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TANGENTIAL LIMITS OF MONOTONE SOBOLEV FUNCTIONS

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Abstract. Tangential limits have been discussed by several authors for harmonic functions with finite Dirichlet integral. This paper deals mostly with tangential limits for monotone functions in the half space of R^n, which are extensions of monotone functions on the one dimensional space R^1. Harmonic functions together with solutions in a wider class of nonlinear elliptic equations are monotone in our sense; of course, the coordinate functions of quasiregular mappings are monotone. We first give the fine limit result for Sobolev functions, and then apply the estimate of the oscillations over balls by the p-th means of partial derivatives over balls.

1. Introduction

Our aim in this paper is to study tangential boundary limits of monotone functions u with finite Dirichlet integral in the half space $R^n_+ = \{x = (x_1, \dots, x_{n-1}, x_n) : x_n > 0\}$, $n \geqq 2$. We say that u has finite Dirichlet integral if

$$\int_{R^n_+} |\operatorname{grad} u(x)|^n \, dx < \infty. \tag{1}$$

Further we say that a continuous function u on R^n_+ is monotone (in the sense of Lebesgue) if

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u$$

hold for any relatively compact open set G in R^n_+, where $\overline{G} = G \cup \partial G$ (see Vuorinen [22], [23]). Harmonic functions, (weak) solutions in a wider class of (non)linear elliptic partial differential equations and the coordinate functions of quasiregular mappings are monotone (see e.g. Gilbarg–Trudinger [5], Heinonen–Kilpeläinen–Martio [6], Reshetnyak [19], Serrin [20] and Vuorinen [23]). For $\gamma \geqq 1$, $\xi \in \partial R^n_+$ and $a > 0$, consider the set

$$T_\gamma(\xi; a) = \{x = (x_1, \dots, x_n) \in R^n_+ : |x - \xi|^\gamma < a x_n\}.$$

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If $\lim_{x\to\xi, x\in T_\gamma(\xi,a)} u(x) = \ell$ for every $a > 0$, then u is said to have a T_γ-limit ℓ at ξ; u is said to have a nontangential limit at ξ if it has a T_1-limit at ξ. We say further that u has a T_∞-limit ℓ at $\xi \in \partial R^n_+$ if

$$\lim_{x\to\xi, x\in T_\gamma(\xi,a)} u(x) = \ell$$

for every $\gamma > 1$ and $a > 0$ (cf. [14]). To evaluate the size of exceptional sets, we use the capacity

$$C_{1,p}(E; G) = \inf \|f\|_p{}^p,$$

where G is an open set in R^n and the infimum is taken over all nonnegative measurable functions f such that $f = 0$ outside G and

$$\int |x-y|^{1-n} f(y)\, dy \geqq 1 \qquad \text{for every } x \in E;$$

see [8] for the basic properties of capacity. Since $C_{1,p}(E; R^n) = 0$ for any set E when $p \geqq n$, we write $C_{1,p}(E) = 0$ simply if

$$C_{1,p}(E \cap G; G) = 0 \qquad \text{for every bounded open set } G.$$

In case $1 < p < n$, $C_{1,p}(E) = 0$ if and only if $C_{1,p}(E; R^n) = 0$.

Our main aim in this paper is to establish the following theorem.

Theorem 1. *If u is a monotone function on R^n_+ satisfying* (1)*, then u has a finite T_∞-limit at every boundary point except for a set $E \subset \partial R^n_+$ such that $C_{1,n}(E) = 0$.*

The nontangential case for harmonic functions has been dealt by many mathematicians (cf. Beurling [1], Carleson [2], Gavrilov [4], Wallin [24] and the author [11]). Miklyukov [10] discussed the nontangential limits for quasiregular mappings with finite Dirichlet integral. Recently, Manfredi and Villamor [7] have proved the existence of nontangential limits for monotone functions on the unit ball. The present tangential limit result for harmonic functions was obtained by Cruzeiro [3].

It is well-known (through an application of change of variables) that the coordinate functions of bounded quasiconformal mappings defined on R^n_+ have finite Dirichlet integral, so that Theorem 1 gives the following result (see Väisälä [21] and Vuorinen [23] for the definition of quasiconformal mappings).

