Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 333–348

# NEGATIVELY CURVED GROUPS HAVE THE CONVERGENCE PROPERTY I

**Eric M. Freden**

Brigham Young University, 292 TMCB
Provo, UT 84602, U.S.A.; freden@sofya.math.byu.edu

**Abstract.** It is known that the Cayley graph $\Gamma$ of a negatively curved (Gromov-hyperbolic) group $G$ has a well-defined boundary at infinity $\partial\Gamma$. Furthermore, $\partial\Gamma$ is compact and metrizable. In this paper I show that $G$ acts on $\partial\Gamma$ as a convergence group. This implies that if $\partial\Gamma \simeq \mathbf{S^1}$, then $G$ is topologically conjugate to a cocompact Fuchsian group.

## 0. Introduction

The theory of convergence groups was first introduced by Gehring and Martin [7] as a natural generalization of Möbius groups. All discrete quasiconformal groups display the convergence property and at first glance this fact would seem to imply a much larger class of groups. However, the work of Tukia [14], Gabai [5], et al, shows that every convergence group acting on $S^1$ is in fact conjugate to a Möbius group by a homeomorphism of $S^1$ (compare with 3.3 of [8]). Although this is false in the case of $S^2$, all known counterexamples share the same construction technique [10].

This paper deals exclusively with *discrete* (in the compact-open topology) convergence groups acting on metric spaces. Let $(X, \mathrm{d})$ be such a metric space. In most of the literature, $X$ is either $S^n$ or $\bar{B}^n$ although many results can be generalized. We say that $G \subset \mathrm{Homeo}(X)$ is a *convergence group* if given any sequence $\{g_m\}$ of distinct group elements, there exist (not necessarily distinct) points $x, y \in X$ and a subsequence $\{g_n\}$ such that

$$\begin{aligned} g_n(z) &\to x \quad \text{locally uniformly on } X \setminus \{y\}, \text{ and} \\ g_n^{-1}(z) &\to y \quad \text{locally uniformly on } X \setminus \{x\}. \end{aligned}$$

(Here "locally uniformly" means uniformly on compact subsets, i.e. if $C \subset X \setminus \{y\}$ is compact and $U$ is a neighborhood of $x$, then $g_n(C) \subset U$ for all sufficiently large $n$.) Tukia has termed the above criteria (CON) and I will do the same.

1991 Mathematics Subject Classification: Primary 20F32; Secondary 57S05.

This research was supported in part by the Geometry Center, University of Minnesota, an STC funded by the NSF, DOE, and Minnesota Technology, Inc.; and by an NSF research grant; also by Brigham Young University.

The idea of negatively curved groups is due to Gromov [8]. Other synonyms in current use are *Gromov hyperbolic, word hyperbolic* or merely *hyperbolic*. The fact that Gromov came up with a good idea has become evident in the last few years—the entire research area of geometric group theory has exploded with the introduction of negative curvature. All negatively curved groups are finitely presented. A nice argument showing that negative curvature is a group invariant can be found in [13], see also [3]. The word and conjugacy problems can always be solved for negatively curved groups [1], [8]. More recently Sela [12] has announced that the isomorphism problem is also solvable in negatively curved groups. Since negatively curved groups are in some sense generic [11], the unsolvability of the above problems is the exception rather than the rule for finitely presented groups.

This paper links negatively curved groups and convergence groups. Some results in this area are already known. For instance, let $M$ denote a closed riemannian $n$-manifold with all sectional curvatures less than some negative constant. By Toponogov's comparison theorem [4], the universal cover $\widetilde{M}$ has thin triangles. The group $G = \pi_1(M)$ acts properly discontinuously, cocompactly and by isometry on both its Cayley graph $\Gamma$ and $\widetilde{M}$; hence they are quasi-isometric [2], [6]. Gromov has shown that negative curvature is a quasi-isometry invariant [6], [8], therefore $G$ is negatively curved. Martin and Skora [10] show that $G$ also acts as a convergence group on the boundary at infinity of $\widetilde{M}$. The main theorem in this paper generalizes the above via combinatorial methods. Pekka Tukia has also recently proved Theorem 3.4 using other methods [15].

In Section 1, I review definitions and some known results. Section 2 contains some technical lemmas. The main theorem is proved in Section 3 along with some related material. I acknowledge with gratitude the encouragement of Jim Cannon. Eric Swenson developed some of the techniques and groundwork used in Section 1. Dave Gabai, Steve Humphries, Gaven Martin, Bernard Maskit, and the referee provided useful comments. I am indebted to Brigham Young University for a stimulating research environment as well as for travel support.

