $[-e_3,e_3]$ and in this order. More precisely, let $S^2_+=\{x\in S^2 : x_3\geq 0\}$, let ω_p be the Möbius transformation of $\overline{R^3}$ that keeps S^1 fixed, maps S^2_+ into $\overline{B^3}$, and for which S^2_+ and $\omega_p S^2_+$ form a dihedral angle π/p. Then $\omega_p^{j-1}S^2_+\cup\omega_p^j S^2_+$ bounds a subdomain U'_{p+2-j} of B^3, $j=1,\dots,p$. We let u_{p+1} be the midpoint of the part of the x_3-axis which lies in U'_{p+1}. Then set $u_j=\omega_p^{p+1-j}(u_{p+1})$, $U_j=U'_j\setminus\{u_j\}$, $j=2,\dots,p+1$, so that the x_3-coordinates of the points u_j increase with j. Let $f\colon R^3\to R^3$ be the mapping given in [R9] that omits the points $u_2,\dots,u_{p+1}$. Then each $f^{-1}\partial U_j$ becomes approximated by $|M_j|$, which is the space of a union M_j of certain 2-complexes constructed by the cave refinement operation [R9, pp. 213, 240]. The union $M_1\cup\dots\cup M_{p+1}$ is denoted by M_∞. The set $R^3\setminus|M_\infty|$ has $2+6(p-1)$ components $V_1,V_2,V_j(h)$, $j=3,\dots,p+1$, $h=0,\dots,5$, and $M_j=\partial V_j$, $j=1,\dots,p+1$, if we also write

$$V_j=\bigcup_{h=0}^{5}V_j(h)\quad\text{for}\quad j\geq 3.$$

On each $|M_j|$ we also have a certain map complex denoted by G_j (see [R9, 4.1 and 8.2]). In V_1 we have level surfaces $\nu^{2i}|N_1|$, $i=0,1,2,\dots$, where N_1 is obtained from M_1 by moving vertices slightly (see [R9, 4.2 and 8.2]). Similarly we are given level surfaces in the other domains $V_2,V_j(h)$, $j=3,\dots,p+1$, $h=0,\dots,5$.

7.2. μ-inheriting. The idea now is to modify parts of M_∞ according to the sizes of the distances $|v_j-v_k|$, $j,k\geq 3$. More precisely, in the new form there will be space between some of sets V_1, V_2, $V_j(h)$, $j=3,\dots,p+1$. This space is obtained by moving vertices, applying homotheties, and performing more cave refinements. Decisive in this procedure will be the relative sizes of the sets $E^*(\mu,q)$ defined in Section 6.

In accordance with the notation in the preceding section we let $r_{\mu,q}$ be the minimal radius such that $E(\mu,q)$ is contained in the ball $\overline{B^3}(b_q,r_{\mu,q})$. Suppose $E^*(\mu,q)\cap E^*(\lambda,l)\neq\emptyset$ for some $\lambda<\mu$ and $b_l\in B_{j_{\lambda-1}}$, and let here $\tilde\mu$ be the maximal λ such that $\lambda<\mu$. Note that $E^*(\tilde\mu,l)\cap E^*(\tilde\mu,m)=\emptyset$ if $m\neq l$, so given $E^*(\mu,q)$ we get a uniquely determined maximal $\tilde\mu<\mu$ and $b_{\tilde q}\in B_{j_{\tilde\mu-1}}$ such that $E^*(\mu,q)\cap E^*(\tilde\mu,\tilde q)\neq\emptyset$. We have $r_{\tilde\mu,\tilde q}/r_{\mu,q}\geq d_{j_{\tilde\mu}}/pd_{j_\mu}>c/p$. Therefore there exists a largest positive integer i such that $\exp\nu^{2i}\leq r_{\tilde\mu,\tilde q}/r_{\mu,q}$ and we denote this by $i_{\mu,q}$. Notice that q is never 1 or 2 in $i_{\mu,q}$. We order these as $i_1\geq i_2\geq\dots\geq i_{k+1}$ with each pair μ,q (such that $i_{\mu,q}$ is defined) corresponding to exactly one index in the sequence (i_l).

We say that b_m is *μ-inherited* from b_q if there are sequences $b_q=b_{q_0}$, $b_{q_1},\dots,b_{q_l}=b_m$ and $\mu_0,\dots,\mu_l$ with $\mu\leq\mu_0$ such that $b_{q_{s+1}}\in E(\mu_s,q_s)$, $s=0,\dots,l-1$. Let the set of m's such that b_m is μ-inherited from b_q be $I(\mu,q)$ and set $I^*(\mu,q)=I(\mu,q)\cup\{q\}$. The corresponding sets of b_m's are denoted by $D(\mu,q) = \{b_m : m \in I(\mu,q)\}$ and $D^*(\mu,q) = D(\mu,q) \cup \{b_q\}$. It follows from the definitions that $D^*(\mu,q) \subset D^*(\lambda,r)$ or $D^*(\mu,q) \supset D^*(\lambda,r)$ whenever $D^*(\mu,q) \cap D^*(\lambda,r) \neq \emptyset$. It also follows that if J is the minimal interval on X_3 that contains $\varphi D^*(\mu,q)$, then $v_j \in \varphi D^*(\mu,q)$ for every v_j in J.

7.3. Level surfaces, map complexes, first layer. Let $v_2, v_{k_3}, \dots, v_{k_{p+1}}$ be the order of the points on φB_{p+1} listed in the positive direction of X_3 and let $\sigma\colon \{1,\dots,p+1\} \to \{1,\dots,p+1\}$ be the permutation with $\sigma(1)=1$, $\sigma(2)=2$, $\sigma(m)=k_m$, $m \geq 3$. Each set $J^*(\mu,q) = \sigma^{-1} I^*(\mu,q)$ is then of the form $\{m_1, m_1+1, \dots, m_1+l\}$.

We start with μ,q with largest $i_{\mu,q}$, also denoted by i_1. Suppose first that there is only one such pair μ,q. Set

$$\begin{aligned} M(\mu,q) &= \bigcup\{M_j : j \in J^*(\mu,q)\},\\ M_c(\mu,q) &= \bigcup\{M_j : j \notin J^*(\mu,q)\},\\ M_-(\mu,q) &= M(\mu,q) \cap M_c(\mu,q), \end{aligned}$$

so that the space $|M_-(\mu,q)|$ is the boundary of

$$V(\mu,q) = \operatorname{int}\Big(\bigcup\{\overline{V}_j : j \in J^*(\mu,q)\}\Big).$$

We now move $M(\mu,q)$ into $V(\mu,q)$ as in [R9, 4.2], where the level surface $|N_1|$ is obtained by moving vertices of M_1. More precisely, we move the vertices of $M_-(\mu,q)$ slightly so that $M_-(\mu,q)$ is replaced by a union $M_+(\mu,q)$ of 2-complexes whose space $|M_+(\mu,q)|$ lies in $V(\mu,q)$ and is approximately at distance $\nu^{-1/2}\, d(A)$ from $|M_-(\mu,q)|$ near each 2-simplex A in $M_-(\mu,q)$. The vertices of $M(\mu,q) \setminus M_-(\mu,q)$ are kept fixed. For $j \notin J^*(\mu,q)$ V_j, M_j, and G_j are also kept fixed in this procedure. For $j \in J^*(\mu,q)$ V_j, M_j, and G_j now appear as V_j', M_j', and G_j'. Also $M(\mu,q)$ and $V(\mu,q)$ will change to $M'(\mu,q)$ and $V'(\mu,q)$. We also write

$$V_c(\mu,q) = \operatorname{int}\big(\bigcup\{\overline{V}_j : j \notin J^*(\mu,q)\}\big) = R^3 \setminus \overline{V}(\mu,q).$$

Between $V_c(\mu,q)$ and $V'(\mu,q)$ we have thus opened a *layer* $Y(\mu,q) = R^3 \setminus \big(\overline{V}_c(\mu,q) \cup \overline{V}'(\mu,q)\big)$. Its boundary consists of $X_-(\mu,q) = \overline{Y}(\mu,q) \cap \overline{V}_c(\mu,q)$ and $X_+(\mu,q) = \overline{Y}(\mu,q) \cap \overline{V}'(\mu,q)$. They are the spaces of unions of 2-complexes induced by $M_c(\mu,q)$ and $M'(\mu,q)$, which we denote by $M_-(\mu,q)$ and $M_+(\mu,q)$ respectively. With the principles in [R9, 3.2–3.4] we also define map complexes $G_-(\mu,q)$ and $G_+(\mu,q)$ on $|M_-(\mu,q)| = X_-(\mu,q)$ and $|M_+(\mu,q)| = X_+(\mu,q)$ respectively.