Corollary 1. *Every coordinate function of bounded quasiconformal mappings on R^n_+ has a finite T_∞-limit at every boundary point except for a set $E \subset \partial R^n_+$ such that $C_{1,n}(E) = 0$.*

This corollary gives an affirmative answer to the open problem given by Vuorinen [23, 15.16].

The key of proving our theorem mentioned above is the fact that if u is monotone on the ball $B(x,r)$ centered at x with radius r and $n-1 < p \leqq n$, then

$$|u(x) - u(y)|^p \leqq M r^{p-n} \int_{B(x,r)} |\operatorname{grad} u(z)|^p \, dz$$

whenever $y \in B(x, r/2)$, with a positive constant M independent of r (see [7, Remark, p. 9] and [23, Section 16]). In the author's paper [12], we used this inequality to study the existence of nontangential limits of weak solutions for nonlinear Laplace equations.

We also give an improvement of the result by Manfredi and Villamor [7] concerning the Lindelöf-type theorem.

Theorem 2. *Let u be a monotone function on R^n_+ satisfying*

$$\int_{R^n_+} |\operatorname{grad} u(x)|^p \, dx < \infty \tag{2}$$

for $n-1 < p \leqq n$. Consider the set

$$E_p = \left\{ \xi \in \partial R^n_+ : \limsup_{r\to 0} r^{p-n} \int_{R^n_+ \cap B(\xi,r)} |\operatorname{grad} u(y)|^p \, dy > 0 \right\}.$$

If $\xi \in \partial R^n_+ - E_p$ and u has a finite limit along a rectifiable curve in R^n_+ ending to ξ, then u has a nontangential limit at ξ.

It is easy to see that E_p has $(n-p)$-dimensional Hausdorff measure zero; in case $p = n$, the exceptional set in this discussion is empty as was also pointed out in [7, Theorem 2], because $E_n = \emptyset$. Note further that $C_{1,p}(E_p) = 0$.

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2. Preliminary lemmas

Throughout this paper, let M denote various constants independent of the variables in question.

To represent Sobolev functions in the integral form, we use the kernel functions

$$k_j(x,y) = \begin{cases} (x_j - y_j)|x-y|^{-n}, & y \in B(0,1), \\ (x_j - y_j)|x-y|^{-n} - (-y_j)|y|^{-n}, & y \in R^n - B(0,1). \end{cases}$$

Lemma 1 (cf. [16, Lemma 3]). *Let u be a continuous function on R^n_+ satisfying (2). Then there exist functions $u_j \in L^n(R^n)$ and a constant C for which*

$$u(x) = \sum_{j=1}^{n} \int_{R^n} k_j(x,y) u_j(y) \, dy + C$$

for every $x \in R^n_+$.

In view of Lemma 1, we have for $x \in B(0,N) \cap R^n_+$, $N > 0$,

$$u(x) = \sum_{j=1}^{n} \int_{B(0,2N)} (x_j - y_j)|x-y|^{-n} u_j(y)\, dy + v_N(x) \tag{3}$$

with a continuous function v_N on $B(0,N)$. For a function u satisfying (1), set

$$A_1 = \left\{ \xi \in \partial R^n_+ : \int_{B(\xi,1)} |\xi - y|^{1-n} f(y)\, dy = \infty \right\}$$

and

$$A_2 = \left\{ \xi \in \partial R^n_+ : \limsup_{r \to 0} [\log(1/r)]^{n-1} \int_{B(\xi,r)} f(y)^n\, dy > 0 \right\},$$

where $f = \sqrt{|u_1|^2 + \cdots + |u_n|^2}$.

The following is easy.

