Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 20, 1995, 379–385

A NOTE ON THE OSCILLATION THEORY OF CERTAIN SECOND ORDER DIFFERENTIAL EQUATIONS

Shupei Wang

University of Joensuu, Department of Mathematics
P. O. Box 111, FIN-80101 Joensuu, Finland

Abstract. We consider the complex oscillation theory of the second order linear differential equation $f'' + (e^{P(z)} + Q(z))f = 0$, where $P(z)$ and $Q(z)$ are polynomials of degrees $n \geq 1$ and $m \geq 0$, respectively. The situation for the case $n = 1$ is clear. For the case $n \geq 2$, a result of Bank and Langley [4] shows that if $m < 2(n-1)$, then for any non-trivial solution f of the equation, its exponent of the convergence of the zero sequence $\lambda(f)$ equals to infinity. The same result was also proved by them [5] for the case $m > 2(n-1)$, provided some additional conditions were assumed on P and Q. In this paper, a general result for the case $m > 2(n-1)$ is obtained. We show that, in this case, $\lambda(f) = (m+2)/2$ or $\lambda(f) = \infty$ holds for any non-trivial solution f of the equation. This improves a former result of the author [8]. Moreover, we also obtain a result for the case $m = 2(n-1)$. Examples show that this result is sharp.

1. Introduction and results

We consider the differential equation of the form

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function. First of all, it follows from the elementary theory of differential equations that all solutions of (1.1) are entire functions, and that the zeros of any non-trivial solution are simple.

In the study of oscillation theory for solutions to the equation (1.1), the case where $A(z)$ in (1.1) is a transcendental entire function of finite order has received much attention since 1982. In this case, any non-trivial solution of (1.1) is of infinite order of growth (in the sense of Nevanlinna). One of the main problems is to find conditions on $A(z)$ so that every solution $f \not\equiv 0$ of (1.1) satisfies $\lambda(f) = \infty$, where $\lambda(f)$ denotes as usual the exponent of convergence for the zeros of f. Our starting point is the following Theorem A which is a modified version of [1, Lemma 8.2].

1991 Mathematics Subject Classification: Primary 34A20; Secondary 30D35.

The author is indebted to the University of Joensuu and the Centre for International Mobility (CIMO) in Finland for financial support.

Theorem A. *Suppose that $A(z)$ is a transcendental entire function of finite order. Then the equation (1.1) admits two linearly independent zero-free solutions if and only if $A(z)$ can be represented as*

$$A(z) = e^{P(z)} - \tfrac{1}{16}\big(P'(z)\big)^2 + \tfrac{1}{4}P''(z), \tag{1.2}$$

where $P(z)$ is a non-constant polynomial.

Actually, Theorem A focused the interest to the oscillation theory of differential equations of the form

$$f'' + \big(e^{P(z)} + Q(z)\big)f = 0, \tag{1.3}$$

where $P(z)$ and $Q(z)$ are polynomials of degrees $n \geq 1$ and $m \geq 0$, respectively. In the case of Theorem A, $m = 2(n-1)$. The following earlier results demonstrate the importance of this relation. The case $n = 1$ was settled completely by Bank, Laine and Langley [2] and Langley [7]. In fact, then we get

Theorem B. *Consider the equation*

$$f'' + \big(e^{z} + Q(z)\big)f = 0, \tag{1.4}$$

where $Q(z)$ is a polynomial.

(1) *If $Q(z)$ is non-constant, then $\lambda(f) = \infty$ for any non-trivial solution f of* (1.4).
(2) *If $Q(z)$ is a constant, say K, then the equation (1.4) admits a solution $f \not\equiv 0$ with $\lambda(f) < \infty$ if and only if $K = -(2p+1)^2/16$ for some integer $p \geq 0$.*

The remaining case $n \geq 2$ was considered by Bank and Langley in [4] and [5]. The following result is a special case of the Theorem in [4].

Theorem C. *If $m < 2(n-1)$, then $\lambda(f) = \infty$ for any non-trivial solution f of* (1.3).