Next we apply the homothety $x \mapsto \nu^{2\omega_1} x$ where $\omega_1 = i_1 - i_2$. We shall perform a certain amount of cave refinement constructions on both $\nu^{2\omega_1}|M_c(\mu,q)|$ and $\nu^{2\omega_1}|M'(\mu,q)|$ as follows. On $\nu^{2\omega_1}|M_c(\mu,q)|$ we perform successively cave refinements guided by the map complexes $\nu^{2\omega_1} G_j$, $j \notin J^*(\mu,q)$, and $\nu^{2\omega_1} G_-(\mu,q)$.

The number of cave refinement steps is determined by a principle similar to the one in [R9, 3.4]. In connection with these cave refinements we also get a number of level surfaces as follows. Let V be a component of V_j for some $j \notin J^*(\mu,q)$. When two steps of cave refinement are performed on $\nu^{2\omega_1}|M_c(\mu,q)|$, $\nu^{2\omega_1}M_j$ is replaced by a new union P_j of 2-complexes and $\nu^{2\omega_1}V_j$ is replaced by a set A_j. A level surface in A_j is obtained by slight movement of those vertices of P_j that lie on ∂A_j into A_j as in [R9, 4.2]; cf. the construction of $M'(\mu,q)$ above. The next finer level surface is obtained similarly after two more cave refinements etc. In the same way we get level surfaces replacing V_j by $Y(\mu,q)$ above. The cave refinements on $\nu^{2\omega_1}|M'(\mu,q)|$ together with level surfaces are obtained similarly. Recall that these constructions always bring along also map complexes on the level surfaces.

In addition to the level surfaces constructed in connection with the cave refinements described above we take the level surfaces $\nu^{2i}|N_1|$, $\nu^{2i}|N_2|$, $\nu^{2i}|N_j(h)|$, $j=3,\dots,p+1$, $h=0,\dots,5$, $i=\omega_1+1,\omega_1+2,\dots$, together with corresponding map complexes (see [R9, 4.3]).

7.4. Next layers. After completing the constructions above, objects such as M_j, V_j etc. have been transformed into ones which we shall denote by adding (1) as superscript. After this first step we thus have for example $M_j^{(1)}$, $V_j^{(1)}$, $G_j^{(1)}$ for all j, $Y^{(1)}(\mu,q)$, $M_-^{(1)}(\mu,q)$, $M_+^{(1)}(\mu,q)$, $G_-^{(1)}(\mu,q)$, and $G_+^{(1)}(\mu,q)$.

Now we proceed as follows. Let $i_2=i_{\lambda,r}$ and assume $i_2>i_3$. Suppose first that $J^*(\mu,q)\cap J^*(\lambda,r)=\emptyset$. This time we define

$$M_c^{(1)}(\lambda,r)=M_-^{(1)}(\mu,q)\cup M_+^{(1)}(\mu,q)\cup\bigcup\{M_j^{(1)}:j\notin J^*(\lambda,r)\},\tag{7.5}$$

i.e., $M_c^{(1)}(\lambda,r)$ contains all that is not indexed by some $j\in J^*(\lambda,r)$. Then set

$$M_-^{(1)}(\lambda,r)=M_c^{(1)}(\lambda,r)\cap M^{(1)}(\lambda,r).\tag{7.6}$$

Note that since $J^*(\mu,q)\cap J^*(\lambda,r)=\emptyset$, $M^{(1)}(\lambda,r)$ does not meet $M_+^{(1)}(\mu,q)$, but it may meet $M_-^{(1)}(\mu,q)$. If not otherwise stated, notations such as $M^{(1)}(\lambda,r)$ are defined formally as for the pair μ,q by means of the $M_j^{(1)}$, that is,

$$M^{(1)}(\lambda,r)=\bigcup\{M_j^{(1)}:j\in J^*(\lambda,r)\},$$

and similarly for other objects. The space $|M_-^{(1)}(\lambda,r)|$ is also now the boundary of $V^{(1)}(\lambda,r)$ and we move $M^{(1)}(\lambda,r)$ into $V^{(1)}(\lambda,r)$ as before. For $j\in J^*(\lambda,r)$, $V_j^{(1)},M_j^{(1)}$, and $G_j^{(1)}$ will change to $V_j^{(1)\prime}$, $M_j^{(1)\prime}$ and $G_j^{(1)\prime}$. The objects $M^{(1)}(\lambda,r)$ and $V^{(1)}(\lambda,r)$ will change to $M^{(1)\prime}(\lambda,r)$ and $V^{(1)\prime}(\lambda,r)$. Set $V_c^{(1)}(\lambda,r)=R^3\setminus\overline{V^{(1)}}(\lambda,r)$ and $Y^{(1)}(\lambda,r)=R^3\setminus\big(\overline{V_c^{(1)}}(\lambda,r)\cup\overline{V^{(1)}}{}'(\lambda,r)\big)$.

As before, the boundary of $Y^{(1)}(\lambda, r)$ consists of two spaces of unions of 2-complexes, namely, $M_-^{(1)}(\lambda, r)$ and $M_+^{(1)}(\lambda, r)$, which are induced by $M_c^{(1)}(\lambda, r)$ and $M^{(1)\,\prime}(\lambda, r)$. On $|M_-^{(1)}(\lambda, r)|$ and $|M_+^{(1)}(\lambda, r)|$ we have map complexes $G_-(\lambda, r)$ and $G_+(\lambda, r)$.

Now we apply the homothety $x \mapsto \nu^{2\omega_2}x$ where $\omega_2 = i_3 - i_2$. The original union $M_\infty = M_1 \cup \cdots \cup M_{p+1}$ has been transformed to three separate parts, namely, $\nu^{2\omega_2}M^{(1)}(\mu, q), \nu^{2\omega_2}M^{(1)\,\prime}(\lambda, r)$, and the rest, which is $\nu^{2\omega_2}\bigl(M_c^{(1)}(\lambda, r) \setminus M^{(1)}(\mu, q)\bigr)$. We perform cave refinements on each of these parts and construct level surfaces by the same principles as before. If T is a level surface from the first step, it transforms to a level surface $\nu^{2\omega_2}T$. In particular, the level surfaces $\nu^{2i}|N_1|$ etc., $i = \omega_1 + 1, \omega_1 + 2, \ldots$, are transformed to level surfaces $\nu^{2i}|N_1|$ etc., $i = \omega_1 + \omega_2 + 1, \omega_1 + \omega_2 + 2, \ldots$. The resulting objects are then provided with a superscript (2).

Suppose next that $J^*(\mu, q) \cap J^*(\lambda, r) \neq \emptyset$ and consider first the case $J^*(\mu, q) \subset J^*(\lambda, r)$. We write

$$\begin{aligned}
&\widetilde{M}^{(1)}(\lambda, r) = M_-^{(1)}(\mu, q) \cup M_+^{(1)}(\mu, q) \cup \bigcup\{M_j^{(1)} : j \in J^*(\lambda, r)\},\\
&M_c^{(1)}(\lambda, r) = \bigcup\{M_j^{(1)} : j \notin J^*(\lambda, r)\},\\
&M_-^{(1)}(\lambda, r) = M_c^{(1)}(\lambda, r) \cap \widetilde{M}^{(1)}(\lambda, r).
\end{aligned}$$

The space $|M_-^{(1)}(\lambda, r)|$ is thus the boundary of

$$\widetilde{V}^{(1)}(\lambda, r) = \operatorname{int}\left(\overline{Y^{(1)}}(\mu, q) \cup \bigcup\{\overline{V_j^{(1)}} : j \in J^*(\lambda, r)\}\right).$$

We move $\widetilde{M}^{(1)}(\lambda, r)$ as above into $\widetilde{V}^{(1)}(\lambda, r)$ which then changes to $\widetilde{V}^{(1)\prime}(\lambda, r)$. Set $V_c^{(1)}(\lambda, r) = R^3 \setminus \overline{\widetilde{V}^{(1)}}(\lambda, r)$ and $Y^{(1)}(\lambda, r) = R^3 \setminus \bigl(\overline{V_c^{(1)}}(\lambda, r) \cup \overline{\widetilde{V}^{(1)}}(\lambda, r)\bigr)$. The procedure is now similar to the one in the case $J^*(\mu, q) \cap J^*(\lambda, r) = \emptyset$.

Suppose then $J^*(\lambda, r) \subset J^*(\mu, q)$. Here we proceed formally as in the case $J^*(\lambda, r) \cap J^*(\mu, q) = \emptyset$, namely, we define $M_-^{(1)}(\lambda, r)$ by (7.6) where $M_c^{(1)}(\lambda, r)$ is given by (7.5). The continuation is clear.

If $i_1 = i_2 > i_3$, the moves such as $M(\mu, q) \to M'(\mu, q)$ are done for both pairs μ, q and λ, r before the homothety, which in this case is $x \mapsto \nu^{2(i_3 - i_1)}x$.

After we have finished the second step, namely, the constructions for the pair λ, r, we provide the new objects with superscript (2), for example $M_j^{(2)}$ etc.

