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EXACT COEFFICIENT ESTIMATES FOR UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSION

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Abstract. We give here a complete solution of the coefficient problem for normalized univalent functions on the unit disk, with k-quasiconformal extension for a small k, and derive an explicit bound for k.

1. Introduction

While the coefficient problem is completely solved in the class of all normalized univalent functions on the disk [dB], the question remains open for functions with quasiconformal extension.

The strongest result here is established for the functions with k-quasiconformal extension where k is small; see [Kr2].

Let S be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ univalent in the unit disk $\Delta = \{|z| < 1\}$. The class $S(k)$ consists of $f \in S$ admitting k-quasiconformal extension onto the whole Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, with additional normalization $\tilde{f}(\infty) = \infty$. Let

$$f_1(z) = \frac{z}{(1-ktz)^2}, \qquad |z| < 1,\ |t| = 1,$$
$$f_{n-1} = \{f_1(z^{n-1})\}^{1/(n-1)} = z + \frac{2kt}{n-1} z^n + \cdots, \qquad n = 3, 4, \ldots .$$

Consider on S a functional F of the form

$$F(f) = a_n + H(a_{m_1}, a_{m_2}, \ldots, a_{m_s}),$$

where $a_j = a_j(f)$; $n, m_j \geqq 2$ and H is a holomorphic function of s variables in an appropriate domain of $\mathbf{C}^s$. We assume that this domain contains the origin $\mathbf{0}$ and that $H, \partial H$ vanish at $\mathbf{0}$.

The mentioned result of [Kr2] is:

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Theorem 1. *For any functional of the above form there exists a* $k(F) > 0$ *such that, for* $k \leqq k(F)$,

$$\max_{S(k)} |F(f)| = |F(f_{n-1})| \tag{1}$$

for some $|t| = 1$.

As a corollary, one immediately gets for $f \in S(k)$ the sharp estimate

$$|a_n| \leqq \frac{2k}{n-1} \tag{2}$$

for $k \leqq k_n$, with equality only for the function f_{n-1}. This solves the well-known problem of Kühnau and Niske; see [KuN]. The estimate (2) is interesting only for $n \geqq 3$, because for $n = 2$ there is the well-known bound $|a_2| \leqq 2k$ for all $k \in [0,1]$ with equality for the function f_1.

The purpose of this paper is to improve on Theorem 1, supplementing it with an explicit estimate for the quantity $k(F)$.

2. Statement of results

The main result of the paper is:

Theorem 2. *Let* $\sup_S |F(f)| = M_n$. *Then the equality* (1) *holds for all*

$$k \leqq \frac{1}{2 + (n-1)(M_n + 1)} =: k_0(F). \tag{3}$$

The bound (3) is not sharp and can be improved.

Corollary. *The estimate* (2) *is valid for all*

$$k \leqq \frac{1}{n^2 + 1}. \tag{4}$$

Proof. Take $F(f) = a_n$. Since $M_n = n$, by de Branges's theorem [dB], one immediately deduces from (3) that in this case

$$k_0(F) = \frac{1}{n^2 + 1}.$$

For simplicity, we consider here the functionals F with holomorphic H depending on a finite number of coefficients a_m. The latter condition is not essential; one can take H depending on infinitely many a_m (provided the series expansion of H converges in some complex Banach space). The result shows that the main contribution here is given by the linear term a_m. The estimate (3) determines for which k this is true.

3. Proof of Theorem 2

We shall show that for k satisfying (3) all arguments employed in the proof of Theorem 1 in [Kr2] remain valid after some modification. Actually, we only need to modify the proof of Lemma 1.

On $\Delta^* = \{z \in \widehat{\mathbf{C}} : |z| > 1\}$ we have the Beltrami coefficients $\mu_f = \partial_{\bar z} f / \partial_z f$ of the extensions f^μ of functions $f \in S(k)$; these coefficients range over the ball

$$B(\Delta^*) = \{\mu \in L_\infty(\mathbf{C}) : \mu \mid \Delta = 0, \|\mu\|_\infty < 1\}.$$

Let $B(\Delta^*)_k = \{\mu \in B(\Delta^*) : \|\mu\| \leqq k\}$.

Note that the Beltrami coefficient for f_{n-1} can be taken to be $kt\mu_n$, where $|t| = 1$ and

$$\mu_n(z) = \frac{|z|^{n+1}}{\bar z^{n+1}}. \tag{5}$$

We shall also use the following notations. For a functional L: $S \to \mathbf{C}$ define

$$\hat L(\mu) = L(f^\mu), \qquad \mu \in B(\Delta^*).$$

If L is complex Gateaux differentiable, $\hat L$ is a holomorphic functional on $B(\Delta^*)$. All our functionals have this property.