In order to state the result in [5] where the case $m > 2(n-1)$ was considered, we first make the following definitions.

Definition 1. Let $P(z) = a_n z^n + \dots + a_0$ be a polynomial with $n \geq 1$, $a_n = (\alpha + i\beta) \neq 0$. Set $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. A ray $\arg z = \theta$ is said to be *critical for* $e^{P(z)}$ if $\delta(P, \theta) = 0$.

Definition 2. Let $P(z) = a_n z^n + \dots + a_0$ be a polynomial with $n \geq 0$. A ray $\arg z = \theta$ is said to be *critical for* $P(z)$ if $\arg a_n + (n+2)\theta = 0 \pmod{2\pi}$.

Remark. It is easily seen that a given polynomial $P(z)$ of degree $n \geq 1$ has $(n+2)$ critical rays which form $(n+2)$ sectors of opening $2\pi/(n+2)$. On the other hand,

$$\delta(P,\theta) = \alpha \cos n\theta - \beta \sin n\theta = 0$$

on the rays

$$\arg z = \theta_j := \tilde{\theta} + \frac{j}{n}\pi, \qquad j = 0, 1, \ldots, 2n-1,$$

which form $2n$ sectors of opening π/n for some $\tilde{\theta}$. Hence, there are $2n$ critical rays for $e^{P(z)}$. For later use, we denote by $S_1^+, \ldots, S_n^+$ (respectively $S_1^-, \ldots, S_n^-$) those open sectors where $\delta(P,\theta) > 0$ (respectively $\delta(P,\theta) < 0$), and denote further $S^+ = \bigcup_{i=1}^n S_i^+$, $S^- = \bigcup_{i=1}^n S_i^-$.

With these definitions, we have the following

Theorem 3. *Let $m \geq 2(n-1)$. Suppose that there exists a ray $\arg z = \theta_0$ such that it is critical for $e^{P(z)}$ but not for $Q(z)$. Then $\lambda(f) = \infty$ for any non-trivial solution f of* (1.3).

Remark. For $m > 2(n-1)$, Theorem 3 is equivalent to [5, Theorem 1.1], in the second order case, see also [8, Theorem 3.1.2]. The proof of [8, Theorem 3.1.2], also applies in the case $m = 2(n-1)$. For completeness, we will prove Theorem 3 below.

In the case $m > 2(n-1)$, we first recall [8, Theorem 3.1.1], as

Theorem D. *If $m > 2(n-1)$, then $n < \lambda(f) \leq \infty$ for any non-trivial solution f of* (1.3).

In this paper, we improve Theorem D by proving

Theorem 4. *If $m > 2(n-1)$, then, for any non-trivial solution f of* (1.3), *we have either $\lambda(f) = (m+2)/2$ or $\lambda(f) = \infty$.*

Remark. It remains open whether the case $\lambda(f) = (m+2)/2$ can really occur in Theorem 4.

Moreover, we also obtain a result for the case $m = 2(n-1)$.

Theorem 5. *If $m = 2(n-1)$, then, for any non-trivial solution f of* (1.3), *we have either f is zero-free, or $\lambda(f) = n$, or $\lambda(f) = \infty$.*

Remark. The following examples, along with [8, Theorem 3.1.4], show that all cases in Theorem 5 can occur.

Example 1. Let $q \geq 3$ be an odd number. Then the equation

$$f'' + \left(e^z - \tfrac{1}{16}q^2\right)f = 0$$

admits two linearly independent solutions f_1, f_2 with the property that $\lambda(f_1) = \lambda(f_2) = 1$. In this example, $P(z) = z$, $Q(z) = -q^2/16$, $n = 1$ and $b_m = -q^2/16$. Hence, $m = 2(n-1)$.