## 1. Preliminaries

Let $(X, \mathrm{d})$ be a metric space, with $a, b \in X$. A *path* connecting $a$ and $b$ is the image of a continuous function $\alpha\colon [0,1] \to X$ satisfying $\alpha(0) = a$ and $\alpha(1) = b$. The *length* of this path is defined by

$$\mathrm{length}(\alpha) = \sup \sum_{i=1}^{n} \mathrm{d}\big(\alpha(x_{i-1}), \alpha(x_i)\big)$$

where the supremum is taken over all finite partitions $\{0 = x_0, x_1, \dots, x_n = 1\}$ of $[0,1]$. $X$ is a *path metric* space if for any $a, b \in X$ there exists a path $\alpha$ connecting $a$ and $b$ with $\mathrm{d}(a,b) = \mathrm{length}(\alpha)$. Any such path realizing the distance between endpoints is referred to as *geodesic*. If there is no ambiguity it is easier to denote a geodesic path between $a$ and $b$ by $\overline{ab}$. A *ray* is the image of a continuous map $R\colon [0,\infty) \to X$. (It is convenient to refer to both the map and its image as $R$, and to refer to $R(t)$ merely as $t$ if the context is clear.) The ray $R$ is geodesic if every finite sub-segment of $R$ is geodesic, i.e. $R$ is an isometry onto its image. Geodesic lines are defined similarly.

Let $(X, \mathrm{d})$ be a path metric space. A (geodesic) *triangle* $\Delta(a,b,c) \subset X$ consists of three distinct points $a,b,c \in X$ (called *vertices*) and three geodesic segments $\overline{ab}$, $\overline{bc}$, and $\overline{ac}$ (called *edges*). It is not true in general that such a triangle is determined by its vertices (consider $X = S^2$).

**Definition 1.1.** Let $\delta \geq 0$. A path metric space $(X, \mathrm{d})$ is *negatively curved* $(\delta)$ or $\delta$*-hyperbolic* if for each geodesic triangle $\Delta(a,b,c) \subset X$ and for each $x \in \overline{ab}$ it is true that

$$\mathrm{d}(x, \overline{bc} \cup \overline{ac}) < \delta.$$

The triangle $\Delta(a,b,c)$ is said to be $\delta$*-thin* (see Figure 1.1).

Figure 1.1

In most of the literature the above inequality is not strict—however, several of the lemmas proved below are shorter using strict inequality. It is easy to see that the two definitions are equivalent.

The thin triangles condition has been attributed to Rips. There are many equivalent definitions of negative curvature, for example, exponential divergence of rays [2], an inequality with respect to the "overlap" or generalized inner-product [8], and a linear inequality relating area to perimeter [8]. Thin triangles seems to be the most intuitive of the above and will be used in this paper. Let $\varepsilon > 0$. Examples of negatively curved path metric spaces are trees which have $\varepsilon$-thin triangles and $H^n$, hyperbolic $n$-space, which has thin triangles with $\delta = \log(1+\sqrt{2}\,) + \varepsilon$.

*Exercise* 1.2. If $X$ has thin triangles ($\delta$), then quadrilaterals are $2\delta$-thin (refer to Figure 1.1).

**Definition 1.3.** Let $G$ denote a group with finite generating set $C$, closed with respect to inverses. Let $\Gamma = \Gamma(G, C)$ denote the *Cayley graph* of $G$ with respect to the given generating set. This is a simplicial 1-complex with one vertex for each group element. The directed edge set is $E = \{(h, c, hc) : h \in G, c \in C\}$ where $h, hc$ represent the initial and terminal vertices respectively, and $c$ is the label of the segment in between (see Figure 1.2).

Figure 1.2

It is not hard to see that $\Gamma$ is homogeneous—every vertex looks like every other vertex. It is often convenient to pick a specific vertex as an origin. Usually this vertex will be denoted as 0 and will correspond to the group identity. The group $G$ acts on $\Gamma$ by left multiplication. If $h$ is a vertex, $(h, c, hc)$ is a directed edge and $g \in G$, then $g \cdot h = gh$ represents another vertex and $g \cdot (h, c, hc)$ is the directed edge $(gh, c, ghc)$. Depending on the context, a word $c_1 c_2 \ldots c_k$ can represent

i) a group element in $G$
ii) the vertex of $\Gamma$ labelled $c_1 c_2 \ldots c_k$
iii) the edge path from 0 to the vertex in ii)
iv) an edge path from any vertex $h$ to the vertex $hc_1 c_2 \ldots c_k$.

Consider each edge as being isometric to the unit interval. In this way $\Gamma$ becomes a locally compact path metric space. The distance function is referred to as the *word metric*. A minimal representation for a group element $h$ becomes a geodesic edge path from 0 to $h$; relators correspond to closed loops in $\Gamma$. By definition, $G$ acts on $\Gamma$ freely and properly discontinuously as a group of isometries. For each generator $c \in C$, consider the set of open half edges emanating from 0. The union of these edges, along with the vertex 0 forms a fundamental domain $D$ for the group action, as can be seen by checking the following conditions:

i) $g(D) \cap D = \emptyset$ for all $g$ except the identity,
ii) every $z \in \Gamma$ is $G$ equivalent to a point in $D$,
iii) the "sides" of $\overline{D}$ are paired by elements of $G$,
iv) if $K \subset \Gamma$ is compact, $g(\overline{D}) \cap K = \emptyset$ except for finitely many $g \in G$.