7.7. Graphs and the general step. We are now ready to give a description of the general procedure. It will be sufficient to give a definition of an object like $M_-^{(1)}(\lambda, r)$ in (7.6) for the general step. That will then define the move, and the rest of the step is accomplished by principles as described above. For this it is convenient to let each step correspond to a connected graph which is a tree with the following properties. Its sides are open, the endsides are attached to one vertex and the other sides to two vertices. Each vertex is attached to at least three sides. The sides, vertices, and pairs (A, w), where A is a side that is attached to the vertex w, will correspond to the objects in the construction in a specific way. To describe this we will list the correspondence after the first step where the various objects are denoted by the addition of the superscript (1). An endside A corresponds to some $V_j^{(1)}$ and an inner side to $Y^{(1)}(\mu, q)$. Let the correspondence be denoted by ψ; hence in the first case $\psi(A) = V_j^{(1)}$. For a pair (A, w), where A is an endside with $\psi(A) = V_j^{(1)}$, we set $\psi(A, w) = |M_j^{(1)}|$, which is the boundary of $V_j^{(1)}$. If in (A, w) A is an inner side with $\psi(A) = Y^{(1)}(\mu, q)$, we take $\psi(A, w)$ to be either $|M_-^{(1)}(\mu, q)|$ or $|M_+^{(1)}(\mu, q)|$ depending in an obvious way on which of the two possible vertices w is. Finally, to a vertex w corresponds $\psi(w)$, which will be the union of all $\psi(A, w)$ where A runs over the sides A that are attached to w.

Figure 1.

Let us use a specific example to make the idea easier to understand. Let

$p = 7$, $J^*(\mu, q) = \{3, 4, 5\}$, $J^*(\lambda, r) = \{6, 7\}$, $i_1 > i_2 > i_3$. In Figure 1 we see four consecutive stages. The graph in (a) represents the original V_j's and M_j's. A number j at a side corresponds to V_j. In (b) the first step is performed and in addition to the endsides the correspondences are

$$\psi(A_1) = Y^{(1)}(\mu, q), \quad \psi(A_1, w_1) = |M_-^{(1)}(\mu, q)|, \quad \psi(A_1, w_1') = |M_+^{(1)}(\mu, q)|.$$

In (c) the second step is performed and we have the correspondences

$$\begin{aligned} \psi(A_2) &= Y^{(2)}(\mu, q), & \psi(A_2') &= Y^2(\lambda, r), \\ \psi(A_2, w_2) &= |M_-^{(2)}(\mu, q)|, & \psi(A_2, w_2') &= |M_+^2(\mu, q)|, \\ \psi(A_2', w_2) &= |M_-^2(\lambda, r)|, & \psi(A_2', w_2'') &= |M_+^2(\lambda, r)|. \end{aligned}$$

We make the following observations. To obtain the graph in (b) from that in (a) we take the minimal connected subgraph Γ containing the sides corresponding to $J^*(\mu, q) = \{3, 4, 5\}$. This subgraph has the vertex w_0 in common with the closure $\widetilde{\Gamma}$ of the complement of Γ. To obtain (b) from (a) we move Γ away from $\widetilde{\Gamma}$ and connect them with a side A_1 at vertices w_1 and w_1' corresponding to w_0. We note that (c) is obtained from (b) with the same principle. From (b) we also can read the space $|M_-^{(1)}(\lambda, r)|$. If now Γ_1 is the graph corresponding to $J^*(\lambda, r) = \{6, 7\}$ and $\widetilde{\Gamma}_1$ is the closure of the complement, they have the uniquely determined common vertex w_1. We form the union of all $\psi(A, w) = \psi(A, w_1)$ where $A \in \Gamma_1$, which gives $|M^{(1)}(\lambda, r)|$, and the union of all $\psi(A, w)$ where $A \in \widetilde{\Gamma}_1$, which gives $|M_c^{(1)}(\lambda, r)|$ as in (7.5). The intersection of these two gives $|M_-(\lambda, r)|$ as in (7.6).

Let us assume that $i_3 = i_{\kappa,s} > i_4$ and $J^*(\kappa, s) = \{3, 4, 5, 6, 7\}$. Then the common vertex in (c) for the two subgraphs is w_2 and the graph is changed to the one in (d).

In the general case too the splitting of the graph corresponding to a given stage into two subgraphs is uniquely determined by the next J^*-set, say $J^*(\gamma, s)$, according to the rules described above. This splitting gives then $M_-(\gamma, s)$, which in turn determines the move and so on. If the i_l's coincide for several consecutive indices l, the corresponding moves, homotheties, and cave refinements are performed simultaneously for those indices (cf. the case $i_1 = i_2 > i_3$ above). The last homothety is $x \mapsto \nu^{2i_k}$. Observe that in the end the layer $Y(\mu, q)$ has been modified to a layer $Y^{(k)}(\mu, k)$ whose "width" is roughly ν^{2i_1} times that of $Y(\mu, q)$, the "width" of the next layer $Y^{(1)}(\lambda, r)$ is multiplied roughly by ν^{2i_2} etc. For notational convenience we have used the pairs μ, q and λ, r to have special values attached to i_1 and i_2. In what follows we shall regard these pairs as free variables.

8. Mapping with arbitrary omitted points

We are now ready to describe the final steps in the proof of Theorem 6.1. By Section 6 we may assume that the points are $v_1 = \infty$ and $v_2, \dots, v_{p+1}$ which lie on the x_3-axis X_3. In addition, $v_2 = -e_3/2$. We shall use notation from Sections 6 and 7. The set $\{v_2, \dots, v_{p+1}\}$ is obtained as the image of B_{p+1} under the quasiconformal mapping φ. The order on X_3 is $v_2, v_{k_3}, \dots, v_{k_{p+1}}$ and σ is a permutation such that $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(m) = k_m$, $m \geq 3$.

Let $E(\mu, q) \neq \emptyset$. Let $d_1, \dots, d_s$ be the points of $\varphi E^*(\mu, q) \subset \varphi B_{p+1}$ in the positive order on X_3. Recall that $|d_l - d_{l-1}|$ is constant for all l. In the beginning of Section 7 we described the domains U_j' bounded by spheres and containing u_j. Now we do a similar thing for the points d_l in place of the u_j's. As a result, we obtain domains $D_l' \ni d_l$ bounded by spheres. We suppose that $E^*(1,2) \neq B_{p+1}$, for otherwise the construction reduces to the original one in [R9].

Following the notation in Section 7.2 we let $\tilde{\mu}, \tilde{q}$ be the uniquely determined pair such that $\tilde{\mu}$ is maximal for the property $\tilde{\mu} < \mu$ and $E^*(\mu, q) \cap E^*(\tilde{\mu}, \tilde{q}) \neq \emptyset$ provided $i_{\mu,q}$ is defined, which means that $E^*(\mu, q)$ meets some $E^*(\lambda, r)$ with $\lambda < \mu$. Let the points of $\varphi^* E^*(\tilde{\mu}, \tilde{q})$ be $f_1, \dots, f_t$, again in increasing order on X_3. Similarly we get domains $F_m' \ni f_m$ bounded by spheres. Since $E^*(\mu, q) \cap E^*(\tilde{\mu}, \tilde{q}) \neq \emptyset$, $d_\sigma = f_\tau$ for uniquely determined σ and τ, and the union of the sets D_l' is contained in F_τ'. In Figure 2 we see the location of these points and domains for $s = 4$, $t = 3$, $\sigma = 4$, $\tau = 2$. Note because of our choice of c, the picture is far from being in the right scale.

To get an idea of the quasiregular map g to be constructed we begin by describing how different parts of R^3 from the construction in Section 7 will roughly be mapped. Each layer $Y^{(k)}(\mu, q)$ will be approximately mapped onto the ring domain $F_\tau' \setminus \bigcup_l \overline{D_l}\,'$, with the notation above, such that the boundary part $|M_+^{(k)}(\mu, q)|$ corresponds to $\partial \bigcup_l \overline{D_l}\,'$ and the boundary part $|M_-^{(k)}(\mu, q)|$ to $\partial F_\tau'$. Each domain $V_j^{(k)}$ will be approximately mapped onto a domain like $D_l = D_l' \setminus \{d_l\}$, where $d_l = v_{k_j}$ provided D_l' does not contain similar domains of smaller category. This is equivalent to saying that $I^*(\mu + 1, k_j)$ contains only k_j.

The level sets in a layer $Y^{(k)}(\mu, q)$ are mapped onto certain spheres Σ concentric to $\partial \bigcup_l \overline{D_l}\,'$ such that $\bigcup_l \overline{D_l}\,'$ remains inside each Σ. The level sets in a set $V_j^{(k)}$ are mapped onto certain spheres Σ in D_l' with center d_l, where D_l' is in the same meaning as above. The mapping on the level sets is constructed with the same principles as in [R9, Section 4].