Example 2. Let n be a positive integer. Then, by Theorem A above, the equation

$$f'' + A(z)f = 0$$

with

$$A(z) = -\tfrac{1}{4}\big(e^{-2z^n} + n^2 z^{2(n-1)} + 2n(n-1)z^{n-2}\big)$$

admits two linearly independent zero-free solutions. In fact, we can rewrite $A(z)$ in the form

$$A(z) = e^{P(z)} - \tfrac{1}{16}\big(P'(z)\big)^2 + \tfrac{1}{4}P''(z),$$

where

$$P(z) = -2z^n - \log(-4).$$

We see immediately that $m = 2(n-1)$.

2. Two lemmas

Lemma 1 below, which can be deduced from Lemma 1 in [7], plays a key role in the proof of our theorems. Recalling Definition 2, we obtain

Lemma 1. *Let* $\arg z = \theta_0$ *be a critical ray for* $Q(z) = b_m z^m + b_{m-2} z^{m-2} + \cdots + b_0$, *where* $b_m \neq 0$ *and* $m \geq 2$. *Let* $Q_0(z)$ *be an entire function. Suppose that there exists* $\alpha > 0$ *such that in* $\{\,\theta_0 - \alpha < \arg z < \theta_0 + \alpha\,\}$,

$$Q_0(z) = b_m z^m + O(|z|^{m-2}).$$

Then there exists a path Γ_{θ_0} *tending to infinity in* $\{\,\theta_0 - \alpha < \arg z < \theta_0 + \alpha\,\}$ *such that on* Γ_{θ_0} *we have* $\arg z \to \theta_0$ *and all solutions of*

$$f'' + Q_0(z)f = 0$$

tend to zero as $z \to \infty$ *along* Γ_{θ_0}.

The following lemma is an easy modification of [2, Lemma 3]. Recalling the notions S^+ and S^- introduced in the remark below Definition 2, we have

Lemma 2. *Let* $P(z)$ *be a polynomial of degree* $n \geq 1$, *and let* $\varepsilon > 0$ *be a given constant. Let* $B(z) \not\equiv 0$ *be analytic for all* z *of sufficiently large modulus, and of order less than* n. *Consider the function* $A(z) := B(z)\exp\big(P(z)\big)$ *on a ray* $re^{i\theta}$. *Then there exists a set* $E_0 \subset [0, 2\pi)$ *with linear measure zero, such that*

(1) *If* $\theta \in S^+ \setminus E_0$, *there exists an* $r(\theta)$ *such that for* $r \geq r(\theta)$,

$$\big|A(re^{i\theta})\big| \geq \exp\big((1-\varepsilon)\delta(P,\theta)r^n\big).$$

(2) *If* $\theta \in S^-$, *there exists an* $r(\theta)$ *such that for* $r \geq r(\theta)$,

$$\big|A(re^{i\theta})\big| \leq \exp\big((1-\varepsilon)\delta(P,\theta)r^n\big).$$

3. Proofs of the theorems

First of all, we may assume, by a suitable transformation, that $Q(z) = b_m z^m + b_{m-2}z^{m-2} + \cdots + b_0$. We suppose that the equation (1.3) admits a non-trivial solution f_0 such that $\lambda(f_0) < \infty$. Therefore, by the Hadamard factorization theorem, we can write f_0 in the form

$$f_0(z) = \pi(z)e^{h(z)}, \tag{3.1}$$

where $h(z)$ is an entire function and $\pi(z)$ is the canonical product formed with the zeros of f_0, hence $\sigma(\pi) = \lambda(\pi) = \lambda(f_0) < \infty$, where and in what follows, $\sigma(f)$ denotes the order of growth of f. Moreover, for the function $h(z)$ in (3.1), we can infer from [6, pp. 96–98], see also [8, Chapter 3], that there exists a rational function $\widetilde{Q}(z)$ such that for any $\theta \in S^-$, we have

$$h'(re^{i\theta}) = \widetilde{Q}(re^{i\theta}) + O(r^{-1}), \tag{3.2}$$

as $r \to \infty$, while for any $\theta \in S_q^+$, $1 \le q \le n$,

$$h'(re^{i\theta}) = c_q \cdot e^{P(re^{i\theta})/2} + \widetilde{Q}(re^{i\theta}) + O(r^{-2}), \tag{3.3}$$

as $r \to \infty$, where c_q is a constant satisfying ${c_q}^2 + 1 = 0$.