For $\mu \in L_\infty(\Delta^*)$, $\varphi \in L_1(\Delta^*)$ we define

$$\langle \mu, \varphi \rangle = -\frac{1}{\pi} \iint_{\Delta^*} \mu\varphi \, dx\, dy \qquad (z = x + iy).$$

For small k, the functions $f^\mu \in S(k)$ can be represented by

$$f^\mu(\zeta) = \zeta - \frac{\zeta^2}{\pi} \iint_{\Delta^*} \frac{\mu(z)\, dx\, dy}{z^2(z-\zeta)} + O(\|\mu\|^2), \tag{6}$$

where the estimate of the remainder term is uniform on compact subsets of $\mathbf{C}$ (see e.g. [Kr1, Ch. 2]); this easily implies

$$\hat F(\mu) = \left\langle \mu, \frac{1}{z^{n+1}} \right\rangle + O_n(\|\mu\|^2)$$

and hence

$$\|\hat F'(\mathbf{0})\| = \sup\left\{ \left| \left\langle \mu, \frac{1}{z^{n+1}} \right\rangle \right| : \|\mu\| \leqq 1 \right\} = \frac{1}{\pi} \iint_{\Delta^*} \frac{dx\, dy}{|z|^{n+1}} = \frac{2}{n-1}.$$

Now, applying the Schwarz lemma to the function

$$h_\mu(t) = \hat{F}(t\mu) - \hat{F}'(\mathbf{0})t\mu\colon \Delta \to \mathbf{C},$$

where $\mu \in B(\Delta^*)$ is fixed, we get

$$|\hat{F}(\mu) - \hat{F}'(\mathbf{0})\mu| \leqq \big(M_n + \|\hat{F}'(\mathbf{0})\|\big)\|\mu\|^2 = \Big(M_n + \frac{2}{n-1}\Big)\|\mu\|^2. \tag{7}$$

Consider the auxiliary functional

$$\hat{F}_p(\mu) = \hat{F}(\mu) + (p-1)\xi\Big\langle \mu, \frac{1}{z^{p+1}}\Big\rangle, \tag{8}$$

where $p \neq n$ is fixed and $|\xi| < \frac{1}{2}$. Then

$$\sup_{B(\Delta^*)} |\hat{F}_p(\mu)| \ < M_n + 1 \tag{9}$$

and, similarly to (7),

$$\Big|\hat{F}_p(\mu) - \hat{F}'(\mathbf{0})\mu - (p-1)\xi\Big\langle \mu, \frac{1}{z^{p+1}}\Big\rangle\Big| \leqq \Big(M_n + 1 + \frac{2}{n-1}\Big)\|\mu\|^2. \tag{10}$$

We shall require that

$$\Big(M_n + 1 + \frac{2}{n-1}\Big)\|\mu\|^2 < \frac{1}{n-1}\|\mu\| \tag{11}$$

or, equivalently,

$$\|\mu\| \leqq \frac{1}{2 + (n-1)(M_n+1)} = k_0(F). \tag{3'}$$

Consider now any function f_0 in $S(k)$ maximizing $|F|$ over $S(k)$ (the existence of such functions follows from compactness). Let μ_0 be an extremal dilatation of f_0, i.e.

$$\|\mu_0\|_\infty = \inf\{\|\mu\|_\infty \leqq k : f^\mu \mid \Delta = f_0 \mid \Delta\}.$$

Note that $\|\mu_0\|_\infty = k$ by the maximum modulus principle. Suppose that $\mu_0 \neq kt\mu_n$, where $|t| = 1$, and μ_n is defined by (5). We show that this leads to contradiction for k satisfying (3). First of all, we may establish the following important property of extremal maps:

Lemma 1. *If k satisfy* (3), *then for all* $2 \leqq p \neq n$,

$$\left\langle \mu_0, \frac{1}{z^{p+1}} \right\rangle = 0.$$

Proof. Note that, from (6),

$$\left\langle \mu_0, \frac{1}{z^{p+1}} \right\rangle = \lim_{\tau\to\infty} \frac{a_p(f^{\tau\mu_0})}{\tau}.$$

Consider the classes $S(\tau k_0)$ where $k_0 = k_0(F)$ is defined in (3) and $0 < \tau < 1$.

It follows from (6) that, as $\tau \to 0$,

$$\max\{|\hat{F}(\mu)| : \|\mu\| \leqq \tau k_0\} = \frac{\tau k_0}{\pi} \iint_{\Delta^*} \frac{dx\,dy}{|z|^{n+1}} + O_n(\tau^2) = |\hat{F}(\tau\mu_0)| + O_n(\tau^2). \tag{12}$$

A similar calculation for functional (8) implies

$$\max_{B(\Delta^*)_k} |\hat{F}_p(\mu)| = \frac{\tau k_0}{\pi} \iint_{\Delta^*} \left| \frac{1}{z^{n+1}} + \frac{(p-1)\xi}{z^{p+1}} \right| dx\,dy + O_n(\tau^2), \tag{13}$$

where the remainder term estimate follows from (10) and depends (as in (12)) only on M_n and k_0.

