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QUASICONFORMAL GROUPS ACTING ON B^3 THAT ARE NOT QUASICONFORMALLY CONJUGATE TO MÖBIUS GROUPS

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Abstract. We construct a quasiconformal group acting on the unit ball in 3-space which is not quasiconformally conjugate to any Möbius group. F. Gehring and P. Palka first asked if each quasiconformal group acting on Euclidean n-space is quasiconformally conjugate to a Möbius group. The answer to the question is positive for $n = 2$ (D. Sullivan and P. Tukia) and is negative for higher dimensions (P. Tukia). A similar question for groups leaving the unit ball in n-space invariant was answered negatively for $n > 3$; but the question has remained open for $n = 3$.

1. Introduction

The purpose of this paper is to prove the following result.

1.1 Theorem. *There exists a quasiconformal group acting on B^3 that is not quasiconformally conjugate to any Möbius group.*

The above theorem answers a question raised by G. Martin in [M] about the existence of such groups. F. Gehring and B. Palka first asked whether each quasiconformal group acting in $\overline{\mathbf{R}}^n$ is conjugate to a Möbius group via a quasiconformal map [GP]. For $n = 2$, the question was answered affirmatively by D. Sullivan [S] and by P. Tukia [T1]. For $n \geq 3$, the answer to the question is negative. P. Tukia [T2] constructed a quasiconformal group not quasiconformally conjugate to a Möbius group. Later, G. Martin modified Tukia's construction to provide a discrete group with the above property [M]. Moreover, for $n \geq 4$ there are quasiconformal Fuchsian groups (discrete groups that leave a ball invariant) which are not quasiconformally conjugate to any Möbius group [M, Theorem 2.4]. But the problem of existence of a quasiconformal group acting on B^3 which is not quasiconformally conjugate to any Möbius group remained open. The basic idea in Tukia's construction is to use the rigidity of quasiconformal maps in higher dimensions to construct a quasiconformal group of the form $G = f\Gamma f^{-1}$ where Γ is a Möbius group and f is not a quasiconformal map and show that G is not quasiconformally conjugate to any Möbius group. We follow the same idea. Our construction can be best described as the "cylindrical version" of Tukia's construction.

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After some preparations in Section 2, we prove the main result in Section 3 and in Section 4 we present an example of a finite quasiconformal reflection group acting in B^3 which is not conjugate to any Möbius group via a quasiconformal map. The group constructed in Section 3 is not discrete and the group presented in Section 4 is finite. I do not know if there is a non-elementary discrete group G for Theorem 1.1.

2. Preliminaries

2.1. Notation. As usual, $\mathbf{R}^n$ denotes the Euclidean n-space and $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. For a set $A \subset \overline{\mathbf{R}}^n$ we write ∂A, $\bar{A}$ for the boundary and the closure of A, respectively. The ball centered at x or radius r is denoted by $B^n(x,r)$ and

$$\mathbf{R}^1 = \mathbf{R}, \qquad B^n = B^n(0,1), \qquad S^{n-1} = \partial B^n, \qquad H^n = \{x \in \mathbf{R}^n : \ x_n > 0\}.$$

When working in $\mathbf{R}^2$ we sometimes use the complex notation $z = x_1 + ix_2$.

2.2. Möbius and quasiconformal groups. Let X be a non-empty set. If G is any group of permutations of X, then the G-orbit of x is

$$G(x) = \{g(x) : g \in G\}.$$

The fixed points of G is denoted by $\mathrm{Fix}(G)$. Two subgroups G_1 and G_2 of G are conjugate in G if for some $h \in G$, $G_2 = hG_1h^{-1}$. Clearly $G_2(hx) = h\bigl(G_1(x)\bigr)$.

A *Möbius transformation* acting in $\overline{\mathbf{R}}^n$ is a finite composition of reflections in spheres and planes. The group of Möbius transformations acting on $\overline{\mathbf{R}}^n$ is called *general Möbius group* and is denoted by $GM(\overline{\mathbf{R}}^n)$. The *Möbius group* $M(\overline{\mathbf{R}}^n)$ is the subgroup of $GM(\overline{\mathbf{R}}^n)$ consisting of all orientation preserving Möbius transformations. Finally, we call G a *quasiconformal group* if it is a K-quasiconformal group for some K.

We need the next two lemmas for the proof of Theorem 1.1.

2.3. Lemma. *Suppose that Γ_1 is a subgroup of $M(\bar{\mathbf{R}}^3)$. If Γ_1 fixes more than two points in $\overline{\mathbf{R}}^3$, then Γ_1 is conjugate to a group of rotations about a line L.*

Proof. After conjugating with a Möbius transformation, we may assume that Γ_1 fixes the points 0, ∞ and x_0. Let $g \in \Gamma_1$ and let L be the line passing through 0 and x_0. Then g fixes L pointwise because $g(L) = L$ and g fixes three points in L. Since g fixes 0 and ∞, g is a similarity by [A, II, Lemma 1] and since g fixes L pointwise g is an isometry.