Lemma 2. *Let u be a continuous function on R^n_+ satisfying (1). Then*

$$C_{1,n}(A_1) = 0.$$

If $h(r) = [\log(2+1/r)]^{1-n}$, then A_2 has zero Hausdorff measure with respect to the measure function h, that is,

$$H_h(A_2) = 0.$$

In view of [9], we find

Lemma 3. *Let u be a continuous function on R^n_+ satisfying (1). Then*

$$C_{1,n}(A_2) = 0.$$

Lemma 4 (cf. [14, Theorem 2′ and Remark 1]). *Let u be a continuous function on R^n_+ satisfying (1). If $\xi \in \partial R^n_+ - (A_1 \cup A_2)$, then there exists a set $E \subset R^n_+$ such that*

(i) $\lim_{x \to \xi, x \in T_\gamma(\xi,a) - E} u(x)$ *exists and is finite for any $\gamma > 1$ and any $a > 0$;*
(ii) $\lim_{j \to \infty} j^{n-1} C_{1,n}(E_j; B_j) = 0$,

where $E_j = \{x \in E : 2^{-j} \leqq |x| < 2^{-j+1}\}$ and $B_j = \{x \in R^n : 2^{-j-1} < |x| < 2^{-j+2}\}$.

Proof. Let $\xi \in B(0,N) \cap \partial R^n_+$. If we note (3), then we may assume that

$$u(x) = \sum_{j=1}^{n} \int_{B(0,2N)} (x_j - y_j)|x-y|^{-n} u_j(y)\, dy$$

for every $x \in R^n_+$. The right-hand side is denoted by $\overline{u}(x)$ for $x \in R^n$, at which the integrals are convergent in absolute value. For $x \in R^n_+$, write

$$U_1(x) = \sum_{j=1}^{n} \int_{B(0,2N)-B(\xi,2|x-\xi|)} (x_j - y_j)|x-y|^{-n} u_j(y)\, dy,$$

$$U_2(x) = \sum_{j=1}^{n} \int_{B(\xi,2|x-\xi|)-B(x,x_n/2)} (x_j - y_j)|x-y|^{-n} u_j(y)\, dy,$$

$$U_3(x) = \sum_{j=1}^{n} \int_{B(x,x_n/2)} (x_j - y_j)|x-y|^{-n} u_j(y)\, dy.$$

Since $\xi \notin A_1$, $\overline{u}(\xi)$ is finite, so that we apply Lebesgue's dominated convergence theorem to see that

$$\lim_{x \to \xi} U_1(x) = \overline{u}(\xi).$$

As before, set

$$f = \sqrt{|u_1|^2 + \cdots + |u_n|^2}.$$

By Hölder's inequality we have for $x \in B(\xi, 1/2) \cap R^n_+$

$$|U_2(x)| \leqq M\biggl(\bigl[\log(4|x-\xi|/x_n)\bigr]^{n-1} \int_{B(\xi,2|x-\xi|)} f(y)^n\, dy \biggr)^{1/n},$$

so that, since $\xi \notin A_2$,

$$\lim_{x \to \xi,\, x \in T_\gamma(\xi,a)} U_2(x) = 0.$$

For a sequence $\{b_j\}$ of positive numbers, we consider the sets

$$E_j = \bigl\{ x : 2^{-j} \leqq |x| < 2^{-j+1}, |U_3(x)| \geqq b_j^{-1} \bigr\}, \qquad j = 1, 2, \ldots,$$

and

$$E = \bigcup_{j=1}^{\infty} E_j.$$

Note that for $x \in E_j$,

$$|U_3(x)| \leqq \int_{B(x,x_n/2)} |x-y|^{1-n} f(y)\, dy \leqq \int_{B_j} |x-y|^{1-n} f(y)\, dy.$$

Consequently we have by the definition of capacity

$$C_{1,n}(E_j; B_j) \leqq {b_j}^n \int_{B_j} f(y)^n \, dy.$$

Since $\xi \notin A_2$, we can find a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} b_j = \infty$ but

$$\lim_{j\to\infty} {b_j}^n j^{n-1} \int_{B_j} f(y)^n \, dy = 0.$$

Now it follows that

$$\lim_{j\to\infty} j^{n-1} C_{1,n}(E_j; B_j) = 0.$$

On the other hand, we see that

$$\limsup_{x\to 0, x\in R^n_+ - E} |U_3(x)| \leqq \limsup_{j\to\infty} b_j^{-1} = 0.$$

Thus Lemma 4 is established.