Evidently $\Gamma/G$ is a bouquet of circles, with one circle for each generator. Any compact subset of $\Gamma$ is contained in the union of finitely many edges and vertices. Bearing this in mind, the following is immediate.

*Exercise* 1.4. If $V, W \subset \Gamma$ are compact sets, then $g(V) \cap W = \emptyset$ for all but finitely many $g \in G$.

Call a metric space *proper* if the closure of every metric ball is compact (i.e. the Heine–Borel theorem holds).

**Definition 1.5.** A finitely generated group $G$ is *negatively curved*, or *word hyperbolic* if the corresponding Cayley graph $\Gamma$ is negatively curved.

**Definition 1.6.** Let $(X, \mathrm{d})$ be a proper path metric space with $\delta$-thin triangles for some fixed $\delta > 0$. Two geodesic rays $R, S\colon [0, +\infty) \to X$ are *equivalent*, written $R \sim S$, if

$$\limsup_{t \to +\infty} \mathrm{d}\big(R(t), S(t)\big) < +\infty.$$

Another way to say this is that $R$ strays at most a bounded distance from $S$ and vice versa. Indicate by $[R]$ the equivalence class containing $R$.

*Exercise* 1.7. In fact, the bounded distance mentioned above is (asymptotically) at most $2\delta$. (Hint: quadrilaterals are $2\delta$-thin.)

One might worry about rays that start at different places. In fact, this is usually not a problem. Given any ray $R$ and point $x \in X$ there is a ray $S$ starting from $x$ that is equivalent to $R$ (see I.2 of [13]). Therefore if $x$ and $y$ are distinct points of $X$, there is a bijection between the set of ray classes starting at $x$ and those starting at $y$.

**Definition 1.8.** The boundary at infinity $\partial X$ is defined as the set of equivalence classes of (geodesic) rays.

It is necessary to put a topology on $\partial X$ that is independent of ray base points. The following is a generalization of classical hyperbolic geometry.

**Definition 1.9.** If $R$ is a ray in $X$ (or more generally any closed set) and $x \in X$, define a relation (multi-valued "function") by

$$\mathbf{p}_R(x) = \big\{ r \in R : \mathrm{d}(x, r) = \mathrm{d}(x, R) \big\}.$$

The set $\mathbf{p}_R(x)$ is called the *closest point projection* of $x$ into $R$ (see Figure 1.3).

Figure 1.3

Observe that if $p \in \mathbf{p}_R(x)$ and $t \in \overline{xp}$ then $p \in \mathbf{p}_R(t)$ also. In the special case $X = \Gamma$, a negatively curved Cayley graph, if $x$ is a point and $R$ a geodesic (segment, ray, line) there can be at most finitely many points (all necessarily vertices) in $\mathbf{p}_R(x)$. Furthermore, $\mathbf{p}_R$ is a "continuous" relation, in the sense that if $x$ is sufficiently close to $y$, then $\mathbf{p}_R(x)$ is in a neighborhood of $\mathbf{p}_R(y)$.

**Definition 1.10.** Let $R$ be a ray and $r \in R$. Define the *halfspace determined by $R$ and $r$* as

$$H(R,r) = \big\{x \in X : \mathrm{d}(x, R[r,+\infty)) \leq \mathrm{d}\big(x, R[0,r)\big)\big\}.$$

Set $H^-(R,r) = X \setminus H(R,r)$. Call $H^-(R,r)$ the *complementary halfspace*.

A point is in $H^-(R,r)$ if *all* of its projections into $R$ lie in the initial segment $R[0,r)$. If at least one of its closest projections lies on the subray $R[r,+\infty)$, then the point is in $H(R,r)$, see Figure 1.4. Although a halfspace and its complement are defined differently, they are almost indistinguishable in the large. Any theorem proved about halfspaces is true (or has an analog) for complementary halfspaces. The halfspaces yield neighborhoods for $\partial X$ in the following way.

Figure 1.4

**Definition 1.11.** Given a halfspace $H(R,r)$, define an (open) *disk at infinity* by

$$D(R,r) = \big\{[S] : S \text{ is a ray, and } \liminf_{s\to+\infty} \mathrm{d}\big(S(s), H^-(R,r)\big) = +\infty\big\}.$$

Let $D^-(R,r)$ signify $X \setminus D(R,r)$; see Figure 1.5.

Figure 1.5

Swenson [13] gives a nice argument showing that the disks at infinity form a base for the same topology on $\partial X$ used by Gromov, et al, and verifies independently that $\partial X$ is compact, metrizable, and finite dimensional. The compactification $\overline{X} = X \cup \partial X$ can be given a global metric which induces the original topology on $X$ and agrees with that of $\partial X$. I prefer, however, to ignore the global metric in favor of combinatorial arguments dealing with halfspaces and disks. This is particularly relevant in the case that $X$ is the Cayley graph $\Gamma$ of a negatively curved group.