Observe that we can recognize the final graph described in Section 7 from the configuration of the sets like D_l and F_m etc. as follows. In Figure 3 part of the graph is drawn by dotted line segments and it represents a case where only one of the sets $I^*(\mu + 1, k_j)$ contains more than one point for $v_{k_j} = d_l$, $l = 1, \dots, 4$, namely, the one corresponding to d_4.

It remains to define the map g between the level sets and around the bound-

Figure 2.

aries of layers $Y^{(k)}(\mu,q)$ and sets $V_j^{(k)}$. Between the level surfaces the principles in [R9, Section 6] apply almost without changes. The construction around the boundaries of the sets $Y^{(k)}(\mu,q)$ and $V_j^{(k)}$ can also be copied without much change from [R9, Section 7] with the remarks from [R9, Section 8]. In particular, the map complex $G_-^{(k)}(\mu,q)$ on the boundary part $|M_-^{(k)}(\mu,q)|$ of $Y^{(k)}(\mu,q)$ has the effect on the construction like the map complex $G_j^{(k)}$ on $\partial V_j^{(k)}$. A similar remark applies to the map complex $G_+^{(k)}(\mu,q)$.

We pay special attention to the definition of g in $g^{-1}U$, where U is a round neighborhood of $v_1=\infty$ or $v_2=-e_3/2$. As follows from the construction in Section 7, after the final step we have level surfaces $\nu^{2i}|N_1|$ etc. for integers $i\geq w_1+\cdots+w_k+1=i_1-i_{k+1}+1$, in particular, for $i\geq i_1$. Since a_1 and a_2 have a special role among the points a_j (in particular, the indices 1 or 2 never occur for q in $i_{\mu,q}$ (see 7.2)), we may construct g so that it has the following additional property. For $j=1,2$ let X_j be the closed set in $V_j^{(k)}$ which is bounded by $\nu^{2i_1}|N_j|$ and which does not touch $|M_j^{(k)}|$. In X_1 we define g exactly as in [R9, Section 8] (in [R9] the map is called f) and in X_2 we take the map from [R9] followed by the obvious translation $x\mapsto v_2-u_2+x$. With this normalization

Figure 3.

we have that if g' is the corresponding map for another sequence $a'_1, \dots, a'_{p+1}$ of points such that the mutual spherical distances attain the minimum for the pair a'_1, a'_2 and if X'_j corresponds to X_j, $j = 1, 2$, then g and g' coincide in $X_j \cap X'_j$, $j = 1, 2$.

This completes the proof of Theorem 6.1. We shall in the following sections not only use the result of Theorem 6.1, but also the actual construction because we need to know the behavior of the mapping to a certain extent.

9. Mappings of cylinders

In this and the next section we shall apply ideas from [R2] to complete the proof of Theorem 1.7. The main idea is to define mappings in infinite cylinders with square base by means of mappings given by Theorem 6.1 and glue such mappings together along the faces of the cylinders. The cylinders will fill R^3 and the mappings in various cylinders are chosen according to the amount of defect the points a_j are given. Let $\kappa\colon R^3 \to R^3$ be the radial stretch map which maps each ball $B^3(t)$ onto the cylinder $Q(t) = B^2(t) \times \,]-t, t[$. We shall construct a map f which satisfies the statements in Theorem 1.7 with respect to the exhaustion of R^3 by the cylinders $Q(t)$. The required map for Theorem 1.7 is then $f \circ \kappa$. The main difference from the constructions in [R2] is that the mapping given by Theorem 6.1 is of complicated nature whereas a mapping of a cylinder in [R2] is essentially a composition of two Zorich maps (see [Z] or [R7, p. 222]). Here special attention must be paid on the gluing process.

9.1. Quasiconformal maps of half cylinders. We start by defining some auxiliary quasiconformal maps. We let H_i^+ (H_i^-) be the half space $\{x \in R^3 : x_i > 0\}$ ($\{x \in R^3 : x_i < 0\}$) and we also write $H^+ = H_3^+$, $H^- = H_3^-$. We cut out the closed half plane $P = \{x \in R^3 : x_2 = -x_1/\sqrt{3},\ x_1 \ge 0\}$ from R^3 and let φ_1 be the folding $\varphi_1\colon H_2^+ \to R^3 \setminus P$ given by $(r, \varphi, x_3) \mapsto (r, 2\varphi - \pi/6, x_3)$ in cylindrical coordinates. Let $\varphi_2\colon H_2^+ \to H_3^+$ be the rotation about the x_1-axis such that $\varphi_2(e_2) = e_3$. We consider sequences $w = (w_1, \dots, w_{p+1})$ of $p+1$ distinct points in $\overline{R^3}$ such that $\sigma(w_1, w_2)$ is minimal among $\sigma(w_j, w_k)$, $j \ne k$, and denote the map, given by Theorem 6.1 and constructed in Sections 6–8, by F_w. Write $\eta_w = F_w \circ \varphi_1 \circ \varphi_2^{-1}$. The maps h and φ from Section 6 are denoted by h_w and φ_w and the map g from Section 8 by g_w.

If $(k, m) \in \mathbf{Z} \times \mathbf{Z}$, we set

$$B_{km} = \{x \in R^3 : k < x_1 < k+1,\ m < x_2 < m+1\}$$

and $B_{km}^+ = B_{km} \cap H^+$, $B_{km}^- = B_{km} \cap H^-$. Let C^+ be the cylinder $\{x \in H^+ : x_1^2 + x_2^2 < 1\}$ and write $C_t^+ = \{x \in C^+ : x_3 < t\}$. We translate B_{00}^+ by $\varphi_3\colon x \mapsto x - (e_1/2, e_2/2)$ onto $\widetilde{B}_{00}^+$ and rotate $\widetilde{B}_{00}^+$ by $\varphi_4(r, \varphi, x_3) = (r, \varphi - \pi/4, x_3)$ onto $\widehat{B}_{00}^+ = \{x \in H^+ : |x_1| + |x_2| < 1/\sqrt{2}\}$. We map C^+ onto $\widehat{B}_{00}^+$ by φ_5, which is the radial stretching $\varphi_5(r, \varphi, x_3) = (r', \varphi, x_3)$ defined by $(r'/r, \varphi, x_3) \in \partial\widetilde{B}_{00}^+$ for $r > 0$. Let $\varphi_6\colon C^+ \to H^+$ be a quasiconformal map such that

(1) for $x_3 \ge 1$, $\varphi_6(r, \varphi, x_3) = (\varrho, \varphi, \theta)$, $\varrho = e^{x_3}$, $\theta = \pi r/2$, where we have used cylindrical and spherical coordinates such that θ is the angle between the x_3-axis and the radius vector;

(2) φ_6 induces the identity on the disk $B^2 \subset R^2 = \partial H^+$;

(3) φ_6 induces $(1, \varphi, x_3) \mapsto (e^{x_3}, \varphi, \pi/2)$ on the boundary part $\{x \in \partial C^+ : 0 < x_3\}$.

Set $\psi = \varphi_6 \circ \varphi_5^{-1} \circ \varphi_4 \circ \varphi_3$. Then $\eta_w \circ \psi$ is a quasiregular map of B_{00}^+ onto $\overline{R^3} \setminus \{w_1, \dots, w_{p+1}\}$. Let u_{km}^+ (u_{km}^-) be the map of B_{km}^+ (B_{km}^-) onto B_{00}^+ obtained by repeated reflection with respect to the faces of the cylinders and let $u\colon H_3^+ \to H_3^+$ be the reflection with respect to the x_2x_3-plane. Set

$$\begin{aligned} \varphi_{km}^{\pm} &= \psi \circ u_{km}^{\pm} && \text{if } u_{km}^{\pm} \text{ is orientation preserving,} \\ \varphi_{km}^{\pm} &= u \circ \psi \circ u_{km}^{\pm} && \text{if } u_{km}^{\pm} \text{ is orientation reversing.} \end{aligned}$$

We write φ_{km} for the map of B_{km} defined by $\varphi_{km} | B_{km}^{\pm} = \varphi_{km}^{\pm}$. Then $\eta_w \circ \varphi_{km}$ is a quasiregular map of B_{km} onto $\overline{R^3} \setminus \{w_1, \dots, w_{p+1}\}$. Let (s_1, t_1) and (s_2, t_2) be two vertices of the base of some B_{km}^+ such that $(s_1, t_1) - (1, 0) \in 2\mathbf{Z} \times 2\mathbf{Z}$ and $(s_2, t_2) - (0, 1) \in 2\mathbf{Z} \times 2\mathbf{Z}$. Then, according to the construction, $\eta_w \circ \varphi_{km}(s_j, t_j, x_3) \to w_j$ as $|x_3| \to \infty$, $j = 1, 2$. We set $\Lambda_1 = (1, 0) + 2\mathbf{Z} \times 2\mathbf{Z}$, $\Lambda_2 = (0, 1) + 2\mathbf{Z} \times 2\mathbf{Z}$.