If P is a hyperplane perpendicular to L, then $g(P)$ is a plane perpendicular to L again containing $P \cap L$. Hence $g(P) = P$ and g in P is an isometry fixing $P \cap L$. Therefore g in P is either a rotation of P about the point $P \cap L$, or it is a reflection in a line L_1 through $P \cap L$ in P.

If g in P is a rotation, we choose a rotation g_0 of $\mathbf{R}^3$ about L which agrees with g on P. Then $g_0^{-1}g = \mathrm{id}$ in P and hence $g_0^{-1}g = \mathrm{id}$ in $\mathbf{R}^3$ or $g_0^{-1}g$ is a reflection in P by [B; Theorem 3.2.4]. But both g_0 and g are sense preserving and g is a rotation about L as desired.

It remains to see what happens if g is a reflection about L_1 in P. In that case, g fixes the plane P_1 containing the lines L and L_1 because it fixes both L_1 and L pointwise. Therefore g is a reflection in P_1 which is impossible. □

2.4. Lemma. *Let Γ_1 and L be as in Lemma 2.3 and let Γ_2 be a subgroup of $M(\overline{\mathbf{R}}^3)$. If there is $x_0 \in \overline{\mathbf{R}}^3$ so that $\mathrm{Fix}(g) = \{x_0\}$ for each $g \in \Gamma_2$, $g \neq \mathrm{id}$ and if $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is abelian, then $\Gamma_2 \,|\, L$ is conjugate to a group of translations along L and each element of Γ_2 is a combination of a translation and a rotation along L.*

Proof. Let $g_2 \in \Gamma_2$, $g_2 \neq \mathrm{id}$ and let $g_1 \in \Gamma_1$. Then $g_1(x_0) = g_1 g_2(x_0) = g_2 g_1(x_0)$ and hence $g_1(x_0) \in \mathrm{Fix}(\Gamma_2) = \{x_0\}$ and thus $x_0 \in \mathrm{Fix}(\Gamma_1)$. After conjugation, we may assume that $x_0 = \infty$ and $\mathrm{Fix}(\Gamma_1) = L$ is a line passing through ∞. Next if $x \in L$ and $g_2 \in \Gamma_2$, then

$$g_1 g_2(x) = g_2 g_1(x) = g_2(x)$$

for each $g_1 \in \Gamma_1$. Hence $g_2(x) \in \mathrm{Fix}(\Gamma_1) = L$ and $g_2 \,|\, L$ is a translation, or $g_2 \,|\, L = ax + b$. Now if $a \neq 1$, then $g_2 \,|\, L$ has a finite fixed point contrary to our assumption. Therefore $g_2 \,|\, L = x + b$ and $g_2 - b$ fixes L_2 pointwise. Hence $g_2 - b$ is a rotation along L. □

2.5. Maps. We consider K-quasiconformal, η-quasisymmetric and L-bilipschitz maps. For definitions of quasiconformal and quasisymmetric maps see [V1], [V2; 1.3 and 3.1].

2.6. Quasihyperbolic metric. Suppose that $D \subset \mathbf{R}^n$, $D \neq \mathbf{R}^n$ is a domain. The quasihyperbolic metric K_D of D is defined by the element of length $|dx|/d(x, \partial D)$ where $d(x, \partial D)$ is the distance from x to ∂D, see [GP]. If $D \subset \mathbf{R}^2$ is simply connected then

$$h_D/2 \leqq K_D \leqq 2h_D$$

where h_D is the hyperbolic metric of D.

The following result is contained in the proof of [G2; 2.11] for a special case, the general case needs only minor modifications.

2.7. Lemma. *Suppose that D and D' are domains in $\mathbf{R}^n$ with connected boundaries and that $f\colon \overline{D} \to \overline{D}'$ is a homeomorphism such that*

1. *$f(D) = D'$,*
2. *$f|\partial D$ is L-bilipschitz,*
3. *$f|D$ is M-bilipschitz in the quasihyperbolic metrics of D and D'. Then f is L_1-bilipschitz with $L_1 = L_1(L, M, n)$.*

We close this section with an extension result that is essential for the proof of Theorem 1.1.