## 2. Some geometric properties of $\Gamma$

Let $G$ be a negatively curved group with fixed (finite) generating set $C$ and associated Cayley graph $\Gamma$. One can define an extension of the action of $G$ to $\partial\Gamma$ in the obvious way: if $R \subset \Gamma$ is a (geodesic) ray, set $g([R]) = [g(R)]$. Suppose that $S$ is another ray equivalent to $R$. Then by definition there exists $N \in \mathbf{Z}^+$ such that for all $r \in R$ and all $s \in S$, both $\mathrm{d}(r, S) < N$ and $\mathrm{d}(s, R) < N$. But every $g \in G$ is an isometry on $\Gamma$, so $\mathrm{d}\big(g(r), g(S)\big) < N$ and $\mathrm{d}\big(g(s), g(R)\big) < N$. Therefore $g(R)$ and $g(S)$ are equivalent rays, meaning that the action on $\partial\Gamma$ is well-defined.

In light of the above, one may dispense with equivalence classes of rays and use individual representatives. Since $G$ is a group, it follows that $g\colon \partial\Gamma \to \partial\Gamma$ is a bijection. To show that $G$ acts as a group of homeomorphisms, it suffices to show that each $g^{-1}$ is continuous on $\partial\Gamma$. Observe that $g$ maps halfspaces to halfspaces, hence basic disk neighborhoods to basic disk neighborhoods. Thus $g$ is an open map, so $g^{-1}$ is continuous.

**Definition 2.1.** A negatively curved group is *elementary* if $\partial\Gamma$ contains at most two points. It is *non-elementary* otherwise.

Elementary groups are either torsion or virtually cyclic depending on whether $\partial\Gamma$ is empty or contains exactly two points (see, e.g. the discussion following II.17 in [13]). On the other hand, any non-elementary negatively curved group $G$ contains a rank two free subgroup, and hence $\partial\Gamma$ is in fact uncountable (see 8.2 of [8]). In the non-elementary case, $G$ is a discrete subgroup of Homeo($\partial\Gamma$) using the compact-open topology. The argument is a transparent consequence of the convergence conditions (CON) which are established in the proof of Theorem 3.4. The action of $G$ on $\partial\Gamma$ is not necessarily effective (meaning nonidentity elements can act trivially). Let $H$ be the subgroup of $G$ that acts trivially on $\partial\Gamma$. Clearly $H$ is normal, and (CON) shows that $H$ must be finite. The quotient $G_0 = G/H$ acts effectively.

The next four properties are generalizations of hyperbolic geometry applied to $\Gamma$. The first result is "folklore" (a proof can be found on page 19 of [3]). It says that there is a (not necessarily unique) geodesic between every two points at infinity.

**Lemma 2.2.** *If $R, S \subset \Gamma$ are inequivalent rays, then there is a geodesic line $P \colon \mathbf{R} \to \Gamma$ such that $P^- = P(-\infty, 0]$ is equivalent to $R$ and $P^+ = P[0, +\infty)$ is equivalent to $S$.*

**Definition 2.3.** Recall that a subset $S$ of a path metric space is *quasiconvex* ($K$) for some $K \geq 0$, if every geodesic segment $\alpha$ with $\alpha(0), \alpha(1) \in S$ satisfies $\sup_{t\in[0,1]} \mathrm{d}\big(\alpha(t), S\big) < K$.

Figure 2.1

**Lemma 2.4.** *Let $R \subset \Gamma$ be a ray and $r \in R$. If $a, b \in H(R, r)$ and $c \in \overline{ab}$, then $\mathrm{d}\big(c, H(R, r)\big) < 2\delta$. (Half-spaces and complements of half-spaces are quasiconvex ($(2\delta)$.)*

*Proof.* Let $p \in \mathbf{p}_R(b)$ and $q \in \mathbf{p}_R(a)$. Consider the ($2\delta$-thin) quadrilateral $abpq$. The point $c$ must be within $2\delta$ of $\overline{bp} \cup \overline{pq} \cup \overline{qa}$, and all of these segments lie in $H(R, r)$, see Figure 2.1. A similar argument shows that $H^-(R, r)$ is also quasiconvex ($2\delta$). □

**Lemma 2.5.** *Let $R$ be a ray, $r = R(r)$ a point on $R$, and $k$ a positive real number. Then the distance from $H^{-}(R, r)$ to $H(R, r + k\delta)$ exceeds $(k - 12)\delta$. (Halfspaces are "thick", see Figure 2.2).*

Figure 2.2

*Proof.* We may assume that $k > 12$. Let $a \in H^{-}(R, r)$ and $b \in H(R, r+k\delta)$. Choose closest point projections $p \in \mathbf{p}_R(a)$, and $q \in \mathbf{p}_R(b)$ with the latter inside $H(R, r + k\delta)$. Set $p' = R(p + 4\delta)$ and $q' = R(q - 4\delta)$.

*Claim.* $\mathrm{d}(p', \overline{ap}) \geq 2\delta$.