9.2. Deformation maps. In order to glue maps defined in cylinders B_{km} with different w's we introduce some more quasiconformal maps. Let

$$\xi\colon [1,\infty[\times[0,\infty[\to[0,\infty[$$

be the function defined by

$$\begin{aligned} \xi(d,s) &= d^2\Big(\frac{s}{d^2}\Big)^{1/2} && \text{if}\quad 0\le s\le d^2,\\ \xi(d,s) &= s && \text{if}\quad d^2\le s. \end{aligned}$$

Let Δ_α be the interval $[(\alpha-1)\pi/2,\alpha\pi/2]$, $\alpha=1,2,3,4$. For $\mu,\nu\in\{1,2,3,4\}$ we define functions $\sigma_{\mu\nu}$ of $[0,2\pi]$ into $[1,2]$, with the following conditions. Each $\sigma_{\mu\nu}$ is continuous, it is affine on each interval Δ_α, it has the same value at the endpoints 0 and 2π, and it is the smallest such function with $\sigma_{\mu\nu}|\Delta_\mu\cup\Delta_\nu=2$. Notice that here $|\nu-\mu|=2$ implies $\sigma_{\mu\nu}=2$. In fact, we only employ the cases $\mu=\nu$ and $\nu-\mu=1 \pmod 4$, and then define quasiconformal selfmaps $\lambda^q_{\mu\nu}$, $q=0,1,\dots$, of H^+ in spherical coordinates by

$$\lambda^q_{\mu\nu}(\varrho,\varphi,\theta)=\big(\xi\big(2^q\sigma_{\mu\nu}(\varphi),\varrho\big),\varphi,\theta\big). \tag{9.3}$$

We also write

$$\lambda^q(\varrho,\varphi,\theta)=\big(\xi(2^q,\varrho),\varphi,\theta\big). \tag{9.4}$$

In addition to these radial stretching maps we need maps which change the angle φ. Let $\zeta\colon[-\pi/2,\pi/2]\to[-\pi/2,\pi/2]$ be the function whose graph consists of the line segments $[(-\pi/2,-\pi/2),(-\pi/8,\pi/8)]$ and $[(-\pi/8,\pi/8),(\pi/2,\pi/2)]$. We define quasiconformal selfmaps ω and ω^* of H^+ in cylindrical coordinates by

$$\begin{aligned} \omega(r,\varphi,x_3) &= \big(r,\zeta(\varphi),x_3\big) && \text{if}\quad -\pi/2\le\varphi\le\pi/2,\\ \omega(r,\varphi,x_3) &= (r,\varphi,x_3) && \text{if}\quad \pi/2\le\varphi\le 3\pi/2, \end{aligned} \tag{9.5}$$

$$\begin{aligned} \omega^*(r,\varphi,x_3) &= \big(r,\zeta(\varphi-\pi),x_3\big) && \text{if}\quad \pi/2\le\varphi\le 3\pi/2,\\ \omega^*(r,\varphi,x_3) &= (r,\varphi,x_3) && \text{if}\quad -\pi/2\le\varphi\le\pi/2. \end{aligned} \tag{9.6}$$

Notice that ω^* is ω followed by the rotation $(r,\varphi,x_3)\mapsto(r,\varphi+\pi,x_3)$.

9.7. Standard maps. Next we define for a given $w=(w_1,\dots,w_{p+1})$ a positive integer $l=l_w$ as follows. We know from the end of Section 8 that the constructed map g, now call it g_w, is of a normalized form in the domains X_j "outside" the level surfaces $\nu^{2i_1}|N_j|$, $j=1,2$. In Figure 4 we have illustrated schematically how $\varphi_2 \circ \varphi_1^{-1} \nu^{2i_1}|N_j|$, $j = 1, 2$, is seen on the boundary of H^+ (φ_1 and φ_2 are defined in 9.1). In the same picture we have drawn lines L and L', which together with the x_1-axis consist of the union of images (on the boundary) of the x_1-axis under the maps $\omega, \omega^{-1}, \omega^*, \omega^{*-1}$. Far enough from the origin the sets $\varphi_2 \circ \varphi_1^{-1} \nu^{2i_1}|N_j|$, $j = 1, 2$, stay (on the boundary) inside the angular domains in ∂H^+ meeting the x_1-axis and bounded by L and L'. We take l to be the least integer such that this happens outside the disk $B^2(2^l) \subset \partial H^+$. Now we are in a position to define what we will call a *standard map of* B_{km} for w. It is denoted by τ_{km}^w and defined by

$$\tau_{km}^w = \eta_w \circ \lambda^l \circ \varphi_{km}. \tag{9.8}$$

Recall the maps $\eta_w = F_w \circ \varphi_1 \circ \varphi_2^{-1}$ and φ_{km}, from 9.1, and λ^l from (9.4). Here and in the following extensions of maps to the boundary of the domain of definition are used without further notice if the extensions make sense. Thus, for example, the definition (9.8) extends naturally to the base of B_{km}^+. The map f will for pairs (k, m) in certain subsets of $\mathbf{Z} \times \mathbf{Z}$ be defined by the rule (9.8). These subsets will be determined according to the defect numbers in the next section.

9.9. Gluing maps of cylinders. First we shall describe how to glue stepwise maps of the form $\eta_w \circ \lambda^q \circ \varphi_{km}$ with different q's. To this end, assume that for fixed w and q we have defined $f|B_{km}$ as $\eta_w \circ \lambda^q \circ \varphi_{km}$ for (k, m) in $A_0 \subset \mathbf{Z} \times \mathbf{Z}$. Suppose we want to increase q gradually to $r > q$ when (k, m) leaves A_0. We define inductively sets $A_1, A_2, \dots$ such that A_i is the set of pairs $(k, m) \in \mathbf{Z} \times \mathbf{Z}$ for which $\overline{B}_{km} \cap \overline{B}_{st} \neq \emptyset$ for some $(s, t) \in A_{i-1}$. We also write $B_A = \operatorname{int}\bigl(\cup\{\overline{B}_{st} : (s, t) \in A\}\bigr)$, where $A \subset R^2$. For $(k, m) \in A_1 \setminus A_0$ we define f in B_{km} by $\eta_w \circ \lambda_{\mu\nu}^q \circ \varphi_{km}$, where $\lambda_{\mu\nu}^q$ is the function defined in (9.3) such that $\sigma_{\mu\nu}$ is maximal (depending on (k, m)) with the condition that $\eta_w \circ \lambda_{\mu\nu}^q \circ \varphi_{km}$ agrees on the common boundary of B_{km} and B_{A_0}. Then, on the common boundary of B_{A_1} and any B_{km}, $(k, m) \in A_2 \setminus A_1$, the constructed map agrees with $\eta_w \circ \lambda^{q+1} \circ \varphi_{km}$, and so the construction can be continued similarly if needed.

As a second gluing process we describe how to change w. Suppose again that for fixed w and q we have defined $f|B_{km}$ as $\eta_w \circ \lambda^q \circ \varphi_{km}$ for $(k, m) \in A_0$. Let $w' = (w_1, w_2', \dots, w_{p+1}')$ be such that $\sigma(w_1, w_2')/\sigma(w_1, w_2) \in [1/2, 2]$. We assume that $q \geq \max(l_w, l_{w'})$. Let D_1 and D_2 be the domains in ∂H^+ bounded by the lines L and L' (Figure 4) and such that D_1 meets the positive and D_2 the negative x_2-axis. Then $g_w \circ \varphi_1 \circ \varphi_2^{-1}$ and $g_{w'} \circ \varphi_1 \circ \varphi_2^{-1}$ agree on $(D_1 \cup D_2) \cap \complement B^3(2^q)$. We can fix an absolute constant $Q > 1$ and a Q-bilipschitz map $\vartheta\colon \overline{R^3} \to \overline{R^3}$ with respect to the spherical metric such that $\vartheta \circ \eta_w$ and $\eta_{w'}$ agree on $D_1 \cap \complement B^3(2^q)$ and $\vartheta \circ \eta_w$ and η_w agree on $D_2 \cap \complement B^3(2^q)$.

For $(k, m) \in A_1 \setminus A_0$ we define $f|B_{km}$ as one of the following three maps:

(a) $\eta_w \circ \omega \circ \lambda^q \circ \varphi_{km}$,

Figure 4.

(b) $\eta_w \circ \omega^{*-1} \circ \lambda^q \circ \varphi_{km}$,

(c) $\eta_w \circ \omega^{*-1} \circ \omega \circ \lambda^q \circ \varphi_{km}$.