2.8. Lemma. *Suspose that C is a bounded Jordan curve in $\mathbf{R}^2$ and that $f_0\colon S^1 \to C$ is η-quasisymmetric. Then f_0 has an extension to a quasiconformal self map f of $\mathbf{R}^2$ so that*

1. $f|S^1 = f_0$,
2. *f is bilipschitz with respect to quasihyperbolic metrics in both components of $\mathbf{R}^2 \setminus S^1$,*
3. *$f(z) = z$ for $|z| \geq r$, $r = r(\eta, \operatorname{diam} C)$.*

Proof. Performing similarity transformations, we may assume that C is contained in an annulus $A = \{1 < |z| < r_1\}$ where r_1 is a positive constant depending only on η. Let D and D^* be the interior and exterior of C_1 respectively. We first consider extension to interiors.

Let $\varphi\colon \overline{D} \to \overline{B}^2$ be a homeomorphism which φ is conformal in D and $\varphi(0) = 0$. Since C is a quasicircle, φ is η'-quasisymmetric, where η' depends only on the constant of C and $\operatorname{dist}(0, \partial D)$ both of which depend only on η and so does η'. Now $h = f_0 \circ \varphi^{-1}$ is a quasisymmetric map of S^1 and can be extended to a self-homeomorphism h_1 of $\overline{B}^2$ so that h_1 is quasisymmetric in B^2, $h_1(0) = 0$ and h_1 is bilipschitz with respect to quasihyperbolic metrics of B^2 and D, by [GNV; Lemma 2.10]. Furthermore f_0 is bilipschitz in $\{|z| < 1/2\}$ and we may assume that $f_1(z) = z$ for $|z| < r_2$ by a version of annulus theorem [M; Theorem 2.6], where $r_2 = r_2(\eta)$.

We now turn to extension of f_0 to exteriors. Let $T(z) = 1/z$. Then $g = Tf_0T$ is a quasisymmetric map that carries S^1 onto TC, because T is bilipschitz and hence quasisymmetric in A. The map g has an extension to a quasisymmetric map $g_1\colon B^2 \to TD^*$ so that $g_1(z) = z$ in $B^2(r_3)$ and g_1 is bilipschitz with respect to quasihyperbolic metric, as is shown for f_1 above.

Next $f_2 = Tg_1T$ maps $\{|z| > 1\}$ onto D^* and $f_2(z) = z$ for $|z| \geq 1/r_3$. Because T is bilipschitz in $\{r_3 < |z| < 1/r_3\}$ and g_1 is bilipschitz with respect to quasihyperbolic metric, f_2 is bilipschitz with respect to quasihyperbolic metrics of $\{|z| > 1\}$ and D^* whenever z and $f_2(z)$ are both in $\{r_3 < |z| < 1/r^3\}$. But this implies that f_2 is bilipschitz with respect to quasihyperbolic metrics of $\{|z| > 1\}$ and D^*, because $f_2(z) = z$ for $|z| \geq 1/r_3$. Finally

$$f(z) = \begin{cases} f_1(z) & \text{for } |z| \leq 1, \\ f_2(z) & \text{for } |z| > 1 \end{cases}$$

is the desired map. □

3. Proof of Theorem 1.1

We now turn to the proof of Theorem 1.1. For this we begin with the familiar bounded snowflake curve S and construct a K-quasiconformal group G acting on a round cylinder so that (1) $G(a) = S \times \mathbf{R}$ for each $a \in S \times \mathbf{R}$. (2) If Γ is any Möbius group conjugate to G with $\infty \in \operatorname{Fix}(\Gamma)$, then $\Gamma(b)$ is contained in a round cylinder. Appealing to a well established result of Tukia and Väisälä, that $S \times \mathbf{R}$ cannot be imbedded in $\mathbf{R}^2$ or $S^1 \times \mathbf{R}$ by a locally quasisymmetric map, we see that G is not quasiconformally conjugate to a Möbius group.

3.1. The Jordan curve S. Let S_n as shown below. Then $\{S_n\}$ converges to a Jordan curve S, where $\dim_H(S) = \alpha > 1$. We may assume that $H_\alpha(S) = 2\pi$ and $1 \in S$, where $H_\alpha(S)$ is the Hausdorff measure of S.

S_1 S_2 S_3

Choose an orientation on S and define $\theta(x) = H_\alpha\bigl(S(1,x)\bigr)$ where $x \in S$ and $S(1,x)$ is the oriented arc in S from 1 to x. Now $f_0^{-1}(x) = e^{i\theta(x)}$ is a bihölder map from S onto S^1, see [FM; Theorem]. Hence f_0^{-1} and f_0 are quasisymmetric. By Lemma 2.8 there exists a quasiconformal self map f of $\mathbf{R}^2$ so that $f \mid S = f_0$, f is bilipschitz with respect to quasihyperbolic metrics in both components of $\mathbf{R}^2 \setminus S^1$ and $f(z) = z$ for $|z| > r$.