If not, there exists a point $t \in \overline{ap}$ with $\mathrm{d}(t, p') < 2\delta$. Since $p \in \mathbf{p}_R(t)$ (see remarks following 1.9), it follows that $\mathrm{d}(t, p) < \mathrm{d}(t, p') < 2\delta$. But then the triangle inequality says

$$4\delta = \mathrm{d}(p, p') \leq \mathrm{d}(p, t) + \mathrm{d}(t, p') < 2\delta + 2\delta = 4\delta$$

which is most certainly a contradiction (see Figure 2.3). Therefore the claim holds.

By exactly the same argument, each of $\mathrm{d}(p', \overline{bq})$, $\mathrm{d}(q', \overline{bq})$, and $\mathrm{d}(q', \overline{ap})$ is at least $2\delta$. Using thin quadrilaterals (Exercise 1.2) both $p', q'$ are within $2\delta$ of $\overline{ab}$. Let $a', b'$ be respective closest point projections of $p', q'$ into $\overline{ab}$. Then

$$\begin{aligned} k\delta \leq \mathrm{d}(p, q) &= 4\delta + \mathrm{d}(p', q') + 4\delta \\ &< 4\delta + 2\delta + \mathrm{d}(a', b') + 2\delta + 4\delta \\ &< 4\delta + 2\delta + \mathrm{d}(a, b) + 2\delta + 4\delta \\ &= 12\delta + \mathrm{d}(a, b). \end{aligned}$$

The conclusion follows. □

The statement and argument of Lemma 2.4 need to be modified when the two endpoints are at infinity.

Figure 2.3

**Lemma 2.6.** *If* $R \subset \Gamma$ *is a ray,* $r \in R$ *and* $L \subset \Gamma$ *is a* (*geodesic*) *line with endpoints* $L(+\infty)$ *and* $L(-\infty)$ *both inside* $D(R, r+14\delta)$, *then* $L \subset H(R,r)$.

*Proof.* By hypothesis, there exist sub-rays $L^+$ and $L^-$ of $L$ entirely contained in $H(R, r+14\delta)$. Let $a \in L^+$ and $b \in L^-$. Then $\overline{ab}$ (the segment of $L$ between $a$ and $b$) stays within $2\delta$ of $H(R, r+14\delta)$ by quasiconvexity. Lemma 2.5 implies that $H(R, r+14\delta)$ is more than $2\delta$ from $H^-(R,r)$ and the result follows (Figure 2.4). □

Figure 2.4

The last result of this section is proved by a straightforward $2\delta$-thin quadrilaterals argument (see I.12 of [13]).

*Exercise* 2.7. Let $X$ be negatively curved and $R \subset X$ a ray. If $r = R(r) \in R$ and $p \in H(R, r + 8\delta)$, then $\mathrm{d}(p, R) < \mathrm{d}\big(p, H^-(R, r)\big) + 4\delta$.

## 3. Convergence and the Main Theorem

The boundary $\partial X$ of a negatively curved space $X$ was originally defined in terms of sequences of points in $X$ convergent at infinity (see 1.8 of [8]). Here is a very intuitive definition of the latter phrase.

**Definition 3.1.** Let $X$ be a negatively curved space, $\{a_n\} \subset X$ a sequence, and $R \subset X$ a (geodesic) ray. Define $a_n \to [R]$ to mean: given any $r$ $(= R(r))$ there exists a positive integer $N$ such that $a_n \in H(R, r)$ for all $n \geq N$.

The pointwise convergence at infinity of a sequence of functions $\{f_n\}$ on $X$ now makes sense. (Since $\partial X$ is a metric space, one needs no special definition of convergence for a sequence of points in $\partial X$.) Uniform convergence at infinity is defined similarly:

**Definition 3.2.** Let $S \subset X$, $\{f_n\}$ a sequence of functions each mapping $X$ into $X$, and $R \subset X$ a ray. Define $f_n(x) \to [R]$ *uniformly on* $S$ to mean: given any $r$ $(= R(r))$ there exists a positive integer $N$ such that $f_n(S) \subset H(R, r)$ for all $n \geq N$.

**Theorem 3.3.** *Let $R \subset \Gamma$ be a ray based at $0$, and $\{g_m\}$ a sequence of distinct group elements. If $g_m(z_0) \to w = [R]$ for some point $z_0 \in \Gamma$, then $g_m(z) \to w$ for all $z \in \Gamma$. Furthermore, the convergence is uniform on compact subsets of $\Gamma$.*

*Proof.* Let $\varepsilon > 0$ and let $B$ denote the open ball with center $z_0$ and radius $\varepsilon$. Let $r \in R$ and choose $N > (4\varepsilon/\delta) + 8$. By hypothesis there is some $M > 0$ such that $w_m = g_m(z_0) \in H(R, r + N\delta)$ for all $m \geq M$. Set $R^+ = R[r + N\delta)$ as the sub-ray of $R$ from $r + N\delta$ onward. There are two possibilities.