Using the known properties of the norm

$$h_p(\xi) = \iint_{\Delta^*} |z^{-n-1} + (p-1)\xi z^{-p-1}|\,dx\,dy$$

following from the Royden [Ro] and Earle–Kra [EK] lemmas, we deduce from (12), (13) that for small ξ there should be

$$\max_{B(\Delta^*)_{\tau k_0}} |\hat{F}_p(\mu)| = \max_{B(\Delta^*)_{\tau k_0}} |\hat{F}(\mu)| + \tau o_p(\xi) + O_p(\tau^2\xi) + O_n(\tau^2). \tag{14}$$

On the other hand, we have as $\xi \to 0, \tau \to 0$, from (8)

$$\begin{aligned} |\hat{F}_p(\tau\mu_0)| &= |\hat{F}(\tau\mu_0)| + \operatorname{Re} \frac{\hat{F}(\tau\mu_0)}{|\hat{F}(\tau\mu_0|}(p-1)\xi \left\langle \tau\mu_0, \frac{1}{z^{p+1}} \right\rangle + O(\tau^2\xi^2) \\ &= |\hat{F}(\tau\mu_0)| + \tau(p-1)|\xi| \left| \left\langle \mu_0, \frac{1}{z^{p+1}} \right\rangle \right| + O(\tau^2\xi^2) \end{aligned}$$

with suitable choices of $\xi \to 0$. Comparing this with (14), (10), (11), we conclude that $\langle \mu_0, z^{-p-1} \rangle = 0$. The proof of Lemma is completed.

This lemma is one of the central points in the proof of the Theorems 1 and 2. The crucial point in the proof of Lemma 1 is that we now have to check here that simultaneously an *infinite* (countable) number of ortogonality conditions remain valid for all k satisfying (3).

The next part of the proof is similar to [Kr2]. We briefly check that the arguments remain valid for all k.

Consider the Grunsky coefficients of the function $\sqrt{f(z^2)}$ which are defined from the series expansion

$$\log \frac{\big(f(z^2)\big)^{1/2} - \big(f(\zeta^2)\big)^{1/2}}{z-\zeta} = -\sum_{m,n=1}^{\infty} \omega_{mn} z^m \zeta^n,$$

taking the branch of logarithm which vanishes at 1. The diagonal coefficients $\omega_{n-1,n-1}(f)$ are related to the Taylor coefficients of f by

$$\omega_{n-1,n-1} = \tfrac{1}{2} a_n + P(a_2, \dots, a_{n-1}) \tag{15}$$

where P is a polynomial without constant or linear terms (see [Hu]). Moreover, for $f \in S(k)$ there is the well-known bound

$$|\omega_{n-1,n-1}| \leqq \frac{k}{n-1}$$

with equality only for the functions f_{n-1}.

Therefore, the map $\Lambda_{n-1}\colon B(\Delta^*) \to B(\Delta^*)$ defined by

$$\Lambda_{n-1}(\mu) = \{(n-1)\omega_{n-1,n-1}(\mu)\}\mu_n$$

is holomorphic and fixes the disk $\{t\mu_n : |t| < 1\}$. The differential of Λ_{n-1} at $\mu = \mathbf{0}$ can be easily computed from (6), (15). It is an operator $P_n\colon L_\infty(\Delta^*) \to L_\infty(\Delta^*)$ given by

$$P_n(\mu) = \beta_n \langle \varphi_n, \mu \rangle \mu_n, \qquad \varphi_n = \frac{1}{z^{n+1}}.$$

Let us define $P_n(\mu) = \alpha(k)\mu_n$. Since, by assumption, f_0 is not equivalent to f_{n-1}, we have

$$\Big\{\Lambda_{n-1}\Big(\frac{t}{k}\mu_0\Big) : |t| < 1\Big\} \subsetneqq \{|\, t \,| < 1\}.$$

Thus, by the Schwarz lemma,

$$|\alpha(k)| < k. \tag{16}$$

Now consider the function

$$\nu_0 = \mu_0 - \alpha(k)\mu_n$$

and show that ν_0 eliminates integrable holomorphic functions on Δ^*.