The rule for the choice among (a)–(c) is that on the common boundary of B_{A_0} and B_{km} the constructed maps agree and in case of ambiguity (c) has preference to (a) and (b). Next we let A_2' be a set with $A_1 \subset A_2' \subset A_2$ and such that no point of $\Lambda_1 = (1,0) + 2\mathbf{Z} \times 2\mathbf{Z}$ appears on the boundary of $\cup\{\partial H^+ \cap \overline{B}_{s,t} : (s,t) \in A_2'\}$. For $(k,m) \in A_2' \setminus A_1$ we define $f|B_{km}$ with the same rule as above with A_0 replaced by A_1. In fact, this time only (c) will be the choice among (a)–(c). We now observe that on $\partial B_{A_2'}$ the constructed map agrees with

$$\text{(d)} \qquad \vartheta \circ \eta_w \circ w^{*-1} \circ \omega \circ \lambda^q \circ \varphi_{km}$$

if $\partial B_{A_2'} \cap \partial B_{km} \neq \emptyset$. We use (d) as a definition for $(k,m) \in A_3 \setminus A_2'$.

Next we strive for changing the role of Λ_1 and Λ_2. For this we stepwise replace $\omega^{*-1} \circ \omega$ in (d) by $\omega^{*-1}, \omega, \mathrm{id}, \omega^*, \omega^{-1}, \omega^* \circ \omega^{-1}$. Preference in the final step is given to $\omega^* \circ \omega^{-1}$. We get that on ∂B_{A_7} the constructed map agrees with

$$\text{(e)} \qquad \eta_{w'} \circ \omega^* \circ \omega^{-1} \circ \lambda^q \circ \varphi_{km}$$

if $\partial B_{A_7} \cap \partial B_{km} \neq \emptyset$. By replacing $\omega^* \circ \omega^{-1}$ in (e) stepwise by ω^*, ω^{-1}, and id so that id has the preference we get that on ∂B_{A_9} the constructed map agrees with $\eta_{w'} \circ \lambda^q \circ \varphi_{km}$ if $\partial B_{A_9} \cap \partial B_{km} \neq \emptyset$. This completes the gluing process for changing w to w'. Keeping w_2 fixed we can similarly move w_1. As a result, given two arbitrary w and w' we can after a finite number of steps move from a map $\eta_w \circ \lambda^q \circ \varphi_{km}$ to $\eta_{w'} \circ \lambda^q \circ \varphi_{st}$ provided $q \geq \max(l_w, l_{w'})$. Recall the definition of l_w from 9.7.

In the next section we consider gluing procedures also for certain parts of the initial set A_0. Such modifications will always be clear from the context.

10. Completion of the proof of Theorem 1.7

10.1. Defect numbers and sectors. Now let $\delta_1, \delta_2, \dots$ be as in 1.7, let p be the integer such that

$$p < \beta = \sum_i \delta_i \leq p + 1,$$

and assume that $p \geq 2$. (The case $0 \leq p \leq 1$ is considered in [R2].) We arrange the indexing so that $\delta_1 \geq \delta_2 \geq \delta_3 \geq \dots$. Set

$$s_k = \sum_{i=1}^{k} \delta_i, \qquad s_0 = 0,$$

and $D_i = [s_{i-1}, s_i[$, $i \geq 1$. Let α: $[0, p+1[\to [0,1[\times \{1, \dots, p+1\}$ be defined by $\alpha(s) = (s - j + 1, j)$ when $j - 1 \leq s < j$, $j = 1, \dots, p+1$, and let π_1: $[0,1[\times \{1, \dots, p+1\} \to [0,1[$ be the projection. Each set $D'_i = \alpha D_i$ is either one or two intervals. Each fibre $\pi_1^{-1}(t)$ contains either p or $p+1$ points of the sets D'_i and all indices i are different. Observe that each $\delta_i \leq 1$. The projections $\pi_1 D'_i$ define a set of intervals in $[0,1[$ and the endpoints of these intervals define a set of subintervals. We denote these subintervals in the part $[0, \beta - p[$ of $[0,1[$ by $\gamma_\mu = [t_{\mu-1}, t_\mu[$, $\mu = 1, 2, \dots$, and in the part $[\beta - p, 1[$ by $\omega_\nu = [u_{\nu-1}, u_\nu[$, $\nu = 1, \dots, \nu_0$. The indexing is such that $t_0 < t_1 < \cdots$, $u_0 < u_1 < \cdots < u_{\nu_0}$. If $\beta = p + 1$, the intervals $[u_{\nu-1}, u_\nu[$ are missing. If there are only finitely many positive terms in $\sum_i \delta_i$, the set of intervals γ_μ is finite. For each γ_μ (ω_ν) there is a set I_μ (J_ν) of $p+1$ (p) indices i such that $\pi_1^{-1}(t) \cap D'_i \neq \emptyset$, $t \in \gamma_\mu$ $(t \in \omega_\nu)$. We define sectors of R^2 by means of the intervals γ_μ and ω_ν by setting (in polar coordinates)

$$\Gamma_\mu = \{(t, \varphi) \in R^2 : \varphi / 2\pi \in \gamma_\mu\}, \quad \mu \geq 1,$$
$$\Omega_\nu = \{(t, \varphi) \in R^2 : \varphi / 2\pi \in \omega_\nu\}, \quad \nu = 1, \dots, \nu_0.$$

10.2. Sectors and the quasiregular map. In cylinders B_{km}, whose base $B_{km} \cap R^2$ is contained in some sector Γ_μ, we mostly define f by a standard map $\tau_{km}^{w^\mu}$ given in formula (9.8). Here $w^\mu = \{a_i : i \in I_\mu\}$. Exceptions of this rule appear only for cylinders B_{km} with base near $\dot{\Gamma}_\mu$, the boundary of Γ_μ with respect to R^2. A precise description of this procedure is given below.

For a sector Ω_ν we add to the set $\{a_i : i \in J_\nu\}$ points as follows. We pick a point a_0 with a spherical distance $\delta > 0$ from $\{a_i : i \in \bigcup_\nu J_\nu\}$. We let (z_κ) be a sequence in $\overline{B}_\sigma(a_0, \delta/2) \setminus \{a_0\}$ converging to a_0 such that $1/2 \leq d(z_{\kappa+1}, a_0)/d(z_\kappa, a_0) < 1$, $\kappa = 1, 2, \ldots$. In (10.11) we shall give more precise conditions on the sequence (z_κ). We also decompose Ω_ν into sets $\Omega_{\nu\kappa}$ defined by

$$\Omega_{\nu\kappa} = \{x \in \Omega_\nu : C2^\kappa \leq d(x, \dot{\Omega}_\nu) < C2^{\kappa+1}\},$$

where C is large enough for the gluing process described below. For cylinders B_{km} with base in $\Omega_{\nu\kappa}$ we mostly define f by a standard map $\tau_{km}^{w^{\nu\kappa}}$, where $w^{\nu\kappa} = \{a_i : i \in J_\nu\} \cup \{z_\kappa\}$. Again exceptions appear only for cylinders B_{km} with base near the boundaries $\dot{\Omega}_{\nu\kappa}$.

To work out these principles in detail we do the following. We start the construction by defining $f|B_{km}$ to be the standard map $\tau_{km}^{w^{11}}$ for $(k,m) \in \Omega_{11}$. Then we use the gluing method from 9.9 to change from the standard maps $\tau_{km}^{w^{11}}$ to standard maps $\tau_{km}^{w^{12}}$ when (k,m) leaves Ω_{11} and enters Ω_{12}. Note that at this point we have not defined f in cylinders B_{km} with $(k,m) \in R^2 \setminus \Omega_1$. In the next step we use the gluing process to change the standard maps $\tau_{km}^{w^{12}}$ to standard maps $\tau_{km}^{w^{13}}$ when (k,m) leaves Ω_{12} and enters Ω_{13}. We continue similarly and, as a result, f is defined for cylinders B_{km}, $(k,m) \in \Omega_1$. Because of the special choice of the sequence (z_κ) each gluing stage is completed after M steps, M independent of κ (see the end of Section 9). Since C is large, we obtain the standard maps $\tau_{km}^{w^{12}}$ for $(k,m) \in \Omega_{12}$ well before the common boundary of Ω_{12} and Ω_{13}. Since the width of $\Omega_{1\kappa}$ increases with κ, the corresponding condition in later stages is satisfied too.

Next we perform the gluing process from the standard maps $\tau_{km}^{w^{11}}$ to standard maps $\tau_{km}^{w^{21}}$ when (k,m) leaves Ω_1. After a finite number of steps we obtain a (minimal) neighborhood Ω_1' of Ω_1 (in R^2) such that $f|B_{km}$ is defined for $(k,m) \in \Omega_1'$ and it is the standard map $\tau_{km}^{w^{21}}$ when $\overline{B}_{km}$ meets the boundary of $B_{\Omega_1'}$. Recall the notation from 9.9. We define $f|B_{km}$ as the standard map $\tau_{km}^{w^{21}}$ for $(k,m) \in \Omega_{21} \setminus \Omega_1'$. We can then continue the definition for the cylinders B_{km} in the rest of B_{Ω_2} as we did for Ω_1 above.