*Case* 1. $\mathrm{d}(w_m, R^+) \geq 4\delta + \varepsilon$ (the distance from $w_m$ to $R^+$ is "large").
We know $w_m \in H(R, r + N\delta) \subset H(R, r + 8\delta)$. Using 2.7 we have

$$\mathrm{d}(w_m, R) - 4\delta < \mathrm{d}\big(w_m, H^-(R, r)\big),$$

so

$$\varepsilon = (4\delta + \varepsilon) - 4\delta \leq \mathrm{d}(w_m, R) - 4\delta < \mathrm{d}\big(w_m, H^-(R, r)\big).$$

Therefore $g_m(B) = B(w_m, \varepsilon) \subset H(R, r)$.

*Case* 2. $\mathrm{d}(w_m, R^+) < 4\delta + \varepsilon$ (the distance from $w_m$ to $R^+$ is "small").
In this case, $g_m(B)$ is in the $4\delta + 2\varepsilon$ neighborhood of $R^+$. Let $z \in g_m(B)$, so $\mathrm{d}(z, R^+) < 4\delta + 2\varepsilon$. Suppose that $z \notin H(R, r)$. Then $z$ must be closer to the segment $\overline{0r}$ than to $R^+$, in particular $\mathrm{d}(z, \overline{0r}) < 4\delta + 2\varepsilon$. Thus

$$N\delta = \mathrm{d}(\overline{0r}, R^+) \leq \mathrm{d}(\overline{0r}, z) + \mathrm{d}(z, R^+) < (4\delta + 2\varepsilon) + (4\delta + 2\varepsilon) = 8\delta + 4\varepsilon,$$

contradicting our choice of $N > (4\varepsilon/\delta) + 8$. It follows that $z$ lies in $H(R, r)$ and hence $g_m(B) \subset H(R, r)$.

Set $Z = \{z \in \Gamma : g_m(z) \to [R]\}$. By the above, $Z$ is open. But $Z$ is also closed: If $\{z_i\} \subset Z$ with $z_i \to z$, then for large $i$, $z \in B(z_i, \varrho)$ for some fixed $\varrho > 0$. By the previous paragraph, $z \in Z$, meaning $Z = \Gamma$. Uniform convergence on closed balls (hence on compact sets) is immediate. □

**Theorem 3.4.** *$G$ acts as a convergence group on $\partial\Gamma$ (compare with* 8.1.G *of* [8]).

*Proof.* Suppose $G$ is elementary. If $\partial\Gamma = \emptyset$ then the theorem is vacuously true. If $\partial\Gamma$ consists of two points, then every $g \in G$ either fixes or interchanges these points, and again the theorem is vacuously true. Assume that $G$ is non-elementary.

Let $\{g_m\}$ be a sequence of distinct group elements and pick any vertex 0 as an origin. Without loss of generality we may assume that $0 \in \Gamma$ corresponds to the identity of $G$ so that $g_m(0) = g_m \cdot id = g_m$, and that each $g_m$ is a minimal representative as a word in the generating set of $G$. From a geometric viewpoint this means that $g_m$ regarded as an edge path from 0 is a geodesic segment. By passing to a subsequence if necessary, we may suppose that $\mathrm{d}(0, g_m) \geq 2m + 1$ for each $m \in \mathbf{Z}^+$. Let $\mathscr{S}_1$ be the (finite) set of edges having 0 as a vertex. It is evident that infinitely many of the edge paths $g_m$ pass through some edge $s_1 \in \mathscr{S}_1$. Pass to this corresponding subsequence $\{g_{1,m}\}$, and pick out and save a shortest element $h_1$ from this subsequence.

Let $v_1$ denote the other vertex of $s_1$. Let $\mathscr{S}_2$ be the collection of edges having $v_1$ as a vertex. Infinitely many of the edge paths $g_{1,m}$ pass through some edge $s_2 \in \mathscr{S}_2$ other than $s_1$. Let $v_2$ denote the other vertex of $s_2$ and pass to the corresponding subsequence $\{g_{2,m}\}$. Pick out and save a shortest word $h_2$ (distinct from $h_1$) from this new subsequence. Proceed recursively to obtain an edge path $S = s_1 s_2 s_3 \cdots$ and a diagonal subsequence $\{h_i\}$ of the original sequence. By construction, each path $\overline{0h_i}$ has an initial segment lying on $S$ of length at least $i$.

Note that $S$, being a nested increasing limit of geodesic segments, is a geodesic ray with initial point 0. Let $s \in S$. Then for large $i$, a shortest path from the vertex $h_i = h_i(0)$ to 0 passes through $s$. This implies that $h_i \in H(S, s)$ for all sufficiently large $i$, i.e. $h_i(0) \to [S]$. Repeat the above construction with respect to the sequence $\{h_i^{-1}\}$ to obtain a geodesic ray $T$, and subsequence $\{g_n\}$ such that $g_n^{-1}(0) \to [T]$.

The strategy is to show that given any half-spaces $H(S, s)$ and $H(T, t)$ about $S$, $T$ respectively, we can find an $N$ such that $g_n(H^-(T, t)) \subset H(S, s)$ for all $n \geq N$. Since for any neighborhood $U$ of $[S]$ and compact $K \subset \partial\Gamma \setminus [T]$ we can find $s$ and $t$ far enough from 0 so that $K \subset D^-(T, t)$ and $D(S, s) \subset U$, the above sentence implies $g_n \to [S]$ uniformly on $K$. Similarly, $g_n^{-1} \to [T]$ locally uniformly on $\partial\Gamma \setminus [S]$, establishing (CON).