From Lemma 1 and the mutual orthogonality of the powers z^m, $m \in \mathbf{z}$,

$$\left\langle \nu_0, \frac{1}{z^{p+1}} \right\rangle = 0$$

for $p = 2, 3, \dots, p \neq n$. To establish that

$$\left\langle \nu_0, \frac{1}{z^{n+1}} \right\rangle = 0,$$

consider the conjugate operator

$$P_n^*(\varphi) = \beta_n \langle \mu_n, \varphi \rangle \varphi_n, \qquad \varphi_n = \frac{1}{z^{n+1}},$$

which maps $L_1(\Delta^*)$ onto $L_1(\Delta^*)$ and fixes the subspace $\{\lambda\varphi_n : \lambda \in \mathbf{C}\}$. The definition of ν_0 implies $P_n(\nu_0) = 0$. Thus, for some λ,

$$\langle \nu_0, \varphi_n \rangle = \lambda \langle \nu_0, P_n^* \varphi_n \rangle = \lambda \langle P_n \nu_0, \varphi_n \rangle = 0.$$

Now consider in $L_1(\Delta^*)$ the subspace $A_1(\Delta^*)$ of functions φ which are holomorphic on Δ^* and satisfy the condition $\varphi(z) = O(|z|^{-3})$ as $|z| \to \infty$. Let

$$A_1(\Delta^*)^\perp = \{\mu \in L_\infty(\Delta^*) : \langle \mu, \varphi \rangle = 0 \text{ for all } \varphi \in A_1(\Delta^*)\}.$$

Since the functions $\varphi_n = 1/z^{n+1}$, $n = 2, 3, \dots,$ form a complete set in $A_1(\Delta^*)$, we have proved that $\nu_0 \in A_1(\Delta^*)^\perp$.

Now we use the well-known properties of extremal quasiconformal maps (see e.g. [Ga], [Kr1], [RS]). First of all, since μ_0 is extremal for f_0,

$$\|\mu_0\|_\infty = \inf\{|\langle \mu_0, \varphi \rangle| : \varphi \in A_1(\Delta^*),\ \|\varphi\| = 1\};$$

moreover, such an equality is necessary and sufficient for $\mu \in B(\Delta^*)$ to be extremal for f^μ. Hence, for any $\nu \in A_1(\Delta^*)^\perp$,

$$\|\mu_0\|_\infty = \inf\{|\langle \mu_0 + \nu \rangle| : \varphi \in A_1(\Delta^*),\ \|\varphi\| = 1\} \leqq \|\mu_0 + \nu\|_\infty.$$

Thus we have

Lemma 2. *If f_0 is extremal,*

$$\|\mu_0\|_\infty = k \leqq \|\mu_0 - \nu_0\|_\infty. \tag{17}$$

We may now complete the proof of Theorem 2. By (17)

$$k \leqq \|\mu_0 - \nu_0\|_\infty = \|\alpha(k)\mu_n\|_\infty = |\alpha(k)|,$$

which contradicts (16). Hence f_0 is equivalent to f_{n-1} and we can take $\mu_0 = kt\mu_n$ for some $|t| = 1$.

4. Complementary remarks and open questions

1) The estimates (1)–(3) also hold in the class $S_k(1)$ of functions $f \in S$ with k-quasiconformal extensions $\tilde{f}$ normalized by $\tilde{f}(1) = 1$.

The proof is similar, only (6) should be replaced with the corresponding representation formula for $f \in S_k(1)$ [Kr1, Ch. 5]:

$$f^{\mu}(\zeta) = \zeta - \frac{\zeta^2(\zeta - 1)}{\pi} \iint\limits_{\Delta^*} \frac{\mu(z)\,dx\,dy}{z^2(z-1)(z-\zeta)} + O(\|\mu\|), \qquad \text{as } \|\mu\| \to 0.$$

2) Similar results are valid for the class $\Sigma(k)$ of functions $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$, $z \in \Delta^*$, with k-quasiconformal extensions to $\widehat{\mathbf{C}}$ which fix the origin.

The next two problems still remain open:

1) Does there exist an estimate of coefficients a_n $(n \geqq 3)$ for $f \in S(k)$ which holds for $k \leqq k_0$ with a single $k_0 > 0$?

2) Can we find exact estimates of coefficients a_n for univalent functions on the disk with quasiconformal extension in the general case when the dilatation $k < 1$ is arbitrary?

For $f \in S(k)$, one gets from (7) the estimate

$$|a_n| \leqq \frac{2k}{n-1} + \Big(n + \frac{2}{n-1}\Big)k^2$$

for any k, $0 \leqq k < 1$, (cf. [KrKu, Part 1, Ch. 2]). Note also that Grinshpan [Gr] established the exact growth order, with respect to n, of the coefficients a_n of $f \in S$ with k-quasiconformal extension, without any additional normalization: $|a_n| \leqq cn^k$.

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