We continue similarly. We find a neighborhood Ω_2' of $\Omega_1' \cup \Omega_2$ such that f is defined in $B_{\Omega_2'}$ and $f|B_{km}$ is the standard map $\tau_{km}^{w^{31}}$ when $\overline{B}_{km}$ meets the boundary of $B_{\Omega_2'}$. Then we continue as before and f is defined in $B_{\Omega_2' \cup \Omega_3}$. When we have treated all Ω_ν's this way, we have f defined in $B_{\Omega'_{\nu_0-1} \cup \Omega_{\nu_0}}$, where Ω'_{ν_0-1} is a neighborhood of $\Omega_1 \cup \cdots \cup \Omega_{\nu_0-1}$. Observe that we have chosen C so large that the gluing processes work.

To proceed we next consider the sector Γ_1. We find a (minimal) neighborhood $\Gamma_0' \subset R^2$ of $\Omega'_{\nu_0-1} \cup \Omega_{\nu_0}$ such that f is defined in $B_{\Gamma_0'}$ and $f|B_{km}$ is the standard map $\tau_{km}^{w^1}$ when $\overline{B}_{km}$ meets the boundary of $B_{\Gamma_0'}$. We extend the definition of f

to $B_{\Gamma_1} \setminus B_{\Gamma_0'}$ by standard maps $\tau_{km}^{w^1}$. Next we find a neighborhood Γ_1' of $\Gamma_0' \cup \Gamma_1$ such that f is defined in $B_{\Gamma_1'}$ and $f|B_{km}$ is the standard map $\tau_{km}^{w^2}$ if $\overline{B}_{km}$ meets the boundary of $B_{\Gamma_1'}$. We extend the definition of f to $B_{\Gamma_2} \setminus B_{\Gamma_1'}$ by the standard maps $\tau_{km}^{w^2}$. We continue similarly through all Γ_μ's. Then f becomes defined in the whole space R^3. This completes the construction of f.

10.3. Value distribution. It remains to show that the constructed map $f\colon R^3 \to \overline{R^3}$ is a realization for the given defect numbers δ_i. For this we need to study the value distribution behavior of the standard maps τ_{km}^{w} where w is w^μ or $w^{\nu\kappa}$. The behavior of f in cylinders B_{km}, where some gluing process is performed, has no effect asymptotically. This is so because the bases of such cylinders are contained in strips whose total angular measure asymptotically tends to zero. Moreover, during the gluing process the map is distorted only by a bounded amount from the corresponding standard maps.

We shall study value distribution of the maps g_w (see 9.7). Recall from the end of Section 8 that $g = g_w$ is of a normalized form in each set X_j bounded by the level surfaces $\nu^{2i_1}|N_j|$, $j = 1, 2$. Fix a level surface $T = \nu^{2i}|N_1|$, where $i \ge i_1$. Recall that on T we are given a map complex G, that g maps each 2-simplex of G injectively onto the upper or lower hemisphere of a sphere $S^2(t_i)$, and that these 2-simplexes appear in adjacent pairs A, B so that $g|A \cup B$ covers the sphere $S^2(t_i)$ once when some part of the boundary of $A \cup B$ is ignored (see [R9, 4.3]). Let $\xi_i(r)$ be the number of these pairs A, B which are contained in $T \cap \overline{B^3}(r)$. Fix $\beta = 1/6$. According to the construction of g there are constants c_1, $c_2 > 0$, depending only on p, and $r_0'' = r_0''(w, i) \ge 1$ such that for $r \ge r_0''$ we have

$$n_g(r, z) \le c_2 r^2 \quad \text{for} \quad z \in \overline{R^3}, \tag{10.4}$$

$$c_1 r^2 \le \xi_i(r) \le c_2 r^2, \tag{10.5}$$

$$c_1 \le d(A) \le c_2 r^\beta \quad \text{if} \quad A \text{ is a 2-simplex of } G \text{ in } \overline{B^3}(r), \tag{10.6}$$

$$\xi_i(r + r^{2\beta}) \le \xi_i(r) + c_2 r^{1+2\beta}. \tag{10.7}$$

For (10.5)–(10.7) we refer to the discussion in [R9, 3.4]. We may assume that $r_0''(w, i)$ increases in i.

Let $v_1, \ldots, v_{p+1}$ be the points on the x_3-axis corresponding to w and omitted by g. Set

$$\sigma_0 = \frac{1}{4} \min_{j \ne k} \sigma(v_j, v_k)$$

and let $0 < \varepsilon < \sigma_0$. To an adjacent pair A, B of 2-simplices of G corresponds a "branched tube" V, which g maps in a one to one way (with proper interpretation on boundary) onto $\overline{R^3} \setminus \{v_1, \ldots, v_{p+1}\}$ (see [R9, Sections 6, 7, 8]). Such branched tubes fill disjointly R^3. Let $Q_\varepsilon = Q_\varepsilon^w = \overline{R^3} \setminus \bigcup_j B_\sigma(v_j, \varepsilon)$. The diameter of the intersection $V\cap g^{-1}Q_\varepsilon$ satisfies

$$\frac{d(V\cap g^{-1}Q_\varepsilon)}{d(A)}\le c(i,\varepsilon,w). \tag{10.8}$$

From (10.6)–(10.8) we get that if r is large enough, then

$$n_g(r,y)\le \xi_i(r+r^{2\beta})\le \xi_i(r)+c_2r^{1+2\beta},\quad y\in Q_\varepsilon,$$

and similarly,

$$\xi_i(r)-c_2r^{1+2\beta}\le \xi_i(r-r^{2\beta})\le n_g(r,y),\quad y\in Q_\varepsilon.$$

With (10.5) we conclude that there exists

$$R=R(i,\varepsilon,w)\ge r_0'' \tag{10.9}$$

such that

$$1-\frac{c_2}{c_1}r^{-4\beta}\le \frac{n_q(r,y)}{\xi_i(r)}\le 1+\frac{c_2}{c_1}r^{-4\beta},\quad r\ge R,\quad y\in Q_\varepsilon. \tag{10.10}$$

We let $s_w\in\,]0,1/2]$ be maximal such that the map g_w is of normalized form in $g_w^{-1}\overline{B}_\sigma(\infty,s_w)$. If $g_{w'}$ is another map, g_w and $g_{w'}$ coincide in $g_w^{-1}\overline{B}_\sigma(\infty,s)=g_{w'}^{-1}\overline{B}_\sigma(\infty,s)$ for $0<s\le\min(s_w,s_{w'})$. Because of the special choice of the sequence (z_κ), we have

$$s_0=\min_{1\le\nu\le\nu_0}\min_{\kappa\ge1}(s_{w^{\nu\kappa}})>0.$$

Suppose the sequence $\Gamma_1,\Gamma_2,\dots$ is infinite. If $\iota\ge1$ is an integer, we let D_ι be the union of all sectors Ω_ν, $\nu=1,\dots,\nu_0$, and Γ_μ, $1\le\mu\le\iota$. If $t\ge1$, we let $P(\iota,t)$ be twice the number of pairs (k,m) for which the base of B_{km} is contained in $D_\iota\cap B^2(t)$ and $f|B_{km}$ is a standard map. Set

$$s_\iota=\min\bigl(s_0,\min_{1\le\mu\le\iota}s_{w^\mu}\bigr).$$

For a minimal $i(\iota)$, each g_w is of normalized form on the level surface $T=\nu^{2i}|N_1|$ and $T\subset g_w^{-1}\overline{B}_\sigma(\infty,s_\iota)$ for $i\ge i(\iota)$, when w is $w^{\nu\kappa}$, $\nu=1,\dots,\nu_0$, $\kappa=1,2,\dots$, or w^μ, $1\le\mu\le\iota$. We let G_ι be the set of these w's. We recall from (9.8) that a standard map τ_{km}^w is of the form $\eta_w\circ\lambda^l\circ\varphi_{km}$, $l=l_w$, and here $\lambda^l\colon H^+\to H^+$ is the identity outside $B^3(2^{2l})\cap H^+$. Again, by the special choice of the sequence (z_κ) from 10.2,

$$l_\iota=\max_{w\in G_\iota}l_w<\infty.$$

To give a precise condition on the sequence (z_κ) we write $\delta_\iota = \delta 2^{-\iota-4}$, $\iota = 1,2,\dots$. Recall from 10.2 the notation δ for the spherical distance of a_0 to $\{a_i : i \in \bigcup_\nu J_\nu\}$. There is a fixed constant $\gamma > 0$, independent of the particular choice of the sequence (z_κ), such that for $\varepsilon_\iota = \gamma\delta_\iota$ we have $\overline{R^3} \setminus \bigcup_j B_\sigma(w_j,\delta_\iota) \subset h_w^{-1}\varphi_w^{-1}Q^w_{\varepsilon_\iota}$ for all $w = w^{\nu\kappa}$, $\kappa = 1,2,\dots$, $\nu = 1,\dots,\nu_0$. Set

$$\begin{aligned} R_{\iota,\varepsilon} &= \max_{w\in G_\iota} R\big(i(\iota),\varepsilon,w\big), \\ r_\iota &= \max(2^{2l_\iota}, R_{\iota,\varepsilon_\iota}), \\ t_\iota &= \log r_\iota. \end{aligned}$$

We now require that

$$z_\kappa \notin B_\sigma(a_0,2\delta_\iota) \quad \text{if} \quad \Omega_{\nu\kappa} \cap B^2(t_\iota) \neq \emptyset \quad \text{for some} \quad \nu = 1,\dots,\nu_0. \tag{10.11}$$

This condition is fulfilled when (z_κ) tends to a_0 slowly enough and (10.11) is needed to ensure the right covering of a_0 by the map f.