We know that each $g_n$ as a word (edge path) consists of an initial string $s_n \subset S$ of length $n$ and that $s_n$ is the initial part of $s_{n+1}$. Similarly $g_n^{-1}$ has initial string $t_n \subset T$ of length $n$. Since the length of $g_n$ is at least $2n+1$ we know the end of $s_n$ does not involve the start of $t_n^{-1}$, i.e. $g_n = s_n w_n t_n^{-1}$, where $w_n$ is some word of length at least one. Now observe that $s_n w_n t_n^{-1}$ is a geodesic path implies that the segment $\overline{s_n g_n} = \overline{s_n(0) g_n(0)}$ lies inside $H(S, s_n)$, see Figure 3.1.

Figure 3.1

Let $s \in S$. Choose $N$ large enough so that $s_N \geq s + 13\delta$ and $t_N \geq t + 15\delta$. Let $z \in H^-(T,t)$ and suppose by way of contradiction that $g_n(z) \in H^-(S,s)$ for some $n \geq N$. Let $q \in \mathbf{p}_S\big(g_n(z)\big)$ and consider the triangle $g_n(0) q g_n(z)$. By choice of $n$, we know that $\mathrm{d}(s_n, q) \geq 13\delta$. Using thickness of half-spaces (Lemma 2.5)

$\mathrm{d}(s_n, \overline{qg_n(z)}) > \delta$. Therefore thin triangles says that

$$\mathrm{d}\big(s_n, \overline{g_n(0)g_n(z)+}\big) = \mathrm{d}\big(g_n^{-1}(s_n), \overline{0z}\big) = \mathrm{d}\big(t_n w_n^{-1}, \overline{0z}\big) < \delta.$$

However, the vertex $t_n w_n^{-1}$ lies in the half-space $H(T, t_n) \subset H(T, t + 15\delta)$. Using thickness of half-spaces, the distance from $H(T, t_n)$ to $H^-(T, t)$ is more than $3\delta$. Finally, since $H^-(T, t)$ is quasiconvex (Lemma 2.4), we know that the path $\overline{0z}$ strays at most $2\delta$ from $H^-(T, t)$. Hence

$$3\delta < \mathrm{d}\big(t_n w_n^{-1}, H^-(T, t)\big) \le \mathrm{d}\big(t_n w_n^{-1}, \overline{0z}\big) + 2\delta < \delta + 2\delta = 3\delta, \quad \text{a contradiction.}$$

Therefore $g_n(z) \in H(S, s)$ after all, and since $z$ was arbitrary, $g_n$ maps all of $H^-(T, t)$ into $H(S, s)$ as required. □

Recall that if $G$ is a convergence group on a space $Y$, then the *limit set* $\Lambda(G)$ consists of all points $y \in Y$ such that there exists $x \in Y$ and a distinct sequence $\{g_n\} \subset G$ such that $g_n(x) \to y$. The ordinary set $\Omega(G)$ is the complement of the limit set. Since any compact $C \subset (\Gamma \cup \partial\Gamma)$ with $\partial\Gamma \not\subset C$ is contained in some suitably large halfspace (along with the corresponding disk at infinity), the proof of 3.4 yields

**Corollary 3.5.** *$G$ acts as a convergence group on $\overline{\Gamma} = \Gamma \cup \partial\Gamma$, with $\Omega(G) = \Gamma$ and $\Lambda(G) = \partial\Gamma$.*

Gehring and Martin [7] classify elements of convergence groups (acting on $S^n$) as *elliptic*, *parabolic*, or *loxodromic*. Elliptic elements are torsion, parabolics have a unique fixed point on $S^n$, and loxodromics have two fixed points on $S^n$. (Tukia has recently extended almost the entire theory to the category of compact Hausdorff spaces in Section 2 of [15].) Similarly (see 8.1 of [8]), Gromov classifies the elements of any negatively curved group as either *elliptic* (=torsion) or *hyperbolic* (=non-torsion). He shows that a hyperbolic element has two fixed points on $\partial\Gamma$, one being attractive and the other repulsive. In light of 3.4 it is clear that if $G$ is negatively curved then $g \in G$ is hyperbolic if and only if $g$ is loxodromic. I will use the term "loxodromic" exclusively hereafter.

Not every convergence group is negatively curved. The Kleinian group generated by $p_1(z) = z + 1$ and $p_2(z) = z + i$ is an elementary convergence group on $S^2$ isomorphic to $Z \oplus Z$. Such a group cannot be negatively curved. Evidently the presence of (non-accidental) parabolic elements in a convergence group is not compatible with negative curvature. Theorem 3.7 gives the details.