Lemma 5 (cf. [18, Lemma 7.3]. *If* $x \in T_\gamma(\xi, a)$ *and* $x_n < 1/2$, *then*

$$C_{1,n}\big(B(x, x_n/2); B(\xi, 2|x-\xi|)\big) \sim [\log(4|x-\xi|/x_n)]^{1-n}.$$

Proof. For our later use, we shall show only that

$$C_{1,n}\big(B(x, x_n/2); B(\xi, 2|x-\xi|)\big) \geqq M[\log(4|x-\xi|/x_n)]^{1-n}.$$

For this purpose, take a nonnegative function f such that $f = 0$ outside $B(\xi, 2|x-\xi|)$ and

$$\int |z-y|^{1-n} f(y)\, dy \geqq 1 \qquad \text{for every } z \in B(x, x_n/2).$$

Then we have by Fubini's theorem and Hölder's inequality

$$\begin{aligned}
\int_{B(x,x_n/2)} dz &\leqq \int_{B(x,x_n/2)} \left(\int |z-y|^{1-n} f(y)\, dy \right) dz \\
&= \int \left(\int_{B(x,x_n/2)} |z-y|^{1-n}\, dz \right) f(y)\, dy \\
&\leqq M{x_n}^n \int_{B(x,3|x-\xi|)} (x_n + |x-y|)^{1-n} f(y)\, dy \\
&\leqq M{x_n}^n [\log(3|x-\xi|/x_n)]^{(n-1)/n} \|f\|_n,
\end{aligned}$$

which yields the required inequality.

3. Proof of Theorem 1

For $\xi \in \partial R^n_+ - (A_1 \cup A_2)$, take a set E as in Lemma 4. Since u is monotone on R^n_+,

$$|u(x) - u(y)|^n \leqq M \int_{B(x,x_n)} |\operatorname{grad} u(z)|^n \, dz \tag{4}$$

whenever $y \in B(x, x_n/2)$, where $x = (x_1, \dots, x_n) \in R^n_+$ (see [7] and [23]). If $x \in T_\gamma(\xi, a)$, then Lemma 5 implies that $B(x, x_n/2) - E$ is not empty, so that there exists $y(x) \in B(x, x_n/2) - E$ (when x_n is small enough). Then we see from (4) that

$$\lim_{x \to \xi, x \in T_\gamma(\xi,a)} \bigl|u(x) - u\bigl(y(x)\bigr)\bigr| = 0.$$

Hence it follows that

$$\lim_{x \to \xi, x \in T_\gamma(\xi,a)} u(x) = \lim_{x \to \xi, x \in T_\gamma(\xi,a)} u\bigl(y(x)\bigr),$$

so that the limit on the left exists and is finite. Thus $E = A_1 \cup A_2$ has all the required properties, with the aid of Lemmas 2 and 3.

4. Proof of Theorem 2

Manfredi and Villamor [7] proved the following result concerning the existence of nontangential limits.

Theorem 3 ([7]). *Let u be a monotone function on R^n_+ satisfying* (2) *for $n-1 < p < n$. Then u has a nontangential limit at every $\xi \in \partial R^n_+ - E$, where $C_{1,p}(E) = 0$.*

As will be shown soon, we may take the above E as

$$E = A_1 \cup A_{2,p},$$

where A_1 is defined as after Lemma 1, that is,

$$A_1 = \left\{ \xi \in \partial R^n_+ : \int_{B(\xi,1)} |\xi - y|^{1-n} f(y) \, dy = \infty \right\}$$

and

$$A_{2,p} = \left\{ \xi \in \partial R^n_+ : \limsup_{r \to 0} r^{p-n} \int_{B(\xi,r)} f(y)^p \, dy > 0 \right\},$$

where $f(y) = |\operatorname{grad} u(y)|$.

Now we give a proof of Theorem 2. Our end in this direction is to show that the condition that $\xi \notin A_1$ may be replaced by the existence of asymptotic values.