Let $y \in \overline{R^3} \setminus \{a_1,a_2,\dots\}$. We want to show that

$$\lim_{t\to\infty} \frac{n_f\big(Q(t),y\big)}{A_f\big(Q(t)\big)} = 1, \tag{10.12}$$

where $A_f(B)$ is the average of $n_f(B,y)$ over $\overline{R^3}$ for any Borel set $B \subset R^3$. Recall the notation $Q(t)$ from the beginning of Section 9. We let m_σ be the spherical 3-measure on $\overline{R^3}$. If $\iota \ge 1$ is an integer and $\varepsilon > 0$, we have

$$\frac{m_\sigma(h_w^{-1}\varphi_w^{-1}Q^w_\varepsilon)}{m_\sigma(\overline{R^3})} \ge \frac{1}{1+\zeta(\iota,\varepsilon)}, \quad w \in G_\iota, \tag{10.13}$$

where $\zeta(\iota,\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $0 < \varepsilon \le \varepsilon_\iota$. By (10.4), (10.5), (10.10) and (10.13) we obtain

$$\begin{aligned} \frac{P(\iota,\varepsilon)(1-c_2c_1^{-1}e^{-4\beta t})\xi_i(e^t)}{1+\zeta(\iota,\varepsilon)} &\le A_f\big(Q(t)\big) \\ &\le 2\pi(t+2)^2(1+c_2c_1^{-1}e^{-4\beta t})\Big(1+\frac{c_2}{c_1}\zeta(\iota,\varepsilon)\Big)\xi_i(e^t) \end{aligned} \tag{10.14}$$

if $t \ge t_{\iota,\varepsilon} = \max(t_\iota, \log R_{\iota,\varepsilon})$ and $i = i(\iota)$.

Suppose $y \neq z_\kappa$, $\kappa = 1,2,\dots$. For fixed ι we find $\varepsilon_{\iota,y} \in]0,\varepsilon_\iota]$ such that $y \in h_w^{-1}\varphi_w^{-1}Q^w_\varepsilon$ for $0 < \varepsilon \le \varepsilon_{\iota,y}$ and $w \in G_\iota$. Let $0 < \varepsilon \le \varepsilon_{\iota,y}$. Then (10.10) yields

$$\begin{aligned} P(\iota,t)(1-c_2c_1^{-1}e^{-4\beta t})\xi_i(e^t) &\le \eta_f\big(Q(t),y\big) \\ &\le 2\pi(t+2)^2(1+c_2c_1^{-1}e^{-4\beta t})\xi_i(e^t) \end{aligned} \tag{10.15}$$

if $t \geq t_{\iota,\varepsilon}$. From (10.14) and (10.15) we conclude that for given $\gamma > 0$ we can first choose ι, then $\varepsilon \leq \varepsilon_{\iota,y}$, and then $t_\gamma \geq t_{\iota,\varepsilon}$ such that

$$\frac{1}{1+\gamma} \leq \frac{n_f\big(Q(t),y\big)}{A_f\big(Q(t)\big)} \leq 1+\gamma \quad \text{if} \quad t \geq t_\gamma.$$

If $y = z_\kappa$ for some κ, the same conclusion holds because

$$\frac{m_2\big(\Omega_{\nu\kappa} \cap B^2(t)\big)}{m_2\big(B^2(t)\big)} = O\Big(\frac{1}{t}\Big) \quad \text{as} \quad t \to \infty$$

for every $\Omega_{\nu\kappa}$. If the sequence $\Gamma_1, \Gamma_2, \dots$ is finite, the arguments above simplify.

The proof that

$$\lim_{t\to\infty} \frac{n_f\big(Q(t),a_j\big)}{A_f\big(Q(t)\big)} = 1 - \delta_j$$

is accomplished in a similar way by taking into account that the sectors Γ_μ and Ω_ν are chosen according to the given defect numbers δ_j. The proof of Theorem 1.7 is complete.

References

[A] Ahlfors, L.V.: Zur Theorie der Überlagerungsflächen. - Acta Math. 65, 1935, 157–194.

[BI] Bojarski, B., and T. Iwaniec: Analytical foundations of the theory of quasiconformal mappings in R^n. - Ann. Acad. Sci. Fenn. Ser. A I Math. 8, 1983, 257–324.

[D] Drasin, D.: The inverse problem of the Nevanlinna theory. - Acta Math. 138, 1977, 83–151.

[EL] Eremenko, A., and J.L. Lewis: Uniform limits of certain A-harmonic functions with applications to quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 16, 1991, 361–375.

[HR1] Holopainen, I., and S. Rickman: A Picard type theorem for quasiregular mappings of R^n into n-manifolds with many ends. - Rev. Mat. Iberoamericana 8, 1992, 131–148.

[HR2] Holopainen, I., and S. Rickman: Quasiregular mappings of the Heisenberg group. - Math. Ann. 294, 1992, 625–643.

[L] Lewis, J.: Picard's theorem and Rickman's theorem by way of Harnack's inequality. - Proc. Amer. Math. Soc. 122, 1994, 199–206.

[MRV1] Martio, O., S. Rickman, and J. Väisälä: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 448, 1969, 1–40.

[MRV2] Martio, O., S. Rickman, and J. Väisälä: Distortion and singularities of quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 465, 1970, 1–13.

[MR] Mattila, P., and S. Rickman: Averages of the counting function of a quasiregular mapping. - Acta Math. 143, 1979, 273–305.

[M] Miles, J.: Bounds on the ratio $n(r,a)/S(r)$ for meromorphic functions. - Trans. Amer. Math. Soc. 162, 1971, 383–393.

[N1] Nevanlinna, R.: Zur Theorie der meromorphen Funktionen. - Acta Math. 46, 1925, 1–99.

[N2] Nevanlinna, R.: Analytic Functions. - Grundlehren Math. Wiss. 162, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[Pe] Pesonen, M.: A path family approach to Ahlfors's value distribution theory. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 39, 1982, 1–32.

[Re] Reshetnyak, Yu.G.: Space mappings with bounded distortion. - Transl. Math. Monographs 73, Amer. Math. Soc., Providence, R.I., 1989.

[R1] Rickman, S.: Path lifting for discrete open mappings. - Duke Math. J. 40, 1973, 187–191.

[R2] Rickman, S.: A quasimeromorphic mapping with given deficiencies in dimension three. - Sympos. Math. XVIII, 1976, INDAM, Academic Press, London, 535–549.

[R3] Rickman, S.: On the value distribution of quasimeromorphic maps. - Ann. Acad. Sci. Fenn. Ser. A I Math. 2, 1976, 447–466.

[R4] Rickman, S.: Asymptotic values and angular limits of quasiregular mappings of a ball. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 185–196.

[R5] Rickman, S.: On the number of omitted values of entire quasiregular mappings. - J. Analyse Math. 37, 1980, 100–117.

[R6] Rickman, S.: A defect relation for quasimeromorphic mappings. - Ann. Math. 114, 1981, 165–191.

[R7] Rickman, S.: Value distribution of quasiregular mappings. - Proc. Value Distribution Theory, Joensuu 1981, Lecture Notes in Math. 981, Springer-Verlag, Berlin–Heidelberg–New York, 1983, 220–245.

[R8] Rickman, S.: Quasiregular mappings and metrics on the n-sphere with punctures. - Comment. Math. Helv. 59, 1984, 134–148.

[R9] Rickman, S.: The analogue of Picard's theorem for quasiregular mappings in dimension three. - Acta Math. 154, 1985, 195–242.

[S] Sastry, S.: Upper bounds for value distribution of quasimeromorphic maps. - Ann. Acad. Sci. Fenn. Ser. A I Math. 20, 1995, 61–97.

[V] Väisälä, J.: Lectures on n-dimensional Quasiconformal Mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

[Vu] Vuorinen, M.: Conformal Geometry and Quasiregular Mappings. - Lecture Notes in Math. 1319, Springer-Verlag, Berlin–Heidelberg, 1988.

[Z] Zorich, V.A.: The theorem of M.A. Lavrent'ev on quasiconformal mappings in space. - Mat. Sb. 74, 1967, 417–433 (Russian).

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