On the other hand, denote

$$W(z) := \pi(z)e^{\frac{1}{4}P(z)+\int_a^z \widetilde{Q}(t)\,dt}, \tag{3.4}$$

$$G(z) := h(z) - \tfrac{1}{4}P(z) - \int_a^z \widetilde{Q}(t)\,dt. \tag{3.5}$$

Then, it follows immediately from (3.1) that

$$f_0(z) = W(z)e^{G(z)}. \tag{3.6}$$

Choose now a in (3.4) and (3.5) sufficiently large by modulus such that $\widetilde{Q}(z)$ has no poles in $|z| > r_0 = |a|$. Hence $W(z)$ is analytic in $|z| > r_0$. To finish the proofs, we quote two conclusions from [8, pp. 53–54] for later use.

Conclusion (i). For any $q = 1, 2, \ldots, n$, there exists a constant $J_q \neq 0, \infty$ such that

$$\lim_{r\to\infty} W(re^{i\theta}) = J_q, \qquad \theta \in S_q^+ \setminus \widetilde{E}_0, \tag{3.7}$$

where $\widetilde{E}_0$, not depending on q, is a set in $[0, 2\pi)$ with linear measure zero.

Conclusion (ii). $W(z)$ is of order no greater than $(m+2)/2$. (Here m is the degree of $Q(z)$.)

We are ready to finish the proof of our theorems.

Completion of the proof of Theorem 3. Under the assumption of Theorem 3, since $m \geq 2(n-1)$, there must exist two adjoining critical rays for $Q(z)$, say $\arg z = \theta_1$ and $\arg z = \theta_2$, such that $\theta_1 < \theta_0 < \theta_2$ and that $\delta(P, \theta_1) \cdot \delta(P, \theta_2) < 0$. Without loss of generality, we may assume that $\delta(P, \theta) < 0$ in $\theta_1 \leq \theta < \theta_0$, while $\delta(P, \theta) > 0$ if $\theta_0 < \theta \leq \theta_2$.

Consider now the domain Ω_1 bounded by the path Γ_{θ_1} (arising from Lemma 1 with $\alpha < \varepsilon$) and the ray $\arg z = \theta_0 + \varepsilon$, and contained in the sector $\theta_1 - \varepsilon < \arg z < \theta_0 + \varepsilon$. We choose $\varepsilon > 0$ such that $(\theta_0 + 2\varepsilon) < \theta_2$ and $(\theta_0 + \varepsilon) \notin \widetilde{E}_0$. Then $\Omega_1 \cap \{|z| > R\}$ is an unbounded domain contained in a sector of opening $\theta_0 - \theta_1 + 2\varepsilon < \theta_2 - \theta_1 = 2\pi/(m+2)$, provided R ($\geq r_0$) is large enough.

Now $W(z)$ is of order no greater than $(m+2)/2$ by the above conclusion (ii), and by (3.7), $W(z) \to J_q$ ($\neq 0, \infty$) along the ray $\arg z = \theta_0 + \varepsilon$ for some $1 \leq q \leq n$, while $W(z) \to 0$ along the path Γ_{θ_1} by Lemma 1. A standard application of the Phragmén–Lindelöf principle to the domain $\Omega_1 \cap \{|z| > R\}$ and the function $W(z)$ yields a contradiction immediately. This completes the proof of Theorem 3.

Completion of the proof of Theorem 4. Observing Theorem 3, we need only to consider the case that every critical ray for $e^{P(z)}$ is also critical for $Q(z)$. Hence, it follows immediately that $(m+2)$ must be an integer multiple of $2n$.