Proof of Theorem 2. Without loss of generality we may assume that $\xi = 0$. Let C be a rectifiable curve in R^n_+ tending to $\xi = 0$, and assume that u has a finite limit along C. For $2^{-j} \leqq r < 2^{-j+1}$, take a point $C(r)$ on $C \cap \partial B(0,r)$. Letting $e(r) = (0,\dots,0,r)$, we see that

$$|u(e(r)) - u(y)|^p \leqq M r^{p-n} \int_{B(e(r),r/2)} f(z)^p\,dz \tag{5}$$

whenever $y \in B(e(r),r/2)$, where $f(z) = |\operatorname{grad} u(z)|$. Moreover, letting θ denote the angle between the x_n-axis and the vector $\vec{0x}$, we find

$$|u(y) - u(C(r))| \leqq \int_0^\pi f(r\Theta) r\,d\theta$$

along the circular arc $\{r\Theta\}$ through y and $C(r)$, for $y \in B(e(r),r/2) \cap \partial B(0,r)$. By Hölder's inequality we have

$$\begin{aligned} |u(y) - u(C(r))| &\leqq \left(\int_0^\pi \left[r^{1-(n-1)/p} (\sin\theta)^{-(n-2)/p}\right]^{p'} d\theta\right)^{1/p'} \\ &\quad \times \left(\int_0^\pi f(r\Theta)^p r^{n-1} \sin^{n-2}\theta\,d\theta\right)^{1/p} \\ &\leqq M r^{1-(n-1)/p} \left(\int_0^\pi f(r\Theta)^p r^{n-1} \sin^{n-2}\theta\,d\theta\right)^{1/p}, \end{aligned}$$

where $1/p + 1/p' = 1$. Hence it follows that

$$\inf_{y \in B(e(r),r/2)} |u(y) - u(C(r))|^p \leqq M r^{p-n+1} \int_0^\pi f(r\Theta)^p r^{n-1} \sin^{n-2}\theta\,d\theta,$$

so that, by considering polar coordinates with the north pole $C(r)/|C(r)|$, we have

$$\inf_{y \in B(e(r),r/2)} |u(y) - u(C(r))|^p \leqq M r^{p-n+1} \int_{\partial B(0,r)} f(z)^p\,dS(z).$$

By integrating both sides with respect to r, we obtain

$$\int_{2^{-j}}^{2^{-j+1}} \left[\inf_{y \in B(e(r)),r/2)} |u(y) - u(C(r))|^p\right] r^{n-p-1} dr \leqq M \int_{B_j} f(z)^p\,dz.$$

Now we can find r_j such that $2^{-j} \leqq r_j < 2^{-j+1}$ and

$$\inf_{y \in B(e(r_j),r_j/2)} |u(y) - u(C(r_j))|^p \leqq M 2^{j(n-p)} \int_{B_j} f(z)^p\,dz.$$

In view of (5),

$$\big|u\big(e(r_j)\big) - u\big(C(r_j)\big)\big|^p \leqq M 2^{j(n-p)} \int_{B_j} f(z)^p \, dz.$$

If the origin does not belong to $A_{2,p}$, then this implies that $\{u\big(e(r_j)\big)\}$ has a finite limit as $j \to \infty$. Applying (5) again, we see that

$$\big|u(x) - u\big(e(r_j)\big)\big|^p \leqq M 2^{j(n-p)} \int_{B_j \cap T_1(0,2a)} |\operatorname{grad} u(z)|^p \, dz$$

for all $x \in T_1(0,a)$ with $2^{-j} \leqq |x| < 2^{-j+1}$. Therefore u has a nontangential limit at the origin. Thus we have proved that u has a nontangential limit at every $\xi \in \partial R^n_+ - A_{2,p}$, and the proof of Theorem 2 is completed.

Theorem 4. *Let u be a monotone function on R^n_+ satisfying* (2) *for $n-1 < p < n$, and let $\gamma > 1$. Then there exists $E_\gamma \subset \partial R^n_+$ such that $H_{\gamma(n-p)}(E_\gamma) = 0$ and u has a finite T_γ-limit at every $\xi \in \partial R^n_+ - E_\gamma$.*

In fact we may take $E_\gamma = A_1 \cup A_{2,p,\gamma}$, where

$$A_{2,p,\gamma} = \left\{ \xi \in \partial R^n_+ : \limsup_{r \to 0} r^{\gamma(p-n)} \int_{B(\xi,r)} |\operatorname{grad} u(y)|^p \, dy > 0 \right\}.$$

In the harmonic case, we refer the reader to [15].