3.2. The group G. Let R be the group of all rotations $R_\theta\colon z \to ze^{i\theta}$ in $\mathbf{R}^2$. Then $R_S = fRf^{-1}$ is a group of self maps of $\mathbf{R}^2$ acting transitively on S. Since R_S leaves $B^2(r)$ invariant, we regard R_S as a group of transformations of $B^2(r)$. Next $fR_\theta f^{-1}$ is bilipschitz with respect to hyperbolic metrics in both components of $\mathbf{R}^2 \setminus S$, being a composition of such maps. Moreover, $fR_\theta f^{-1} \mid S = f_0 R_\theta f_0^{-1}$ is bilipschitz by [FM; Corollary] and hence $fR_\theta f^{-1}$ is bilipschitz throughout by Lemma 2.7. Let $G_1 = R_S \times \mathrm{id}$ and let G_2 be the group of translations along the x_3-axis. Then $G = \langle G_1, G_2 \rangle$ is a bilipschitz group of $B^2(r) \times \mathbf{R}$ and we have the following result.

3.3. Theorem. *The group G above is a quasiconformal group of $B^2(r) \times \mathbf{R}$ that is not quasiconformally conjugate to any Möbius group.*

Proof. Suppose that there exists a quasiconformal map h so that $\Gamma = hGh^{-1}$ is a Möbius group. Let $\Gamma_j = hG_jh^{-1}$ where G_1 and G_2 are as in 3.2. By Lemmas 2.3 and 2.4, after conjugation, Γ_1 is a group of rotations about the x_3-axis and Γ_2 is a group of translations along that axis and $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$. Let $h^{-1}(b) = a \in S \times \mathbf{R}$. Then $G(a) = h\bigl(\Gamma(b)\bigr) = S \times \mathbf{R}$, or

$$h^{-1}(S \times \mathbf{R}) = \Gamma(b) \subset S^1(r) \times \mathbf{R},$$

which cannot be, by [T2; Theorem 5]. Hence G is not quasiconformally conjugate to any Möbius group. □

3.4. Completion of the proof of Theorem 1.1. Let g_0 map $B^2(r) \times \mathbf{R}$ onto B^3 quasiconformally. Then

$$G' = g_0 G g_0^{-1}$$

is a quasiconformal group of B^3 that is not quasiconformally conjugate to any Möbius group. □

4. Remarks

Let C be a quasicircle through ∞ that contains a nonrectifiable bounded subarc and let σ be a bilipschitz reflection in σ. Then $\langle \sigma x \times \mathrm{id} \rangle$ is a (sense-reversing) QC group leaving the upper half space in $\mathbf{R}^3$ invariant. This observation is due to Väisälä, and Martio [Remark 4.29, MV].

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References

[A] Ahlfors, L.: Möbius transformations in several dimensions. - University of Minnesota Lecture Notes, 1981.

[B] Beardon, A.: The Geometry of Discrete Groups. - Springer-Verlag, 1983.

[FM] Falconer, K., and T. Marsh: Classification of quasi-circles by Hausdorff dimension. - Nonlinearity 2, 1989, 489–493.

[G1] Gehring, F.: Injectivity of local quasi-isometries. - Comment. Math. Helv. 57, 1982, 202–220.

[G2] Gehring, F.: Extension of quasiisometric embeddings of Jordan curves. - Complex Variables Theory Appl. 5, 1986, 245–263.

[GP] Gehring, F., and B. Palka: Quasiconformally homogeneous domains. - J. Analyse Math. 30, 1976, 172–199.

[GNV] Ghamsari, M., R. Näkki and J. Väisälä: John disks and extension of maps. - Mh. Math. 117, 1994, 63–94.

[M] Martin, G.: Discrete QC groups that are not the quasiconformal conjugates of Möbius groups. - Ann. Acad. Sci. Fenn. Ser. A I Math. 11, 1986, 174–202.

[MV] Martio, O., and J. Väisälä: Elliptic equations and maps of bounded length distortions. - Math. Ann. 282:3, 1988, 423–443.

[S] Sullivan, D.: The ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. - Ann. of Math. Stud. 97, Princeton University Press, 1981, 465–496.

[T1] Tukia, P.: On two dimensional quasiconformal groups. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 73–78.

[T2] Tukia, P.: A quasiconformal group not conjugate to a Möbius group. - Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1981, 149–160.

[V1] Väisälä, J.: Lectures on n-dimensional Quasiconformal Mapping. - Lecture Notes in Math. 229, Springer-Verlag, Berlin–Heidelberg–New York 1971.

[V2] Väisälä, J.: Quasimöbius maps. - J. Analyse Math. 44, 1985/84, 218–234.

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