We first prove that $\sigma(W) = (m+2)/2$. In fact, by the above conclusion (ii), $\sigma(W) \leq (m+2)/2$. Therefore, we need only to show that the assumption $\sigma(W) < (m+2)/2$ will yield a contradiction. To this end, we pick two adjoining critical rays for $Q(z)$, say $\arg z = \theta'$ and $\arg z = \theta''$, such that $\theta' < \theta''$, that $\arg z = \theta''$ is also critical for $e^{P(z)}$ and that $\delta(P, \theta') < 0$. All these can be done since $m > 2(n-1)$.

Note that $\theta'' - \theta' = 2\pi/(m+2)$ and that $\sigma(W) < \frac{1}{2}(m+2)$, we can choose an $\varepsilon > 0$ such that $\sigma(W) < \pi/(\theta'' - \theta' + \varepsilon) < \frac{1}{2}(m+2)$ and $\theta'' + \frac{1}{2}\varepsilon \notin \widetilde{E}_0$. Again, as in the proof of Theorem 3 above, we consider the domain Ω' bounded by the path $\Gamma_{\theta'}$ (arising from Lemma 1) and the ray $\arg z = \theta'' + \frac{1}{2}\varepsilon$. By applying the same argument as in the proof of Theorem 3, we also get a contradiction. Therefore, $\sigma(W) = \frac{1}{2}(m+2)$.

Next, we will prove $\lambda(f_0) = (m+2)/2$. In fact, by (3.4), it follows that $\lambda(f_0) = \lambda(\pi) = \lambda(W) \leq (m+2)/2$, the order of $W(z)$. We now assume that $\lambda(f_0) < \frac{1}{2}(m+2)$. Since $m > 2(n-1)$, it follows from (3.4) that

$$W(z) = W_1(z) e^{\alpha_1 z^{(m+2)/2}}, \tag{3.8}$$

where $W_1(z)$ is an analytic function in $|z| > r_0$ with order less than $\frac{1}{2}(m+2)$, and α_1 is a non-zero constant.

Therefore, by Lemma 2 and the remark below Definition 2, there are $(m+2)$ sectors, each with opening $2\pi/(m+2)$, such that $W(z)$ tends to zero in every second of these sectors and to infinity in the remaining ones. On the other hand, as we know from the above conclusion (i), $W(z)$ tends to a non-zero constant along almost all radii in n sectors of total angular measure π. This results in a contradiction. Hence, $\lambda(f_0) = \frac{1}{2}(m+2)$, and we are done.

Completion of the proof of Theorem 5. Since $m = 2(n-1)$, it follows from the above conclusion (ii) that $\sigma(W) \leq n$. Hence, by (3.4), we have $\lambda(f_0) = \lambda(\pi) \leq n$. If now $\lambda(f_0) < n$, then f_0 must be zero-free by [3, Theorem 3.3]. This completes the proof of Theorem 5.

References

[1] Bank, S.: On the value distribution theory for entire solutions of second-order linear differential equations. - Proc. London Math. Soc. 50, 1985, 505–543.

[2] Bank, S., I. Laine, and J. Langley: On the frequency of zeros of solutions of second-order linear differential equations. - Resultate Math. 10, 1986, 8–24.

[3] Bank, S., I. Laine, and J. Langley: Oscillation results for solutions of linear differential equations in the complex domain. - Resultate Math. 16, 1989, 3–15.

[4] Bank, S., and J. Langley: On the oscillation of solutions of certain linear differential equations in the complex domain. - Proc. Edinburgh Math. Soc. 30, 1987, 455–469.

[5] Bank, S., and J. Langley: On the zeros of the solutions of the equation $w^{(k)} + (Re^P + Q)w = 0$. - Kodai Math. J. 13, 1990, 298–309.

[6] Laine, I.: Nevanlinna Theory and Complex Differential Equations. - W. Gruyter, 1993.

[7] Langley, J.: On complex oscillation and a problem of Ozawa. - Kodai Math. J. 9, 1986, 430–439.

[8] Wang, S.: On the sectorial oscillation theory of $f'' + A(z)f = 0$. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 92, 1994.

Received 13 April 1994