Theorem 5. *Let u be a harmonic function on R^n_+ satisfying* (1) *and $\xi \in \partial R^n_+ - A_2$. If u has a finite limit along a rectifiable curve in R^n_+ ending to ξ, then u has a T_∞-limit at ξ.*

To prove this theorem, let $\xi = 0$. It suffices to note that

$$\big|u(x) - u\big(e(|x|)\big)\big|^n \leqq M \big[\log(2|x|/x_n)\big]^{n-1} \int_{B_j \cap R^n_+} |\operatorname{grad} u(z)|^n \, dz$$

for all $x \in T_\gamma(0,a)$ with $2^{-j} \leqq |x| < 2^{-j+1}$ (see [17, Theorem 1 and its proof]), because Theorem 2 implies that $u\big(e(|x|)\big)$ has a limit at the origin.

5. Remarks

Remark 1. According to [15, Remark 5], for given $\gamma > 1$ and $1 < p \leqq n$, we can find a harmonic function u on R^n_+ satisfying (2) such that

(i) u has a nontangential limit at the origin.

(ii) $\limsup_{x \to 0, x \in T_{\gamma'}(0,a') - T_{\gamma'}(0,a)} u(x) = \infty$ for every $\gamma' > \gamma$ and $a' > a$.

This shows that the assumption $\xi \in \partial R^n_+ - A_2$ is needed in Theorem 5.

Remark 2. We do not know whether Theorem 5 remains valid for general monotone functions or not.

Remark 3. For $x^{(j)} = (2^{-j}, 0, \dots, 0) \in \partial R^n_+$ and $0 < r_j < 2^{-j-1}$, consider the sets

$$B_j = \left[B(x^{(j)}, 2^{-j-2}s_j) - B(x^{(j)}, r_j s_j)\right] - R^n_+, \quad \text{where } s_j = \left(\log \frac{1}{2^j r_j}\right)^{(2-n)/n}.$$

Suppose $\{r_j\}$ is chosen so small that

$$\sum_j \left(\log \frac{1}{2^j r_j}\right)^{1-n} < \infty;$$

if this is the case, $B = \bigcup_j R^n_+ \cap B(x^{(j)}, r_j)$ is called $C_{1,n}$-thin at the origin in the sense of [13]. Taking a sequence $\{a_j\}$ of positive numbers such that

$$\lim_{j\to\infty} a_j = \infty$$

and

$$\sum_j {a_j}^n \left(\log \frac{1}{2^j r_j}\right)^{1-n} < \infty, \tag{6}$$

we now define

$$f(y) = \begin{cases} a_j \left(\log \dfrac{1}{2^j r_j}\right)^{-1} |x^{(j)} - y|^{-1} & \text{when } y \in B_j, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$u(x) = \int_{R^n} \frac{x_n - y_n}{|x-y|^n} f(y)\, dy, \quad x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n).$$

Then, as in [13, Proposition], we can prove:

(i) u is a harmonic function on R^n_+ with finite Dirichlet integral.
(ii) u has a nontangential limit at the origin.
(iii) $\lim_{j\to\infty} u(x^{(j)} + (0, \dots, 0, r_j)) = \infty$.

To show (i) and (ii), we note by (6) that

$$\int f(y)^n\, dy \leqq M \sum_j {a_j}^n \left(\log \frac{1}{2^j r_j}\right)^{-n+1} < \infty$$

and

$$u(0) = \int (-y_n)|y|^{-n} f(y)\,dy$$
$$\leqq M \sum_j a_j \Bigl(\log \frac{1}{2^j r_j}\Bigr)^{-1} 2^{jn} \int_{B_j} (-y_n)|x^{(j)} - y|^{-1}\,dy$$
$$\leqq M \sum_j a_j \Bigl(\log \frac{1}{2^j r_j}\Bigr)^{-n+1} < \infty.$$

Finally we see that for $x \in R^n_+ \cap B(x^{(j)}, r_j)$,

$$u(x) \geqq M a_j \Bigl(\log \frac{1}{2^j r_j}\Bigr)^{-1} \int_{r_j s_j}^{2^{-j-2} s_j} (|x - x^{(j)}| + r)^{1-n} r^{-1} r^{n-1}\,dr \geqq M a_j,$$

which implies that

$$\lim_{x\to 0, x\in B} u(x) = \infty.$$

Remark 4. The examples in Remarks 1 and 3 show that the existence of nontangential limits may not always imply that of tangential limits.

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