**Definition 3.6.** Let $G$ be a (discrete) convergence group acting on a compact metric space $(X, d)$. We say that the limit point $w$ is a *point of approximation* if there is associated with $w$ a sequence $\{g_m\}$ of distinct group elements such that for each $x \in X \setminus \{w\}$ there is some $\varepsilon = \varepsilon(x)$ satisfying $\mathrm{d}\big(g_m(w), g_m(x)\big) \ge \varepsilon$ for all $m$.

As an example, let $G$ be a Kleinian group and $g \in G$ a loxodromic element. The fixed points of $g$ on $S^2$ are both points of approximation. On the other hand, no parabolic fixed point can be a point of approximation [9]. Evidently, every loxodromic fixed point (in the boundary at infinity) of a negatively curved group is a point of approximation. In fact more is true.

**Theorem 3.7.** *Let $G$ be a negatively curved group. Then every $x \in \partial\Gamma$ is a point of approximation* (*compare with* 8.2.J *in* [8]).

*Proof.* Let $x, y \in \partial\Gamma$ be distinct points and let $L$ be any geodesic line with $L(+\infty) = x$ and $L(-\infty) = y$. Pick a vertex on $L$, call it $v_0$. Let $L^+$ denote that part of $L$ between $v_0$ and $x$. Label the succesive vertices of $L$ from $v_0$ tending towards $x$ as $v_1$, $v_2$, $v_3$, .... For each $m$, let $g_m \in G$ be the group element taking $v_m$ to $v_0$. Use the convergence property to obtain a subsequence $\{g_k\}$ and rays $S$ and $T$ such that

$$g_k \to [S] \text{ locally uniformly on } \partial\Gamma \setminus [T]$$

and

$$g_k^{-1} \to [T] \text{ locally uniformly on } \partial\Gamma \setminus [S].$$

We can pass to a further subsequence so that $\{g_j([T])\}$ converges as well. Since $g_j^{-1}(v_0) \to [L^+]$ by construction, it is clear that both $T$ and $L^+$ represent the point $x \in \partial\Gamma$. Because $y \in \partial\Gamma$ was chosen to be distinct from $x = [T]$ we know that $g_j(y) \to [S]$. Suppose for the moment that $g_j([T]) \to [S]$ also. Pick $s \in S$ so that $v_0 \notin H(S, s)$. Then for all sufficiently large $j$ it is true that both $g_j(x) = g_j([T])$ and $g_j(y)$ are inside $D(S, s + 14\delta)$. Lemma 2.6 implies that the geodesic $g_j(L)$ is contained in $H(S, s)$. But then

$$v_0 = g_j(v_j) \in g_j(L) \subset H(S, s),$$

a contradiction. Therefore $g_j(x)$ cannot converge to $[S]$. Since $g_j(w) \to [S]$ for all $w \neq x$, it is clear that $x$ is a point of approximation. □

**References**


[1] Cannon, J.: The combinatorial structure of cocompact discrete hyperbolic groups. - Geom. Dedicata 16, 1984, 123–148.

[2] Cannon, J.: The theory of negatively curved spaces and groups. - In: Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces, edited by T. Bedford, M. Keane, and C. Series, Oxford University Press, 1991, 315–369.

[3] Coornaert, M., T. Delzant, and A. Papadopoulos: Géométrie et théorie des groupes. - Lecture Notes in Math. 1441, Springer-Verlag, 1990.

[4] Cheeger, J., and D. Ebin: Comparison Theorems in Riemannian Geometry. - North-Holland Math. Library 9, North-Holland, 1975.

[5] Gabai, D.: Convergence groups are Fuchsian groups. - Ann. of Math. 136, 1992, 447–510.

[6] Ghys, E., and P. de La Harpe: Infinite groups as geometric objects (after Gromov). - In: Ergodic theory, Symbolic Dynamics, and Hyperbolic Spaces, edited by T. Bedford, M. Keane, and C. Series, Oxford University Press, 1991, 299–314.

[7] Gehring, F.W., and G. Martin: Discrete quasiconformal groups I. - Proc. London Math. Soc. 55, 1987, 331–358.

[8] Gromov, M.: Hyperbolic groups. - In: Essays in Group Theory, edited by S.M. Gersten, MSRI Ser. 8, 1987, 75–263.

[9] Maskit, M.: Kleinian Groups. - Grundlehren Math. Wiss. 287, Spinger-Verlag, 1988.

[10] Martin, G., and R. Skora: Group actions of the 2-sphere. - Amer. J. Math. 111, 1989, 387–402.

[11] Ol'shanskii, A.Yu.: Almost every group is hyperbolic. - Internat. J. Algebra Comput. 1, 1992, 1–17.

[12] Sela, Z.: On the isomorphism theorem for hyperbolic groups and the classification of 3-manifolds. - IHES preprint.

[13] Swenson, E.L.: Negatively curved groups and related topics. - Thesis, Brigham Young University, 1993.

[14] Tukia, P.: Homeomorphic conjugates of Fuchsian groups. - J. Reine Angew. Math. 391, 1988, 1–54.

[15] Tukia, P.: Convergence groups and Gromov's metric hyperbolic spaces. - New Zealand J. Math. 23, 1994, 157–187.

Received 3 March 1994